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On some Lindelöf-like properties

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1. Introduction

S – a set. $\mathfrak{S} \subseteq \mathcal{P}(S)$ is a **texturing** of S (and S is said to be textured by $\mathfrak{S}$) if

(1) $(\mathfrak{S}, \subseteq)$ is a complete lattice containing S and $\emptyset$, and

$\bigwedge_{j \in J} A_j = \bigcap_{j \in J} A_j$, $(A_j \in \mathfrak{S},\ j \in J)$

$\bigvee_{j \in J} A_j = \bigcup_{j \in J} A_j$, $(A_j \in \mathfrak{S},\ j \in J)$ for all finite J.

(2) $\mathfrak{S}$ is completely distributive.

(3) $\mathfrak{S}$ separates the points of S (i.e. given $s_1 \neq s_2$ in S there is $A \in \mathfrak{S}$ containing only one of these two points).

We call $(S, \mathfrak{S})$ a **texture space** (M. Brown, 1980)

- In a texture, arbitrary joins need not coincide with unions, and clearly this will be so if and only if $\mathfrak{S}$ is closed under arbitrary unions. In this case $(S, \mathfrak{S})$ is said to be **plain**.

- A mapping $\sigma : \mathfrak{S} \to \mathfrak{S}$ satisfying $\sigma(\sigma(A)) = A$, $\forall\, A \in \mathfrak{S}$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$, $\forall\, A, B \in \mathfrak{S}$ is called a **complementation** on $(S, \mathfrak{S})$ and $(S, \mathfrak{S}, \sigma)$ is then said to be a **complemented texture**.

Examples

1. For any set X, $(X, \mathcal{P}(X), \pi_X)$ is the complemented *discrete texture* representing the usual set structure of X. Here the complementation $\pi_X(Y) = X \setminus Y$, $Y \subseteq X$, is the usual set complementation.

2. For $I = [0,1]$ define $\mathfrak{I} = \{[0,t] \mid t \in [0,1]\} \cup \{[0,t) \mid t \in [0,1]\}$. $(I, \mathfrak{I}, \iota)$ is a complemented texture, called the *unit interval texture*.

3. The texture $(L, \mathfrak{L})$ is defined by $L = (0,1]$ and $\mathfrak{L} = \{(0,r] \mid r \in [0,1]\}$.

Two classical selection principles

$\mathcal{A}$ and $\mathcal{B}$ – sets of families of subsets of an infinite set X.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$:

For each sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subset A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$:

For each sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

Game Theory

$\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$: in the n-th round ONE chooses a set $A_n \in \mathcal{A}$; TWO responds by a finite $B_n \subset A_n$. The play $(A_1, B_1, \cdots, A_n, B_n, \cdots)$ is won by TWO if and only if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$\mathsf{G}_1(\mathcal{A},\mathcal{B})$: in the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO $b_n \in A_n$. TWO wins a play $(A_1, b_1; \cdots; A_n, b_n; \cdots)$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

2. Selection properties of texture spaces

Let $(S, \mathfrak{S})$ be a texture. A subset $\mathcal{C}$ of $\mathfrak{S}$ is said to be a **cover** of a set $A \subset S$ if $A \subset \bigvee \mathcal{C}$; if $\bigvee \mathcal{C} = S$, then $\mathcal{C}$ is said to be a cover of S. By $\mathbb{C}$ (or $\mathbb{C}_S$ when it is necessary) we denote the family of all covers of S.

Definition. A texture space $(S, \mathfrak{S})$ is said to be **Menger** (**Rothberger**) if S satisfies the selection property $\mathsf{S}_{fin}(\mathbb{C}, \mathbb{C})$ ($\mathsf{S}_1(\mathbb{C}, \mathbb{C})$).

Textures having the Menger property satisfies: each cover of S has a countable subcover. This property will be called the **Lindelöf property**. For this reason we assume that all covers of a texture are countable.

$(S, \mathfrak{S})$ is a texture iff $(S, \mathfrak{S}^c)$ is a C-space (T_0 topological space with a completely distributive lattice of open sets)

Proposition. $(S, \mathfrak{S})$ has the Menger property implies that the topological space $(S, \mathfrak{S}^c)$ has the weak Menger property.

For a texture $(S, \mathfrak{S})$ the symbol $\mathbb{C}_\Omega$ denotes the collection of all covers $\mathcal{C}$ of S with the property:

> For each k and each partition $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$ there is an $i \leq k$ with $\mathcal{C}_i \in \mathbb{C}$.

Theorem The following are equivalent in a texture space $(S, \mathfrak{S})$:

(1) $\mathsf{S}_{fin}(\mathbb{C}, \mathbb{C})$ holds;

(2) $\mathsf{S}_{fin}(\mathbb{C}_\Omega, \mathbb{C})$ holds.

Theorem. For a texture space $(S, \mathfrak{S})$ TFAE:

(1) S has the Menger property $\mathsf{S}_{fin}(\mathbb{C}, \mathbb{C})$;

(2) ONE does not have a winning strategy in the game $\mathsf{G}_{fin}(\mathbb{C}, \mathbb{C})$ on S.

