Reflection theorems for cardinal functions and cardinal arithmetic

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- Reflection of a topological property P: if P is satisfied by X, then P is satisfied by some "small" subspace of X.
- "Small" subspaces: cardinality, first category, closure of discrete subspaces.

Reflection for cardinal functions

Definition (Hodel and Vaughan, 2000)

Let ϕ be a cardinal function, κ an infinite cardinal and S a class of topological spaces.

Definition (ϕ reflects κ for the class S)

If $X \in S$ and $\phi(X) \geq \kappa$, then there exists $Y \subset X$ with $|Y| \leq \kappa$ and $\phi(Y) \geq \kappa$.

If S is the class of all topological spaces, then we can say that "ϕ reflects κ".

In the above definition, the "small" subspaces are those of cardinality $\leq \kappa$.

If $S = \{X\}$, then we will say "ϕ reflects κ for X".

Theorem (Hajnal and Juhász, 1980)

w reflects all infinite cardinals.

In particular, w reflects ω_1: if all subspaces of X of cardinality $\leq \omega_1$ are second-countable, then X is second-countable.

Cardinal function Lindelöf degree

Definition

The Lindelöf degree of a space X ($L(X)$) is the least infinite cardinal κ for which every open cover of X has a subcover of cardinality $\leq \kappa$.

Definition

The linear Lindelöf degree of a space X ($ll(X)$) is the least infinite cardinal κ for which every increasing open cover of X has a subcover of cardinality $\leq \kappa$.

A space X is linearly Lindelöf if and only if $ll(X) = \omega$.

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Theorem (Hodel and Vaughan, 2000)

1. *L reflects every successor cardinal;*
2. *L reflects every singular strong limit cardinal for the class of Hausdorff spaces;*
3. *(GCH + there are no inaccessible cardinals) L reflects all infinite cardinals for the class of Hausdorff spaces.*

Problem

Does L reflect the (strongly or weakly) inaccessible cardinals?

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Given an open cover C of a space X, define
$m(C) := \min\{|S| : S \text{ is a subcover of } C\}$.

Definition

$IS(X) := \{m(C) : C \text{ is an open cover of } X\}$.

Theorem

1. $L(X) = \omega + \sup IS(X)$;
2. $ll(X) = \omega + \sup(IS(X) \cap REG)$.

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Definition (Abraham and Magidor, 2010)

Given cardinals μ, η and λ, with $\mu \geq \eta \geq \lambda \geq \omega$, $cov(\mu, \eta, \lambda)$ is the least cardinality of a $X \subset [\mu]^{<\eta}$ such that, for every $a \in [\mu]^{<\lambda}$, there is a $b \in X$ with $a \subset b$.

Theorem

Let X be a topological space, and κ be a weakly inaccessible cardinal. L reflects κ for X, if for every infinite cardinal $\lambda < \kappa$, there is some $\mu \in IS(X)$, with $\mu > \lambda$, such that $cov(\mu, \mu, \lambda^+) \leq \kappa$.

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Corollary

L reflects every strongly inaccessible cardinal.

Proof: $cov\left(\mu, \mu, \lambda^{+}\right) \leq cov\left(\mu, \lambda^{+}, \lambda^{+}\right) \leq \mu^{\lambda} \leq \kappa$.

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Under GCH, L reflects every weakly inaccessible cardinal.

Definition (wH)

For every weakly inaccessible cardinal κ and every infinite cardinal $\lambda < \kappa$, $|\Theta_{\lambda,\kappa}| < \kappa$, where

$$\Theta_{\lambda,\kappa} = \{\mu \in CARD : \lambda < \mu < \kappa, cov(\mu, \mu, \lambda^+) > \kappa\}.$$

Under wH, L reflects every weakly inaccessible cardinal.
wH follows from GCH, and we will see that the negation of wH (if consistent with ZFC) requires large cardinals.

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Denote by *FIX* the class of all infinite cardinals that are fixed points of the aleph function ($\aleph_\kappa = \kappa$).

If κ is a weakly inaccessible cardinal, then $\kappa \in FIX$ and $|\kappa \cap FIX| = \kappa$.

