On the Lindelöf Property of Products of Spaces $C_p(X)$

Oleg Okunev

Benemérita Universidad Autónoma de Puebla

All spaces are assumed to be Tychonoff.

$C_p(X, Z) = \{f : f$ is a continuous function from X to $Z\}$ with the topology of pointwise convergence.

$C_p(X, Z) \subset Z^X$.

$C_p(X, \mathbb{R}) = C_p(X)$

All spaces are assumed to be Tychonoff.

$C_p(X,Z) = \{f : f$ is a continuous function from X to $Z\}$ with the topology of pointwise convergence.

$C_p(X,Z) \subset Z^X$.

$C_p(X,\mathbb{R}) = C_p(X)$

All spaces are assumed to be Tychonoff.

$C_p(X, Z) = \{f : f$ is a continuous function from X to $Z\}$ with the topology of pointwise convergence.

$C_p(X, Z) \subset Z^X$.

$C_p(X, \mathbb{R}) = C_p(X)$

Problem (A.V. Arhangel'skii, 198?) *Let X be a space such that $C_p(X)$ is Lindelöf. Must $C_p(X) \times C_p(X)$ be Lindelöf? What if X is compact?*

$C_p(X) \times C_p(Y) = C_p(X \oplus Y)$
$C_p(X) \times C_p(X) = C_p(X \oplus X) = C_p(X, \mathbb{R}^2)$.

$C_p(X)$ always has *ccc*, so
$C_p(X)$ is Lindelöf $\iff$ $C_p(X)$ is paracompact.

Still unknown:

If G is a Lindelöf topological group, must $G \times G$ be Lindelöf?

If G is a paracompact topological group, must $G \times G$ be paracompact?

Problem (A.V. Arhangel'skii, 198?) *Let X be a space such that $C_p(X)$ is Lindelöf. Must $C_p(X) \times C_p(X)$ be Lindelöf? What if X is compact?*

$$C_p(X) \times C_p(Y) = C_p(X \oplus Y)$$
$$C_p(X) \times C_p(X) = C_p(X \oplus X) = C_p(X, \mathbb{R}^2).$$

$C_p(X)$ always has *ccc*, so
$C_p(X)$ is Lindelöf $\iff$ $C_p(X)$ is paracompact.

Still unknown:

If G is a Lindelöf topological group, must $G \times G$ be Lindelöf?

If G is a paracompact topological group, must $G \times G$ be paracompact?

Problem (A.V. Arhangel'skii, 198?) *Let X be a space such that $C_p(X)$ is Lindelöf. Must $C_p(X) \times C_p(X)$ be Lindelöf? What if X is compact?*

$C_p(X) \times C_p(Y) = C_p(X \oplus Y)$
$C_p(X) \times C_p(X) = C_p(X \oplus X) = C_p(X, \mathbb{R}^2)$.

$C_p(X)$ always has *ccc*, so
$C_p(X)$ is Lindelöf $\iff$ $C_p(X)$ is paracompact.

Still unknown:

If G is a Lindelöf topological group, must $G \times G$ be Lindelöf?

If G is a paracompact topological group, must $G \times G$ be paracompact?

Problem (A.V. Arhangel'skii, 198?) *Let X be a space such that $C_p(X)$ is Lindelöf. Must $C_p(X) \times C_p(X)$ be Lindelöf? What if X is compact?*

$C_p(X) \times C_p(Y) = C_p(X \oplus Y)$
$C_p(X) \times C_p(X) = C_p(X \oplus X) = C_p(X, \mathbb{R}^2)$.

$C_p(X)$ always has *ccc*, so
$C_p(X)$ is Lindelöf $\iff$ $C_p(X)$ is paracompact.

Still unknown:

If G is a Lindelöf topological group, must $G \times G$ be Lindelöf?

If G is a paracompact topological group, must $G \times G$ be paracompact?

There are spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are Lindelöf, but $C_p(X) \times C_p(Y)$ is not.

First example by A. Leiderman and V. Malykhin (1988);

both X and Y with one non-isolated point, forcing.

K. Tamano, OO (1996): *There are a separable, scattered, σ-compact X and a countable Y such that $C_p(X)$ is Lindelöf, and $C_p(X) \times C_p(Y)$ is not.*

$w(C_p(Y)) = |Y|$,

so in this example $C_p(Y)$ is second-countable.

There are spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are Lindelöf, but $C_p(X) \times C_p(Y)$ is not.

First example by A. Leiderman and V. Malykhin (1988);

both X and Y with one non-isolated point, forcing.