Ramsey theoretic approach

Recall the following notion in Ramsey theory, called the *Baumgartner-Taylor partition relation*. For each positive integer k,

$$\mathcal{A} \rightarrow \lceil \mathcal{B} \rceil_k^2$$

denotes the following statement:

> For each A in $\mathcal{A}$ and for each function $f : [A]^2 \rightarrow \{1, \cdots, k\}$ there are a set $B \in \mathcal{B}$ with $B \subset A$, a $j \in \{1, \cdots, k\}$, and a partition $B = \bigcup_{n<\infty} B_n$ of B into pairwise disjoint finite sets such that for each $\{a, b\} \in [B]^2$ for which a and b are not from the same B_n, we have $f(\{a, b\}) = j$.

Call a texture space $(S, \mathfrak{S})$ *ω-Lindelöf* if each $\mathcal{C} \in \mathbb{C}_\Omega$ has a countable $\mathcal{C}' \subset \mathcal{C}$ with $\mathcal{C}' \in \mathbb{C}_\Omega$.

Theorem. Let $(S, \mathfrak{S})$ be an ω-Lindelöf texture space. Then (1) $\Rightarrow$ (2) below:

(1) ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathbb{C}_\Omega, \mathbb{C})$

(2) For each $k \in \mathbb{N}$ the partition relation $\mathbb{C}_\Omega \to \lceil \mathbb{C} \rceil^2_k$ holds.

A texture space $(S, \mathfrak{S})$ is said to have the **dual-Menger property** if for each sequence $(\mathcal{K}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{K}_n \subset \mathfrak{S}$ and $\bigcap_{n\in\mathbb{N}} \mathcal{K}_n = \emptyset$, there is a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ satisfying: (1) for each n, $\mathcal{F}_n$ is a finite subset of $\mathcal{K}_n$ and (2) $\bigcap_{n\in\mathbb{N}} \bigcap\{F : F \in \mathcal{F}_n\} = \emptyset$.

The **dual-Rothberger property** is defined similarly.

Proposition. For a texture $(S, \mathfrak{S})$ the following statements are equivalent:

(1) $(S, \mathfrak{S})$ has the dual-Menger property;

(2) $(S, \mathfrak{S}^c)$ has the Menger property.

(3) For each sequence $(\mathcal{T}_n : n \in \mathbb{N})$ such that $\mathcal{T}_n \subset \mathfrak{S}$ and $S = \bigvee\{T : T \in \mathcal{T}_n\}$, $n \in \mathbb{N}$, there is a sequence $(\Phi_n : n \in \mathbb{N})$ such that (1) for each n, Φ_n is a finite subset of $\mathcal{T}_n$, and (2) for each $A \in \mathfrak{S}$ there is $B \in \bigcup_{n\in\mathbb{N}} \Phi_n$ with $B \cap (S \setminus A) \neq \emptyset$.

3. Selection properties of ditopological texture spaces

A *dichotomous topology* on $(S, \mathfrak{S})$, or **ditopology** for short, is a pair (τ, κ) of generally unrelated subsets τ, κ of $\mathfrak{S}$ satisfying

(τ_1) $S, \emptyset \in \tau$,

(τ_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$,

(τ_3) $G_i \in \tau,\ i \in I \Rightarrow \bigvee_i G_i \in \tau$,

(κ_1) $S, \emptyset \in \kappa$,

(κ_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$,

(κ_3) $K_i \in \kappa,\ i \in I \Rightarrow \bigcap K_i \in \kappa$.

$(S, \mathfrak{S}, \tau, \kappa)$: **Ditopological texture space**.

Let (τ, κ) be a ditopology on $(S, \mathfrak{S})$ and take $A \in \mathfrak{S}$. The set $\{G_i \mid i \in I\}$ is called an **open cover** of A if $G_i \in \tau$ for all $i \in I$ and $A \subseteq \bigvee_{i \in I} G_i$.

The family $\{F_i \mid i \in I\}$ is called **closed cocover** of A if $F_i \in \kappa$ for all $i \in I$ and $\bigcap_{i \in I} F_i \subseteq A$.

Dicompactness

Let (τ, κ) be a ditopology on the texture $(S, \mathfrak{S})$ and $A \in \mathfrak{S}$.

1. A is called **compact** if whenever $\{G_i \mid i \in I\}$ is an open cover of A then there is a finite subset J of I with $A \subseteq \bigcup_{j \in J} G_j$. The ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is called compact if S is compact.

2. A is called **cocompact** if $\{F_i \mid i \in I\}$ is a closed cocover of A then there is a finite subset J of I with $\bigcap_{j \in J} F_j \subseteq A$. The ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is called cocompact if $\emptyset$ is cocompact.

3. (τ, κ) is called **stable** if every $K \in \kappa$ with $K \neq S$ is compact.

4. (τ, κ) is called **costable** if every $G \in \tau$ with $G \neq \emptyset$ is cocompact.

A ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is called **dicompact** if it is compact, cocompact, stable and costable.

A ditopological texture space $(S, \mathfrak{S}, \tau, \kappa)$ is σ-compact (σ-cocompact) if S is a countable union of compact sets ($\emptyset$ is a countable intersection of cocompact sets).

