A topological Ramsey space of infinite polyhedra and the random polyhedron

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The pigeon hole principle

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For every partition $\mathbb{N} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$ there exists $i \in \{1, \ldots, r\}$ such that $\mathcal{C}_i$ is infinite.

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Infinite version: For every finite coloring of $\mathbb{N}^{[2]}$ there is $A \in \mathbb{N}^{[\infty]}$ such that $A^{[2]}$ is monochromatic.

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Infinite version: For every finite coloring of $\mathbb{N}^{[2]}$ there is $A \in \mathbb{N}^{[\infty]}$ such that $A^{[2]}$ is monochromatic.

Finite version: Given $n, r \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that for every coloring $c : M^{[2]} \to r$, there is $A \in M^{[n]}$ tal que $A^{[2]}$ is monochromatic for c.

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Generalized infinite version: Given $n \in \mathbb{N}$, for every finite coloring of $\mathbb{N}^{[n]}$ there is $A \in \mathbb{N}^{[\infty]}$ such that $A^{[n]}$ is monochromatic.

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Generalized infinite version: Given $n \in \mathbb{N}$, for every finite coloring of $\mathbb{N}^{[n]}$ there is $A \in \mathbb{N}^{[\infty]}$ such that $A^{[n]}$ is monochromatic.

Generalized finite version: Given $m, n, r \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that for every coloring $c : M^{[n]} \to r$, there is $A \in M^{[m]}$ tal que $A^{[n]}$ is monochromatic for c.

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Let $cl(A)$ denote the class of A and define

$$X = \{A \in \mathbb{N}^{[\infty]} : |A \triangle B_{cl(A)}| \text{ is even}\}$$

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Ellentuck's theorem, 1974 X is Ramsey iff X has the Baire property (in Ellentuck's topology). X is Ramsey null iff X is meager (in Ellentuck's topology).

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Topological Ramsey Spaces – Todorcevic, 2010

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$(\mathcal{R}, \leq, r)$ is a **topological Ramsey space** if subsets of $\mathcal{R}$ with the Baire property are Ramsey and meager subsets of $\mathcal{R}$ are Ramsey null.

AXIOMS

(A.1) [Metrization]

(A.1.1) For any $A \in \mathcal{R}$, $r_0(A) = \emptyset$.

(A.1.2) For any $A, B \in \mathcal{R}$, if $A \neq B$ then $(\exists n)\ (r_n(A) \neq r_n(B))$.

(A.1.3) If $r_n(A) = r_m(B)$ then $n = m$ and $(\forall i < n)\ (r_i(A) = r_i(B))$.

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(A.1.3) If $r_n(A) = r_m(B)$ then $n = m$ and $(\forall i < n)\ (r_i(A) = r_i(B))$.

(A.2) [Finitization] There is a quasi order $\leq_{fin}$ on $\mathcal{AR}$ such that:

(A.2.1) $A \leq B$ iff $(\forall n)\ (\exists m)\ \ (r_n(A) \leq_{fin} r_m(B))$.

(A.2.2) $\{b \in \mathcal{AR} : b \leq_{fin} a\}$ is finite, for every $a \in \mathcal{AR}$.

(A.2.3) If $a \leq_{fin} b$ and $c \sqsubseteq a$ then there is $d \sqsubseteq b$ such that $c \leq_{fin} d$.

AXIOMS

(A.3) [Amalgamation] Given a and A with $\mathrm{depth}_A(a) = n$, the following holds:

(i) $(\forall B \in [n, A])\ ([a, B] \neq \emptyset)$.

(ii) $(\forall B \in [a, A])\ (\exists A' \in [n, A])\ ([a, A'] \subseteq [a, B])$.

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(A.4) [Pigeonhole Principle] For every $A \in \mathcal{R}$ and every $\mathcal{O} \subseteq \mathcal{AR}_1$ there is $B \leq A$ such that $\mathcal{AR}_1|B \subseteq \mathcal{O}$ or $\mathcal{AR}_1|B \subseteq \mathcal{O}^c$.

Abstract Ellentuck theorem

Abstract Ellentuck theorem. Any $(\mathcal{R}, \leq, r)$ with $\mathcal{R}$ metrically closed and satisfying (A.1)-(A.4) is a topological Ramsey space.

