A ring-theoretic characterisation of *Oz* spaces

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(joint work with Themba Dube)

A subset S of a topological space X is z-embedded in X in case each zero-set of S is the restriction to S of a zero-set of X.

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Spaces in which special sets are z-embedded, *Can. J. Math.*, **28** *(1976), 673–690,*

Blair calls a Tychonoff space X an *Oz* space if every open set of X is z-embedded.

A useful characterisation is that X is an *Oz* space if and only if every regular-closed subset of X is a zero-set.

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- A frame is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

  holds for all $a \in L$ and $S \subseteq L$.

- Frame homomorphisms are maps that preserve the frame structure.

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- An element a of L is rather below an element b, written $a \prec b$, in case there is an element s, called a separating element, such that $a \wedge s = 0$ and $s \vee b = 1$.
- The frame L is regular if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$.
- An element a is completely below b, written $a \prec\prec b$, if there are elements (x_r) indexed by rational numbers $\mathbb{Q} \cap [0, 1]$ such that $a = x_0$, $x_1 = b$ and $x_r \prec x_s$ for $r < s$.
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- Coz L is the cozero part of L, and is the regular sub-σ-frame consisting of all the cozero elements of L.

- βL is the Stone-Čech compactification of L and it is the frame of regular ideals of Coz L. We denote by $j_L : \beta L \to L$ the join map $J \mapsto \bigvee J$, and the right adjoint of j_L is here denoted by r_L.

- By a point of a frame we mean a prime element, that is, an element $p < e$ such that for any a and b in the frame, $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. We denote by Pt(L) the set of all points of L.

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- An element a of a frame L is called Lindelöf whenever $a = \bigvee S$ implies $a = \bigvee T$ for some countable $T \subseteq S$, and L is Lindelöf whenever $e \in L$ is Lindelöf.

- For any sublattice A of a frame L, an ideal $J \subseteq A$ is called
  1. σ-proper if $\bigvee S \neq e$ for any countable $S \subseteq J$, and
  2. completely proper if $\bigvee J \neq e$,

  the joins understood in L.

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$$\mathfrak{M}(a) = \{M \in \mathrm{Max}(A) \mid a \in M\} \quad \text{and} \quad M(a) = \bigcap \mathfrak{M}(a).$$

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Let L be a completely regular frame, and $\mathcal{R}L$ be the ring of real-valued continuous functions on L with $\mathcal{R}^*L$ as the subring of its bounded elements.

For $I \in \beta L$,

$$\boldsymbol{M}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\} \quad \text{and} \quad \boldsymbol{O}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \prec I\}.$$

For any $a \in L$ we abbreviate $\boldsymbol{M}^{r_L(a)}$ as $\boldsymbol{M}_a$, and remark that

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The maximal ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}^I$, for $I \in \mathrm{Pt}(\beta L)$.

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It is shown in

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that annihilator ideals in $\mathcal{R}L$ are precisely the ideals M_{a^*}, for $a \in L$. In particular, for any $\alpha \in \mathcal{R}L$, $\mathrm{Ann}(\alpha) = M_{(\mathrm{coz}\,\alpha)^*}$.

For any Tychonoff space X, the rings $C(X)$ and $\mathcal{R}(\mathfrak{O}X)$ are isomorphic.

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An ideal of a ring A is a principal z-ideal if it is of the form $M(a)$ for some $a \in A$.

Corollary

An ideal of $\mathcal{R}L$ is an intersection of maximal ideals iff it is of the form $\mathbf{M}^I$, for some $I \in \beta L$.

In the ring $\mathcal{R}L$, principal z-ideals have the following description.

Lemma

Principal z-ideals of $\mathcal{R}L$ are precisely the ideals $\mathbf{M}_c$ for $c \in \mathrm{Coz}\, L$.

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Lemma

Principal z-ideals of $\mathcal{R}L$ are precisely the ideals $\mathbf{M}_c$ for $c \in \operatorname{Coz} L$.

Proof.

We recall that $M(\alpha) = \bigcap\{M \in Max(\mathcal{R}L) \mid \alpha \in M\}$.

Put $\mathcal{P} = \{I \in \mathrm{Pt}(\beta L) \mid \alpha \in M^I\}$, then

$$\mathcal{P} = \{I \in \mathrm{Pt}(\beta L) \mid r_L(\operatorname{coz}\alpha) \leq I\}.$$

Since βL is spatial,

$\bigwedge \mathcal{P} = \bigwedge\{I \in \mathrm{Pt}(\beta L) \mid r_L(\operatorname{coz}\alpha) \leq I\} = r_L(\operatorname{coz}\alpha)$. So,

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Proof.

We recall that $M(\alpha) = \bigcap\{M \in Max(\mathcal{R}L) \mid \alpha \in M\}$.

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We say a ring is an *Oz*-ring if every annihilator ideal of the ring is a principal *z*-ideal.

Proposition

A completely regular frame L is an Oz-frame iff $\mathcal{R}L$ is an Oz-ring.

Proof.

Suppose L is an Oz-frame, and let Q be an annihilator ideal of $\mathcal{R}L$. Then there is an $a \in L$ such that $Q = \boldsymbol{M}_{a^*}$. Since L is an Oz-frame, $a^* \in \operatorname{Coz} L$, hence Q is a principal z-ideal, and therefore $\mathcal{R}L$ is an Oz-ring.
Conversely, suppose $\mathcal{R}L$ is an Oz-ring. For any $a \in L$, $\boldsymbol{M}_{a^*}$ is an annihilator ideal, and so, by hypothesis, there is a $c \in \operatorname{Coz} L$ such that $\boldsymbol{M}_{a^*} = \boldsymbol{M}_c$, so that $a^* = c$. Therefore L is an Oz-frame. □

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The property of being an *Oz*-ring is preserved by ring isomorphisms. Since X is an *Oz*-space if and only if $\mathfrak{O}X$ is an *Oz*-frame, and since $C(X) \cong \mathcal{R}(\mathfrak{O}X)$, we have the following result.

Corollary

A Tychonoff space X is an Oz-space iff $C(X)$ is an Oz-ring. A realcompact space is Oz iff its Stone-Čech compactification is Oz.

Definition

- A subspace S of a topological space X is *C-embedded* in X if every function in $C(S)$ can be extended to a function in $C(X)$.
- An onto frame homomorphism $h: L \to M$ is a *C-quotient* map if for every frame homomorphism $\gamma: \mathfrak{O}\mathbb{R} \to M$ there is a frame homomorphism $\delta: \mathfrak{O}\mathbb{R} \to L$ such that $h \circ \delta = \gamma$.

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Banaschewski shows that a frame homomorphism $h\colon L \to M$ is dense if and only if the ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ is one-one.

Consequently, $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ is an isomorphism if and only if $h\colon L \to M$ is a dense C-quotient map. This yields the following results.

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A frame L is an Oz-frame iff λL is an Oz-frame iff υL is an Oz-frame. A Tychonoff space X is an Oz-space iff υX is an Oz-space.

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Every dense C-embedded subspace of an Oz-space is Oz. If a space has a dense C-embedded subspace which is Oz, then the space itself is Oz.

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