Tukey Domination

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1. Tukey Order
   - Examples and background
   - Cardinals
2. Calibers
3. Application
   - $C_p(X)$
   - More topological version

Tukey Order

Definition

Let D and E be directed posets:

- $\psi : D \to E$ Tukey map iff ψ maps unbounded sets to unbounded sets;
- $\phi : E \to D$ cofinal map iff ϕ maps cofinal sets to cofinal sets.

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Lemma (Tukey)

$\exists$ *Tukey map* $D \to E \quad \iff \quad \exists$ *cofinal map* $E \to D$.

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Lemma (Tukey)

$\exists$ Tukey map $D \to E \iff \exists$ cofinal map $E \to D$.

- Write: $E \geq_T D$.
- Say: E Tukey-dominates D.
- $\geq_T$ is a partial order.

Tukey Order – Examples

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Theorem (Todorcevic)

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$$1, \quad \omega, \quad \omega_1, \quad \omega \times \omega_1, \quad [\omega_1]^{<\omega}.$$

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Theorem (Todorcevic)

(CH) There are $2^{\aleph_1}$ Tukey classes of size $\leq \aleph_1$.

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NULL $\geq_T$ *MGR*

$$\text{add(NULL)} \leq \text{add(MGR)}$$

$$\text{cof(NULL)} \geq \text{cof(MGR)}$$

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Lemma

$\mathcal{K}(M) \geq_T \mathcal{K}(X)$ *iff there is cofinal and order preserving*
$$\phi : (\mathcal{K}(M), \subseteq) \to (\mathcal{K}(X), \subseteq).$$

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Lemma

If $\mathcal{K}(M) \geq_T \kappa$ *and* κ *is regular, then* $\kappa \leq \mathfrak{c}$.

Cardinals continued...

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Let B be Bernstein set of size $\mathfrak{c}$, $B = \{x_\alpha : \alpha < \mathfrak{c}\}$.
Let $B_\kappa = \{x_\alpha : \alpha < \kappa\}$.

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$\phi : \mathcal{K}(B_\kappa) \to \kappa$ defined by $\phi(K) = sup\{\alpha : x_\alpha \in K\}$ works.

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We can also arrange for all $\mathcal{K}(B_\kappa) \geq_T \omega_1$.

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Example: $\omega_1, \kappa \in Spec(B_\kappa)$.

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Q: What are possible consistent versions of $Spec(\mathbb{N}^{\mathbb{N}})$?

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Lemma

$\mathcal{K}(\mathbb{N}^{\mathbb{N}})$ has caliber ω_1 $\iff$ $\omega_1 < \mathfrak{b}$.

Calibers – general

Definition

A poset P has caliber $(\theta, \lambda, \kappa)$ iff any $S \subseteq P$ of size θ can be refined to S' of size λ so that any $S'' \subseteq S'$ of size κ is bounded in P.

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Lemma

If $E \geq_T D$ and E has caliber $(\theta, \lambda, \kappa)$, then so does D.

Lemma (caliber $(\omega_1, \omega, \omega)$)

For any M, $\mathcal{K}(M)$ has caliber $(\omega_1, \omega, \omega)$.

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$\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$ extensively studied by Cascales, Orihuela, Tkachuk.

Reconciling terminology:

- "$\mathcal{K}(M) \geq_T \mathcal{K}(Y)$" corresponds to "$Y$ is M-strongly dominated".

$C_p(X)$ – COT theorems

Theorem (Tkachuk)

If $\mathcal{K}(\mathbb{N}^\mathbb{N}) \geq_T \mathcal{K}(C_p(X))$ *then* X *is countable and discrete.*

Theorem (Cascales, Orihuela, Tkachuk)

Under CH, if X *is compact and* $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$*, then* X *is countable.*

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Question (COT): Is it true in ZFC that if X is compact and $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$, then X is countable?

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Lemma (COT)

Suppose X is compact and $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$. If X is not countable then $\Sigma_*(\mathbb{R}^\kappa)$ is closely embedded into $C_p(X)$ for some uncountable κ and therefore $\mathcal{K}(M) \geq_T \mathcal{K}(\Sigma_*(\mathbb{R}^\kappa))$.

"Yes" to Q (COT)

Theorem

In ZFC, if X is compact and $\mathcal{K}(M) \geq_T \mathcal{K}(C_p(X))$, then X is countable.

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Proof.

Suppose not and X is uncountable.
Then by the last lemma $\mathcal{K}(M) \geq_T \mathcal{K}(\Sigma_*(\mathbb{R}^\kappa))$.

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We know that $\mathcal{K}(M)$ has caliber$(\omega_1, \omega, \omega)$,
so $\mathcal{K}(\Sigma_*(\mathbb{R}^\kappa))$ should also have caliber$(\omega_1, \omega, \omega)$.

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so $\mathcal{K}(\Sigma_*(\mathbb{R}^\kappa))$ should also have caliber$(\omega_1, \omega, \omega)$.

But it does not: we can assume $\kappa = \omega_1$ and construct an uncountable collection of compact sets in $\Sigma_*(\mathbb{R}^{\omega_1})$ so that no countable subcollection is contained in a compact set of $\mathbb{R}^{\omega_1}$. □

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X compact:
$C_p(X)$ Lindelöf Σ $\iff$ X Gul'ko compact.

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Thank You!