On universal topologicals groups

Brice Rodrigue Mbombo

IME-USP
Joint work with Vladimir Pestov

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Universal topological group

- Let $\mathcal{C}$ be a class of topologicals groups.
- $G \in \mathcal{C}$ is universal for $\mathcal{C}$ if for every $H \in \mathcal{C}$ there is an isomorphism between H and a subgroup of G

Ulam's Question, 1935

Ulam

Does there exist a universal topological group with a countable base?

Remind

Teleman

- Every topological group G has a topologically faithful representation on a Banach space B: embedding $G \longrightarrow Iso(B)$
- Every topological group G has a topologically faithful representation on a compact space X: embedding $G \longrightarrow \mathrm{Homeo}(X)$.

1st Proof

$B = RUCB(G)$, $X =$ the unit ball in $B^\star$ with $w^\star$-topology.

$RUCB(G)$=Right Uniformly Continuous Bounded functions $f : G \longrightarrow \mathbb{C}$.

2nd Proof

$X = \mathcal{S}(G)$ and $B = C(X)$

$\mathcal{S}(G)$= the maximal ideal space of the abelian unital $C^\star$-algebra $RUCB(G)$ = the Samuel compactification of $(G, \mathcal{U}_R)$

1st solution: Uspenskij, 1986

$\mathrm{Homeo}([0,1]^{\aleph_0})$ contains all groups with a countable base.

$$G \hookrightarrow \mathrm{Homeo}(X) \hookrightarrow \mathrm{Homeo}(P(X)) = \mathrm{Homeo}([0,1]^{\aleph_0})$$

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2nd solution: Uspenkij, 1990

$Iso(\mathbb{U})$ contains all groups with a countable base, where $\mathbb{U}$ is the universal polish Urysohn space.

$\mathbb{U}$ = the Urysohn universal metric space = the complete separable space, contains isometric copies of all separable spaces, and is ultrahomogeneous =that is every isometry between two finite metric subspace of $\mathbb{U}$ extends to a global isometry of $\mathbb{U}$ onto itself.

Let X be a polish space

Katětov

Construct $X = X_0 \hookrightarrow X_1 \hookrightarrow \ldots$, with $w(X) = w(X_0) = w(X_1) = \ldots$ and $X_\omega = \overline{\bigcup X_n} = \mathbb{U}$ is the Urysohn space.

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Uspenskij

$Iso(X) = Iso(X_0) \hookrightarrow Iso(X_1) \hookrightarrow \ldots$, whence $G \hookrightarrow Iso(X) \hookrightarrow Iso(X_\omega) = Iso(\mathbb{U})$.

Solutions of Ulam Question

3rd solution: Ben Yaacov, 2012

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A Gurarij space is a Banach space $\mathbb{G}$ having the property that for any $\varepsilon > 0$, finite-dimensional Banach space $E \subseteq F$ and an isometric embedding $\varphi : E \longrightarrow \mathbb{G}$ there is a linear map $\psi : F \longrightarrow \mathbb{G}$ extending φ such that in addition, for all $x \in F$, $(1-\varepsilon)\|x\| \leq \|\psi(x)\| \leq (1+\varepsilon)\|x\|$

Solutions of Ulam Question

Ben Yaacov

Let E be a separable Banach space,

- Construct $E = E_0 \hookrightarrow E_1 \hookrightarrow \ldots$, with $w(E) = w(E_0) = w(E_1) = \ldots$ and $E_\omega = \overline{\bigcup E_n} = \mathbb{G}$ is the Gurarij space.

Solutions of Ulam Question

Ben Yaacov

Let E be a separable Banach space,

- Construct $E = E_0 \hookrightarrow E_1 \hookrightarrow ...$, with $w(E) = w(E_0) = w(E_1) = ...$ and $E_\omega = \overline{\bigcup E_n} = \mathbb{G}$ is the Gurarij space.
- $Iso(E) = Iso(E_0) \hookrightarrow Iso(E_1) \hookrightarrow ...$, whence $G \hookrightarrow Iso(E) \hookrightarrow Iso(E_\omega) = Iso(\mathbb{G})$.