By PCF Theory (Shelah), cardinal arithmetic is relatively "well-behaved" for cardinals that are not fixed points:

Lemma (easily follows from other results of the PCF Theory)

If $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$, then $cov(\aleph_\delta, \aleph_\delta, \lambda) < \aleph_{|\delta|^{++++}}$ for any $\lambda < \aleph_\delta$.

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Theorem

$\Theta_{\lambda,\kappa} \subset FIX \setminus REG.$

Corollary

If X is a consistent counterexample (reflection for L and a inaccessible cardinal), then "almost all" elements of $IS(X)$ are singular fixed points.

Corollary

If X is a consistent counterexample, then $ll(X) < L(X)$.

Definition (SSH (Shelah's strong hypothesis))

For all singular cardinals μ, $pp(\mu) = \mu^+$.

Definition (SWH (Shelah's weak hypothesis))

For every infinite cardinal κ, $|\{\mu < \kappa : cf(\mu) < \mu, pp(\mu) \geq \kappa\}| \leq \aleph_0$.

SSH implies SWH and SCH (Singular Cardinal Hypothesis).
SSH is implied by GCH, and also by "$0^\sharp$ does not exist".

Theorem (Shelah)

SSH is equivalent to the following: given cardinals μ and λ, with $\mu \geq \lambda = cf(\lambda) \geq \aleph_1$,

- *$cov(\mu, \lambda, \lambda) = \mu$ if $cf(\mu) \geq \lambda$;*
- *$cov(\mu, \lambda, \lambda) = \mu^+$, otherwise.*

Theorem

SSH implies wH (because $\Theta_{\lambda,\kappa} = \emptyset$ under SSH).

To have any $\Theta_{\lambda,\kappa} \neq \emptyset$, we need "$0^\sharp$ exists".

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- $cov(\mu, \lambda, \lambda) = \mu$ *if* $cf(\mu) \geq \lambda$;
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Reflection theorems for L (Lindelöf degree)

Lemma (follows from other results of Shelah)

If μ is a singular strong limit cardinal, then $2^\mu = cov\left(\mu, \mu, (cf(\mu))^+\right)$.

Theorem (Gitik, 2005)

Let μ be the first fixed point of the aleph function:

$$\mu = \sup\left\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\right\}.$$

Assuming some large cardinals, we can have GCH below μ, and 2^μ arbitrarily large.

Corollary

Let κ be a weakly inaccessible cardinal, and $\lambda < \mu$. It is consistent (assuming large cardinals) that $\mu \in \Theta_{\lambda,\kappa}$.

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Definition (SWH$_\lambda$)

For every infinite cardinal κ, $|\{\mu < \kappa : cf(\mu) < \mu, pp(\mu) \geq \kappa\}| \leq \lambda$.

Theorem (Gitik 2013)

The negation of SWH (SWH$_{\aleph_0}$) is consistent with ZFC.

In the same paper, there is a problem that can be expressed in this way:

Problem

Is the negation of SWH$_\lambda$ consistent with ZFC, when $\lambda > \aleph_0$?

Definition (SWH_λ)

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Is the negation of SWH_λ consistent with ZFC, when $\lambda > \aleph_0$?

Theorem (Shelah)

$pp(\mu) \leq cov\left(\mu, \mu, (cf(\mu))^{+}\right)$ *for every singular cardinal* μ*; and*
$pp(\mu) = cov\left(\mu, \mu, (cf(\mu))^{+}\right)$ *if* μ *is not a fixed point.*

Problem (Shelah)

$pp(\mu) = cov\left(\mu, \mu, (cf(\mu))^{+}\right)$ *for every singular cardinal* μ*?*

Definition ($SWH_{cov,\lambda}$)

For every infinite cardinal κ,

$$\left|\left\{\mu < \kappa : cf(\mu) < \mu, cov\left(\mu, \mu, (cf(\mu))^+\right) \geq \kappa\right\}\right| \leq \lambda.$$

For every λ, SSH$\Rightarrow SWH_{cov,\lambda} \Rightarrow SWH_\lambda$.
$SWH_{cov,\lambda}$ and SWH_λ may be equivalent.

Theorem

$SWH_{cov,\lambda}$ implies wH for every λ smaller than the first weakly inaccessible cardinal.

Considering the problems above, $SWH_{cov,\lambda}$ may be a theorem in ZFC for $\lambda > \aleph_0$.

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References

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