K. Tamano, OO (1996): *There are a separable, scattered, σ-compact X and a countable Y such that $C_p(X)$ is Lindelöf, and $C_p(X) \times C_p(Y)$ is not.*

$w(C_p(Y)) = |Y|$,

so in this example $C_p(Y)$ is second-countable.

Still unknown:

Let X and Y be compact. Suppose $C_p(X)$ and $C_p(Y)$ are Lindelöf. Must $C_p(X) \times C_p(Y)$ be Lindelöf?

Let X be compact and Y countable. If $C_p(X)$ is Lindelöf, must $C_p(X) \times C_p(Y)$ be Lindelöf?

If $C_p(X)$ is Lindelöf, must $C_p(X) \times \omega^\omega$ be Lindelöf?

(equivalent: if $C_p(X)$ is Lindelöf, must $C_p(X \oplus \omega)$ be Lindelöf?)

Still unknown:

Let X and Y be compact. Suppose $C_p(X)$ and $C_p(Y)$ are Lindelöf. Must $C_p(X) \times C_p(Y)$ be Lindelöf?

Let X be compact and Y countable. If $C_p(X)$ is Lindelöf, must $C_p(X) \times C_p(Y)$ be Lindelöf?

If $C_p(X)$ is Lindelöf, must $C_p(X) \times \omega^\omega$ be Lindelöf?

(equivalent: if $C_p(X)$ is Lindelöf, must $C_p(X \oplus \omega)$ be Lindelöf?)

Still unknown:

Let X and Y be compact. Suppose $C_p(X)$ and $C_p(Y)$ are Lindelöf. Must $C_p(X) \times C_p(Y)$ be Lindelöf?

Let X be compact and Y countable. If $C_p(X)$ is Lindelöf, must $C_p(X) \times C_p(Y)$ be Lindelöf?

If $C_p(X)$ is Lindelöf, must $C_p(X) \times \omega^\omega$ be Lindelöf?

(equivalent: if $C_p(X)$ is Lindelöf, must $C_p(X \oplus \omega)$ be Lindelöf?)

Theorem (A. Arhangel'skii, E. Reznichenko, 198?). *Let X be compact zero-dimensional. If $C_p(X)$ is Lindelöf, then $C_p(X)^\omega$ is Lindelöf.*

Theorem (OO, 2011). *If* $\dim X = 0$ *and Z is a locally compact second-countable space, then $C_p(X, Z)$ is a continuous image of a closed subspace of $C_p(X)$.*

$C_p(X)^n = C_p(X, \mathbb{R}^n)$, so

Corllary (OO, 2011). *If* $\dim X = 0$ *and $C_p(X)$ is Lindelöf, then for every $n \in \omega$, $C_p(X)^n$ is Lindelöf.*

Theorem (A. Arhangel'skii, E. Reznichenko, 198?). *Let X be compact zero-dimensional. If $C_p(X)$ is Lindelöf, then $C_p(X)^\omega$ is Lindelöf.*

Theorem (OO, 2011). *If* $\dim X = 0$ *and Z is a locally compact second-countable space, then $C_p(X, Z)$ is a continuous image of a closed subspace of $C_p(X)$.*

$C_p(X)^n = C_p(X, \mathbb{R}^n)$, so

Corllary (OO, 2011). *If* $\dim X = 0$ *and $C_p(X)$ is Lindelöf, then for every $n \in \omega$, $C_p(X)^n$ is Lindelöf.*

Theorem (A. Arhangel'skii, E. Reznichenko, 198?). *Let X be compact zero-dimensional. If $C_p(X)$ is Lindelöf, then $C_p(X)^\omega$ is Lindelöf.*

Theorem (OO, 2011). *If $\dim X = 0$ and Z is a locally compact second-countable space, then $C_p(X, Z)$ is a continuous image of a closed subspace of $C_p(X)$.*

$C_p(X)^n = C_p(X, \mathbb{R}^n)$, so

Corllary (OO, 2011). *If $\dim X = 0$ and $C_p(X)$ is Lindelöf, then for every $n \in \omega$, $C_p(X)^n$ is Lindelöf.*

Open questions.

If $\dim X = 0$ *and* $C_p(X)$ *is Lindelöf, must* $C_p(X)^\omega$ *be Lindelöf?*

If $\operatorname{ind} X = 0$ *and* $C_p(X)$ *is Lindelöf, must* $C_p(X) \times C_p(X)$ *be Lindelöf?*

If $\operatorname{ind} X = 0$ *and* $C_p(X)$ *is Lindelöf, must* $\dim X = 0$*?*

Thank you!