Topological games and Alster spaces

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In honour of Ofelia T. Alas

Definition (Rothberger 1938)

A topological space X is *Rothberger* if, for every sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X, there is a sequence $(U_n)_{n\in\omega}$ satisfying $X = \bigcup_{n\in\omega} U_n$ with $U_n \in \mathcal{U}_n$ for all $n \in \omega$.

Definition (Galvin 1978)

The *Rothberger game* in a topological space X is played according to the following rules: in each inning $n \in \omega$, One chooses an open cover $\mathcal{U}_n$ of X, and then Two chooses $U_n \in \mathcal{U}_n$; the play is won by Two if $X = \bigcup_{n\in\omega} U_n$, otherwise One is the winner.

Theorem (Pawlikowski 1994)

Two $\uparrow$ Rothberger(X) $\Rightarrow$ One $\not\uparrow$ Rothberger(X) $\Leftrightarrow$ X is Rothberger.

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Definition (Hurewicz 1926)

A topological space X is *Menger* if, for every sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X, there is a sequence $(\mathcal{F}_n)_{n\in\omega}$ satisfying $X = \bigcup\bigcup_{n\in\omega}\mathcal{F}_n$ with $\mathcal{F}_n \in [\mathcal{U}_n]^{<\aleph_0}$ for all $n \in \omega$.

Definition (Telgársky 1984)

The *Menger game* in a topological space X is played as follows: in each inning $n \in \omega$, One chooses an open cover $\mathcal{U}_n$ of X, and then Two chooses a finite subset $\mathcal{F}_n$ of $\mathcal{U}_n$; Two wins the play if $\bigcup_{n\in\omega}\mathcal{F}_n$ is a cover of X, otherwise One is the winner.

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Topological games

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Definition (Galvin 1978)

The *point-open game* in a topological space X is played as follows: in each inning $n \in \omega$, One picks a point $x_n \in X$, and then Two chooses an open set $U_n \subseteq X$ with $x_n \in U_n$; the play is won by One if $X = \bigcup_{n\in\omega} U_n$, otherwise Two is the winner.

Theorem (Galvin 1978)

- One $\uparrow$ Rothberger(X) $\Leftrightarrow$ Two $\uparrow$ point-open(X);
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What about the Menger game?

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What about the Menger game?

Theorem (Telgársky 1983-84)

Let X be a regular space. TFAE:

(a) Two $\uparrow$ Menger(X);

(b) One $\uparrow$ compact-open(X);

(c) One $\uparrow$ compact-G_δ(X).

Question (Telgársky 1984)

Is Two $\uparrow$ compact-open equivalent to One $\uparrow$ Menger?

Or, equivalently:

Does the Menger property imply Two $\not\uparrow$ compact-open?

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Playing with k-covers

Definition

An open cover $\mathcal{U}$ of a topological space X is a *k-cover* ($\mathcal{U} \in \mathcal{K}_X$) if every compact subset of X is included in an element of $\mathcal{U}$.

Definition

The game $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ on a topological space X is played as follows: in each inning $n \in \omega$, One chooses $\mathcal{U}_n \in \mathcal{K}_X$, and then Two chooses $U_n \in \mathcal{U}_n$; Two wins the play if $X = \bigcup_{n\in\omega} U_n$, otherwise One is the winner.

Proposition (Telgársky 1983, Galvin 1978)

The game $\mathsf{G}_1(\mathcal{K}, \mathcal{O})$ and the compact-open game are dual.

Corollary

If Two $\not\uparrow$ compact-open(X), then $\mathsf{S}_1(\mathcal{K}_X, \mathcal{O}_X)$ holds — i.e., for every sequence $(\mathcal{U}_n)_{n\in\omega}$ of k-covers of X, there is a sequence $(U_n)_{n\in\omega}$ with $U_n \in \mathcal{U}_n$ for all $n \in \omega$ and $X = \bigcup_{n\in\omega} U_n$.

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Corollary

Two $\not\uparrow$ compact-open $\Rightarrow \mathsf{S}_1(\mathcal{K}, \mathcal{O}) \Rightarrow$ Menger.

Question (Telgársky 1984)

Does the Menger property imply Two $\not\uparrow$ compact-open?

We will now partially answer this question in the negative by giving consistent examples of Menger regular spaces that do not satisfy $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$.

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Lemma

Let X be a topological space such that every compact subspace of X has an isolated point. Then X satisfies $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$ if and only if X is Rothberger.

Example

If $\mathrm{cov}(\mathcal{M}) < \mathfrak{d}$, then any non-Rothberger subspace of $\mathbb{R}$ of size $\mathrm{cov}(\mathcal{M})$ is Menger but does not satisfy $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$.

Example

Any Sierpiński subset of the real line (which exists e.g. under CH) endowed with the Sorgenfrey topology is a Menger regular space that does not satisfy $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$.

Problem

Is it consistent with ZFC that every Menger regular space satisfies $\mathsf{S}_1(\mathcal{K}, \mathcal{O})$?

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Definition (Alster 1988)

A topological space X is *Alster* if every k_δ*-cover* of X — i.e., every G_δ-cover $\mathcal{W}$ of X such that

for every compact $K \subseteq X$ *there is* $W \in \mathcal{W}$ *with* $K \subseteq W$

— has a countable subcover.

Problem (Tamano)

Characterize (internally) the *productively Lindelöf spaces*, i.e. the spaces X such that $X \times Y$ is Lindelöf whenever Y is Lindelöf.

Theorem (Alster 1988)

Alster spaces are productively Lindelöf. Assuming CH, productively Lindelöf regular spaces of weight not exceeding $\aleph_1$ are Alster.

Problem (Alster 1988)

Is every productively Lindelöf (regular) space Alster?