Non-separable case

The question of existence of a universal topological group of a given uncountable weight $\mathfrak{m}$ remains open to the day. In fact, it is open for *any* given cardinal $\mathfrak{m} > \aleph_0$.

The Urysohn-Katětov space

Let $\mathfrak{m}$ be an infinite cardinal such that

$$\sup \{\mathfrak{m}^{\mathfrak{n}} : \mathfrak{n} < \mathfrak{m}\} = \mathfrak{m}, \tag{1}$$

there exists a unique up to an isometry complete metric space $\mathbb{U}_{\mathfrak{m}}$ of weight $\mathfrak{m}$, such that

- $\mathbb{U}_{\mathfrak{m}}$ contains an isometric copy of every other metric space of weight $\leq \mathfrak{m}$
- $\mathbb{U}_{\mathfrak{m}}$ is $< \mathfrak{m}$-homogeneous, that is, an isometry between any two metric subspaces of density $< \mathfrak{m}$ extends to a global self-isometry of $\mathbb{U}_{\mathfrak{m}}$.

In particular, $\mathbb{U}_{\aleph_0}$ is just the classical Urysohn space $\mathbb{U}$.

The ideal candidate

The topological group $Iso(\mathbb{U}_{\mathfrak{m}})$, equipped with the topology of simple convergence, has weight $\mathfrak{m}$, and was a candidate for a universal topological group of weight $\mathfrak{m}$.

SIN and FSIN groups

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- A topological group G is call SIN (Small Invariant Neighbourhoods)if it admits a base at the identity consisting of invariant neighborhoods.
- A topological group G is called functionally balanced, or sometimes FSIN ("Functionally SIN") if every right uniformly continuous bounded function on G is left uniformly continuous.
- Every SIN group is FSIN. The converse implication has been established for:
  - locally compact groups (Itzkowitz),
  - metrizable groups (Protasov),
  - locally connected groups (Megreslishvili, Nickolas and Pestov),

  among others
- It remains an open problem in the general case.

Property (OB)

- A topological group G has *property* (OB) if whenever G acts by isometries on a metric space $(X; d)$ every orbit is bounded.
- Examples of such groups include, among others:
  - the infinite permutation group S_∞ (Bergman),
  - the unitary group $U(\ell^2)$ with the strong opererator topology (Atkin),
  - the isometry group of the Urysohn sphere (that is, a sphere in the Urysohn space) (Rosendal)
  - $\mathrm{Homeo}([0,1]^{\aleph_0})$ (Rosendal).

Theorem

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Corollary

If G is a topological subgroup of $Iso(\mathbb{U}_{\mathfrak{m}})$ of density $< \mathfrak{m}$ having property (OB) which is either metrizable or locally connected, then G is a SIN group.

Some groups with no-embeddings in $Iso(\mathbb{U}_{\mathfrak{m}})$

1. Denote $\mathbb{U}^{\bigcirc}$ the unit sphere in the Urysohn metric space. The group $Iso(\mathbb{U}^{\bigcirc})$ is both metrizable and locally connected (Melleray), has property (OB) (Rosendal) and is not SIN.
2. The group S_∞ is Polish and has the property (OB) (Bergman).
3. The group $\mathrm{Homeo}([0,1]^{\aleph_0})$ is a non-SIN Polish group with property (OB)(Rosendal).
4. In particular the group $Iso(\mathbb{U})$ admits no embedding into $Iso(\mathbb{U}_{\mathfrak{m}})$ as a topological subgroup.

open questions

- Let κ be an uncountable cardinal. Does there exist a universal topological group of weight κ?
- Is $\mathrm{Homeo}([0,1]^\kappa)$ such a group?

Theorem

Every metrizable SIN group of weight $\leq \mathfrak{m}$ *embeds into Iso*$(\mathbb{U}_{\mathfrak{m}})$

Theorem

Every $P_{\mathfrak{m}}$*-groups of weight* $\mathfrak{m}$ *embeds into Iso*$(\mathbb{U}_{\mathfrak{m}})$.