Cardinal invariants of paratopological groups

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Let n be a positive integer. We say that a space X has a G_δ-diagonal of rank n if there exists a countable collection $\{\mathcal{U}_k : k \in \omega\}$ of open covers of X such that $\bigcap\{st^n(\mathcal{U}_k, x) : k \in \omega\} = \{x\}$ for each $x \in X$.

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Example 2 (Arhangel'skii and Burke, 2006).

There exists a Tychonoff paratopological group G with a countable π-base such that the space G is not first countable.

A question of Arhangel'skii and Bella

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The next theorem resolves Problem 1 for regular paratopological groups.

Theorem 3.

Every regular paratopological group of countable π-character has a G_δ-diagonal of infinite rank.

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The previous fact permits us to call G_r the regularization of G.

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Lemma 5.

If H is a dense subgroup of a paratopological group (G, τ) and $\sigma = \tau|_H$, then $\tau_r|_H = \sigma_r$.

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Put $\sigma = \tau|_H$. Since (H, σ) has countable π-character, (H, σ_r) too. By Lemma 5, $\pi\chi(H, \tau_r|_H) \leq \omega$.

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Put $\sigma = \tau|_H$. Since (H, σ) has countable π-character, (H, σ_r) too. By Lemma 5, $\pi\chi(H, \tau_r|_H) \leq \omega$. Since $(H, \tau_r|_H)$ is a dense subgroup of the regular space G_r, the regular paratopological group G_r has countable π-character. Theorem 3 implies that G_r has a G_δ-diagonal of infinite rank. Since $\tau_r \subseteq \tau$, the space (G, τ) has a G_δ-diagonal of infinite rank. □

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Example 8.

There exists a Hausdorff paratopological group with uncountable π-character which contains a second countable dense subgroup.

Thank you!