On the Borel Complexity of Characterizable Subgroups

joint work with D. Impieri

Sao Sebastiao, Brazil, August 16, 2013

Dedicated to Ofelia T. Alas
on the occasion of her 70th birthday

Theorem (Kronecker (a special case))

For every irrational $\alpha \in [0,1]$ the set of all multiples $\{n\alpha : n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ modulo 1.

$\mathcal{Z}$ – infinite strictly increasing sequences $S = (u_n)$ of integers,
$W_S = \{\alpha \in [0,1] : (u_n\alpha) \text{ is uniformly distributed mod } 1\}$
for $S \in \mathcal{Z}$ (where "uniformly distributed" means that

$$\lim_m \frac{|\{n \in \mathbb{N} : 1 \leq n \leq m \quad \text{and} \quad a_n\alpha \in \Delta\}|}{m} = \mu(\Delta)$$

for every subinterval $\Delta \subseteq [0,1]$.)

Theorem (Weyl 1916)

(a) *If $u_n = P(n)$ is a polynomial function of n, then W_S contains all irrational $\alpha \in [0,1]$.*

(b) *W_S has measure 1 for every $S \in \mathcal{Z}$.*

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W_S need not contain all irrational $\alpha \in [0,1]$ in (b):

Example (the sequence of factorials)

If $S = (n!)$, then

$$[0,1] \ni \alpha = e - 2 = \sum_{n=2}^{\infty} \frac{1}{n!} \notin W_S$$

as $\frac{1}{n+1} < n!e < \frac{2}{n+1} \pmod 1$, so $n!e \to 0 \bmod 1$.

Example (The Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$)

If $S = (f_n)$, then $\alpha = \frac{1+\sqrt{5}}{2} \notin W_S$ as $f_n\alpha \to 0 \bmod 1$ ($\alpha - 1 \in [0,1]$)
Indeed, $\alpha = \frac{1}{1+\alpha} = \frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}} =: [0;1,1,\dots]$ with convergents $\frac{f_{n-1}}{f_n}$,

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Replace reals mod 1 by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ written additively, with norm $\|x\|$ = distance to the closest integer for $x \in \mathbb{R}$, Haar measure μ.

Definition (a set of singular points in Weyl's theorem)

Let $\Gamma_S = \{x \in \mathbb{T} : u_n\alpha \to 0\}$ for $S \in \mathcal{Z}$.

Γ_S is related also to trigonometric series (Arbault sets).

Lemma (Properties of the sets Γ_S)

(a) Γ_S *is a (proper) subgroup of* $\mathbb{T}$*;*

(b) $\Gamma_S = \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n>k} \{x \in \mathbb{T} : \|u_n\alpha\| \leq 1/m\}$ *is a Borel set;*

(c) $\mu(\Gamma_S) = 0$.

Proof. (a) – (b) The closed set $F_k = \bigcap_{n>k}\{x \in \mathbb{T} : \|u_n\alpha\| \leq 1/4\}$ has $Int(F_k) = \emptyset$ as $u_n \to \infty$, so by Baire category theorem $\Gamma_S \subseteq \bigcup_{k=1}^{\infty} F_k \neq \mathbb{T}$.

(c) $\mathbb{T} = n\mathbb{T}$ for all $n \in \mathbb{N}$, hence $[\mathbb{T} : \Gamma_S]$ is infinite and $\mu(\Gamma_S) = 0$.

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Being a Borel set of $\mathbb{T}$, Γ_S is either countable or $|\Gamma_S| = \mathfrak{c}$.

Definition (Let $q_n = \frac{u_{n+1}}{u_n}$.)

Theorem (Egglestone 1952: $|\Gamma_S|$ depends on q_n)

(a) $|\Gamma_S| = \mathfrak{c}$ if $q_n \to \infty$;
(b) Γ_S is countable if (q_n) is bounded.

Neither (a) nor (b) are necessary conditions.

Theorem (C. Kraaikamp and P. Liardet 1991)

If $u_n = a_n u_{n-1} + u_{n-2}$ and $u_1 = 1$, then TFAE:
(a) Γ_S is countable;
(b) q_n is bounded;
(c) Γ_S is cyclic.

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Are F_σ-subgroups (of $\mathbb{T}$) characterizable

Question [Kunen, DD]

When F_σ- subgroups H of compact metrizable abelian groups are characterizable ?