TRS of of infinite polyhedra, Mijares - Padilla, 2012/2013

Elements: Let $\mathcal{P}$ be the collection of pairs (X, S_X) such that:

1. $X \in \mathbb{N}^{[\infty]}$,
2. $S_X \subseteq X^{[<\infty]}$ is **hereditary**, i.e., $u \subseteq v \,\&\, v \in S_X \Rightarrow u \in S_X$, and
3. $\bigcup S_X = \bigcup\{u : u \in S_X\} = X$.

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Approximations: $r(n, (X, S_X)) = r_n(X, S_X) = (X \restriction n, S_X \restriction n)$.

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Theorem: (M-Padilla, 2013+) $(\mathcal{P}, \leq, r)$ is a TRS.

Subspaces, Mijares - Padilla, 2012/2013

Elements: Given $k \in \mathbb{N}$, let $\mathcal{P}(k)$ be the collection of pairs (X, S_X) such that:

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Theorem: (M-Padilla, 2013+) For each k, $(\mathcal{P}(k), \leq, r)$ is a TRS.

Ramsey classes

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$(F_j)_{j\in J}$ is a set of **function symbols**

Structures

L-**structure**, $\mathbb{A} =< A, (R_i^{\mathbb{A}})_{i\in I}, (F_j^{\mathbb{A}})_{j\in J} >$:

- A non empty set $A \neq \emptyset$ called the **universe** of the structure;
- a set of relations $(R_i^{\mathbb{A}})_{i\in I}$ where $R_i^{\mathbb{A}} \subseteq A^{n(i)}$ for each $i \in I$; and
- a set of functions $(F_j^{\mathbb{A}})_{j\in J}$ where $F_j^{\mathbb{A}} : A^{m(j)} \longrightarrow A$ for each $j \in J$.

Structures

A **morphism** of L-structures $\mathbb{A} \xrightarrow{\pi} \mathbb{B}$ is a map $A \xrightarrow{\pi} B$
• $\left(a_1, \ldots, a_{n(i)}\right) \in R_i^{\mathbb{A}}$ iff $\left(\pi\left(a_1\right), \ldots, \pi\left(a_{n(i)}\right)\right) \in R_i^{\mathbb{B}}$; for all $a_1, \ldots, a_{n(i)} \in A$.
• $\pi\left(F_j^{\mathbb{A}}\left(a_1, \ldots a_{m(j)}\right)\right) = F_j^{\mathbb{B}}\left(\pi\left(a_1\right), \ldots, \pi\left(a_{m(j)}\right)\right)$ for all $a_1, \ldots, a_{m(j)} \in A$.
When π is injective we say that it is an **embedding**. In particular, we say that $\mathbb{A}$ is a **substructure** of $\mathbb{B}$, and write $\mathbb{A} \leq \mathbb{B}$ whenever $A \subseteq B$ and the inclusion map is an embedding.

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5. Amalgamation property: Given $\mathbb{A}, \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{C}$ and embeddings $\mathbb{A} \xrightarrow{f_i} \mathbb{B}_i$, $i \in \{1, 2\}$, there is $\mathbb{D} \in \mathcal{C}$ and embeddings $\mathbb{B}_i \xrightarrow{g_i} \mathbb{D}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

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IIf $\mathcal{C}$ is Fraïssé , then there is a unique (up to isomorphism) countably infinite *ultrahomogeneous* structure $\mathbb{F}$ such that

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Extreme amenability

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Theorem

(Pestov 2006) Let $\mathbb{F}$ be a countably infinite ultrahomogeneous structure and $\mathcal{C} = \mathrm{Age}(\mathbb{F})$. The polish group $\mathrm{Aut}(\mathbb{F})$ is extremely amenable if and only if $\mathcal{C}$ has the Ramsey property and all the structures of $\mathcal{C}$ are rigid.

Finite polyhedra as a Ramsey class

Consider $L =< (R_i)_{i \in \mathbb{N} \setminus \{0\}} >$, a signature with an infinite number of relational symbols such that for each $i \in \mathbb{N}$ the arity of R_i is $n(i) = i$.