Selection properties and ditopology

Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space and A a subset of S.

A is said to have the *Menger property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of **open** covers of A there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $A \subseteq \bigvee_{n \in \mathbb{N}} \bigvee \mathcal{V}_n$. $(S, \mathfrak{S}, \tau, \kappa)$ is Menger if the set S is Menger. (This is denoted by $\mathsf{S}_{fin}(\theta_S, \theta_S)$; θ_S – the family of open covers of S.)

A is said to have the *co-Menger property* if for each sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of closed cocovers of $\emptyset$ there is a sequence $(\mathcal{K}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{K}_n \subseteq \mathcal{F}_n$ and $\bigcap_{n \in \mathbb{N}} \bigcap \mathcal{K}_n$ is a closed cocover of A. $(S, \mathfrak{S}, \tau, \kappa)$ is co-Menger if $\emptyset$ is co-Menger. (Notation: $\mathsf{S}_{cfin}(\Phi_S, \Phi_S)$; Φ_S is the family of closed co-covers of S.)

The Rothberger and co-Rothberger properties of a ditopological spaces are defined in a similar way.

Proposition. Let $(S, \mathfrak{S}, \tau, \kappa)$ be a ditopological texture space.

a) If $(S, \mathfrak{S}, \tau, \kappa)$ is σ-compact, then $(S, \mathfrak{S}, \tau, \kappa)$ has the Menger property. (b) If $(S, \mathfrak{S}, \tau, \kappa)$ is σ-cocompact, then $(S, \mathfrak{S}, \tau, \kappa)$ has the co-Menger property

Example. There is a ditopological texture space which is Menger (in fact Rothberger), but not compact.

Let $(\mathbb{R}, \mathfrak{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ be the real line with the texture $\mathfrak{R} = \{(-\infty, r] : r \in \mathbb{R}\} \cup \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$, topology $\tau_{\mathbb{R}} = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ and cotopology $\kappa_{\mathbb{R}} = \{(-\infty, r] : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. This ditopological texture space is neither compact (because the open cover $\mathcal{U} = \{(-\infty, n) : n \in \mathbb{N}\}$ does not contain a finite subcover) nor cocompact (because its closed cocover $\{(-\infty, r] : r \in \mathbb{R}\}$ does not contain a finite cocover). But $(\mathbb{R}, \mathfrak{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is Rothberger and co-Rothberger. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $\mathbb{R}$. Write $\mathbb{R} = \cup\{(-\infty, n) : n \in \mathbb{N}\}$. For each n there is some $r_n \in \mathbb{R}$ such that $(-\infty, n) \subset (-\infty, r_n) \in \mathcal{U}_n$. Then the collection $\{(-\infty, r_n) : n \in \mathbb{N}\}$ shows that $(\mathbb{R}, \mathfrak{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is Rothberger.

Proposition. Let $(S, \mathfrak{S}, \sigma)$ be a texture with the complementation σ and let (τ, κ) be a complemented ditopology on $(S, \mathfrak{S}, \sigma)$. Then $S \in \mathsf{S}_{fin}(\theta_S, \theta_S)$ if and only if $\emptyset \in \mathsf{S}_{cfin}(\mathfrak{F}_S, \mathfrak{F}_S)$.

Proposition. Let $(S, \mathfrak{S}, \sigma)$ be a texture with complementation σ and let (τ, κ) be a complemented ditopology on $(S, \mathfrak{S}, \sigma)$. Then for $K \in \kappa$ with $K \neq S$, $K \in \mathsf{S}_{fin}(\theta, \theta)$ if and only if $G \in \mathsf{S}_{cfin}(\mathfrak{F}, \mathfrak{F})$ for $G \in \tau$ and $G \neq \emptyset$.

Operations

For a texture space $(S, \mathfrak{S})$ and a set $A \in \mathfrak{S}$ the texturing $\mathfrak{S}_A := \{A \cap K : K \in \mathfrak{S}\}$ of A is called the *induced texture* on A, and $(A, \mathfrak{S}_A)$ is called a *principal subtexture* of $(S, \mathfrak{S})$.

Proposition. Let $(S, \mathfrak{S}, \sigma, \tau, \kappa)$ be a complemented ditopological texture space. If S is Menger and $A \in \kappa$, then $(A, \mathfrak{S}_A, \tau_A, \kappa_A)$ is also Menger.

Remark. If S is co-Menger and $A \in \tau$, then $(A, \mathfrak{S}_A, \tau_A, \kappa_A)$ is also co-Menger.

An open cover $\mathcal{U}$ of a ditopological texture space $(S, \mathfrak{S}, \sigma, \tau, \kappa)$ is said to be an ω-*cover* if for each finite $F \subseteq S$ there is an element U_F of $\mathcal{U}$ such that $F \subseteq U_F$.

Proposition. The following are equivalent for a ditopological texture space $(S, \mathfrak{S}, \sigma, \tau, \kappa)$:

(1) For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-covers of S there are finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an ω-cover of S;

(2) Each finite power of S has the Menger property.

THANK YOU!