If H contains a compact subgroup K of G with countable torsion quotient H/K (so that H is even a countable union of compact subgroups [KK & DD]). Here "torsion" can be relaxed as the countable subgroup H/K of the compact metrizable group G/K is characterizable.

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Definition

A non empty compact subset K of an infinite compact metrizable abelian group X is called a Kronecker set, if for every continuous function $f : K \to \mathbb{T}$ and $\varepsilon > 0$ there exists a $v \in \widehat{X}$ such that

$$\max\{\|f(x) - v(x)\| : x \in K\} < \varepsilon.$$

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Theorem (Gabriyelyan 2009)

Let X be a compact metrizable abelian group. Then

(a) *$\Gamma_S(X)$ is Polishable for every sequence S of characters of X;*

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When uncountable characterizable subgroups (of $\mathbb{T}$) are F_σ?

Example (uncountable characterizable F_σ-subgroups)

(a) [Gabriyelyan 2012] if $j : \mathbb{R} \hookrightarrow \mathbb{T}^2$ is a dense continuous monomorphism, then $j(\mathbb{R})$ is characterizable.
(b) if a subgroup H of a compact metrizable group G contains a compact subgroup K such that H/K is countable, then H is characterizable.

Item (a) can be generalized as follows:

Theorem (Gabriele Negro, answering a question of Gabriyelyan)

If G is a compact metrizable abelian group, H is a metrizable LCA group and $j : H \hookrightarrow G$ is a continuous monomorphism, then $j(H)$ is characterizable.

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All characterizable subgroups of compact metrizable abelian group are F_σ iff G has finite exponent.

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Example (characterizable, non-F_σ-subgroups of $\mathbb{T}$)

(a) [Bukovský, Kholshevikova, Repický 1994]
Γ_S is not an F_σ-subgroup of $\mathbb{T}$ for $S = (2^{2^n})$.
(b) [Gabriyelyan 2013] Γ_S is not an F_σ-subgroup of $\mathbb{T}$ for $S = (n!)$.

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In both cases $u_n | u_{n+1}$ in $S = (u_n)$ and $q_n = \frac{u_{n+1}}{u_n} \to \infty$.

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Definition (Call a sequence $S = (u_n)$ of positive integers an artitmetic sequence (briefly, an a-sequence) if $u_n | u_{n+1}$ for all but finitely many n.)

Theorem (Impieri, DD 2013)

The following are equivalent for an a-sequence $S = (u_n) \in \mathcal{Z}$:

(a) $\Gamma_S \leq \mathbb{Q}/\mathbb{Z}$;

(b) (q_n) *is bounded;*

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(e) τ_S *is discrete.*

(a) and (b) are specific properties of $\mathbb{T}$, while (c)—(e) can be discussed for every metrizable compact abelian group G in place of $\mathbb{T}$ and (c) $\Leftrightarrow$ (e) holds true in general, (c) $\Leftrightarrow$ (d) is open even in $\mathbb{T}$.

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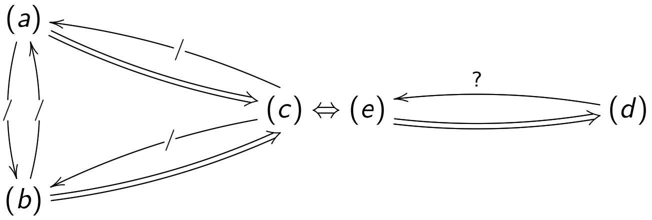
Removing the hypothesis "a-sequence" in the theorem leads to



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Open questions

Question

If Γ_S is an F_σ-set of $\mathbb{T}$ for some $S \in \mathbb{Z}^{\mathbb{N}}$, must Γ_S be necessarily countable?

The answer is positive if S is an a-sequence.

Question

If H is a countable subgroup of $\mathbb{T}$, does there exist a characterizing sequence $S \in \mathbb{Z}^{\mathbb{N}}$ of H with bounded sequence of ratios (q_n)?

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Does every Polishable F_σ-subgroup of $\mathbb{T}$ admit a characterizing sequence?

By Biro's theorem, the answer is negative if we relax "Polishable", by Gabriyelyan's example, the answer is negative if we replace $\mathbb{T}$ be an arbitrary compact metrizable group.

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