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Facts:

- $\mathcal{AP} \subseteq \mathcal{KP}$.

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Facts:

- $\mathcal{AP} \subseteq \mathcal{KP}$.
- For every $\mathbb{A} \in \mathcal{KP}$ there is $(a, S_a) \in \mathcal{AP}$ such that $\mathbb{A} \cong (a, S_a)$. Actually, $\mathcal{KP}$ is the closure of $\mathcal{AP}$ under isomorphisms.

Finite polyhedra as a Ramsey class

Theorem

The class $\mathcal{KP}$ of all finite ordered polyhedra is Ramsey.

Corollary

Let $\mathbb{P} = \mathrm{FLim}(\mathcal{KP})$, the Fraïssé limit of $\mathcal{KP}$. Then, $\mathrm{Aut}(\mathbb{P})$ with the Polish topology inherited from S_∞ is extremely amenable.

The random polyhedron

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(ω, S_ω) is an infinite *random* polyhedron.

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Lemma

Each finite polyhedron can be embedded in ***the*** *infinite random polyhedron.*

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Lemma

*Each finite polyhedron can be embedded in **the** infinite random polyhedron.*

Theorem

Let $\mathbb{P} = \mathrm{FLim}(\mathcal{KP})$, *the Fraïssé limit of* $\mathcal{KP}$. *Then* $\mathbb{P}$ *is an infinite ordered polyhedron which is isomorphic to* (ω, S_ω), *as a polyhedron, and to* $(\mathbb{Q}, \leq)$, *as an ordered set.*

References

- Carlson T. J. , Simpson S. G., *Topological Ramsey theory*. In: Neŝetr̂il, J., Rödl (eds.) Mathematics of Ramsey Theory, pp. 172–183, Springer, Berlin, 1990.
- Carlson T. J., Simpson S. G., *A dual form of Ramsey's theorem*. Advan. in Math. 53, 265–290 (1984).
- Dobrinen, N., Todorcevic, S., *A new class of Ramsey-classification theorems and their applications in the Tukey theory of ultrafilters*. Parts 1 and 2. To appear in TAMS.
- Ellentuck E., *A new proof that analytic sets are Ramsey*. J. Symbol. Logic 39, 163–165 (1974).

References

- Galvin F., Prikry K., *Borel sets and Ramsey's theorem*. J. Symbol. Logic 38, 193–198 (1973).
- Hindman, N., *The existence of certain ultrafilters on* $\mathbb{N}$ *and a conjecture of Graham and Rothschild*, Proc. Amer. Math. Soc., **36**(1973), 341–346.
- Kechris A. S., Pestov V. G. and Todorcevic S., *Fraïssé Limits, Ramsey Theory, and topological dynamics of automorphism groups*, Geom. and Funct. Analysis, Volume **15**, Number 1, 106-189 (2005).
- Milliken K. , *Ramsey's theorem with sums or unions*. J. Combin. Theory Ser. A 18, 276–290 (1975).

References

- Mijares, J. G., *A notion of selective ultrafilter corresponding to topological Ramsey spaces*, Math. Log. Quart. 53(3), 255–267 (2007).
- Mijares, J. G., *Parametrizing the abstract Ellentuck theorem*, *Discrete Mathematics* 307 (2007), 216-225.
- Mijares J. G. , *On Galvin's lemma and Ramsey spaces*. *Ann. Comb.* 16 (2012), no. 2, 319-330.
- Mijares J. G., Nieto J., *A parametrization of the abstract Ramsey theorem*. Divulgaciones Matemáticas. Vol. 16, No. 2, (2008) ,259-274

References

- Mijares J., Padilla, G. *A Ramsey space of infnite polyhedra and the random polyhedron.* arXiv:1209.6421 (2013)
- Pestov, V. *Dynamics of infinite-dimensional groups. The Ramsey-Dvoretzky-Milman Phenomenon*, ULECT **40**, AMS (2006).
- Ramsey F. P. , On a problem of formal logic. Proc. London Math. Soc. Ser. 2. 30, 264-286 (1929).
- Todorcevic S., *Introduction to Ramsey spaces*. Princeton University Press, Princeton, New Jersey, 2010.