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Of course, σ-compact spaces are Alster.

Theorem (Aurichi, Tall 2012)

Alster spaces are Menger.

In view of the fact that
σ-compact $\Rightarrow$ Two $\uparrow$ Menger $\Rightarrow$ Menger,

Question (Tall 2013)

Does Two $\uparrow$ Menger imply Alster? Does the converse hold?

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Alster vs. Two $\uparrow$ Menger

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A selective characterization

Proposition

A topological space X is Alster if and only if it satisfies $\mathsf{S}_1(\mathcal{K}^\delta, \mathcal{O}^\delta)$.

Proposition

The game $\mathsf{G}_1(\mathcal{K}^\delta, \mathcal{O}^\delta)$ and the compact-G_δ game are dual.

Corollary

Two $\not\uparrow$ compact-G_δ $\Rightarrow$ Alster.

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Corollary

If X is regular and Two ↑ Menger(X), then X is Alster.

Example (Telgársky 1983)

There is a regular space in which the compact-G_δ game is undetermined — hence a regular Alster space in which Two $\not\uparrow$ Menger.

Corollary

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How about Two $\uparrow$ Rothberger?

How about Two ↑ Rothberger?

Theorem

Two ↑ Rothberger(X) ⇒ X_δ is Lindelöf (⇒ X is Rothberger).

Sketch of proof.

Let $\sigma : {}^{<\omega}\tau \to X$ be a winning strategy for One in point-open(X).
Now let $\mathcal{W}$ be a cover of X by G_δ subsets. For each $W \in \mathcal{W}$, fix a sequence $(U(W,n))_{n\in\omega}$ of open sets with $W = \bigcap_{n\in\omega} U(W,n)$.
Proceeding by induction on $n \in \omega$, we shall assign to each $s \in {}^n\omega$ an element W_s of $\mathcal{W}$ as follows.
First, pick $W_\emptyset \in \mathcal{W}$ such that $\sigma(\emptyset) \in W_\emptyset$. Now let $n \in \omega$ be such that $W_s \in \mathcal{W}$ has already been defined for all $s \in {}^n\omega$. For each $s \in {}^n\omega$ and each $k \in \omega$, choose $W_{s^\frown k} \in \mathcal{W}$ satisfying $\sigma(t_{s,k}) \in W_{s^\frown k}$, where $t_{s,k} \in {}^{n+1}\tau$ is the sequence defined by $t_{s,k}(i) = U(W_{s\restriction i}, s(i))$ for all $i < n$ and $t_{s,k}(n) = U(W_s, k)$.
Since the strategy σ is winning, it follows that (...) $\{W_s : s \in {}^{<\omega}\omega\} \subseteq \mathcal{W}$ is a cover of X. □

How about Two $\uparrow$ Rothberger?

Theorem

Two $\uparrow$ Rothberger$(X) \Rightarrow X_\delta$ is Lindelöf ($\Rightarrow X$ is Rothberger).

Sketch of proof.

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Two ↑ Rothberger(X) ⇒ X_δ is Lindelöf (⇒ X is Rothberger).

Sketch of proof.

Let $\sigma : {}^{<\omega}\tau \to X$ be a winning strategy for One in point-open(X). Now let $\mathcal{W}$ be a cover of X by G_δ subsets. For each $W \in \mathcal{W}$, fix a sequence $(U(W,n))_{n\in\omega}$ of open sets with $W = \bigcap_{n\in\omega} U(W,n)$. Proceeding by induction on $n \in \omega$, we shall assign to each $s \in {}^n\omega$ an element W_s of $\mathcal{W}$ as follows.

First, pick $W_\emptyset \in \mathcal{W}$ such that $\sigma(\emptyset) \in W_\emptyset$. Now let $n \in \omega$ be such that $W_s \in \mathcal{W}$ has already been defined for all $s \in {}^n\omega$. For each $s \in {}^n\omega$ and each $k \in \omega$, choose $W_{s^\frown k} \in \mathcal{W}$ satisfying $\sigma(t_{s,k}) \in W_{s^\frown k}$, where $t_{s,k} \in {}^{n+1}\tau$ is the sequence defined by $t_{s,k}(i) = U(W_{s\restriction i}, s(i))$ for all $i < n$ and $t_{s,k}(n) = U(W_s, k)$.

Since the strategy σ is winning, it follows that (...) $\{W_s : s \in {}^{<\omega}\omega\} \subseteq \mathcal{W}$ is a cover of X. □

How about Two $\uparrow$ Rothberger?

Theorem

Two $\uparrow$ Rothberger(X) $\Rightarrow X_\delta$ is Lindelöf ($\Rightarrow X$ is Rothberger).

Sketch of proof.

Let $\sigma : {}^{<\omega}\tau \to X$ be a winning strategy for One in point-open(X). Now let $\mathcal{W}$ be a cover of X by G_δ subsets. For each $W \in \mathcal{W}$, fix a sequence $(U(W,n))_{n\in\omega}$ of open sets with $W = \bigcap_{n\in\omega} U(W,n)$. Proceeding by induction on $n \in \omega$, we shall assign to each $s \in {}^n\omega$ an element W_s of $\mathcal{W}$ as follows.

First, pick $W_\emptyset \in \mathcal{W}$ such that $\sigma(\emptyset) \in W_\emptyset$. Now let $n \in \omega$ be such that $W_s \in \mathcal{W}$ has already been defined for all $s \in {}^n\omega$. For each $s \in {}^n\omega$ and each $k \in \omega$, choose $W_{s^\frown k} \in \mathcal{W}$ satisfying $\sigma(t_{s,k}) \in W_{s^\frown k}$, where $t_{s,k} \in {}^{n+1}\tau$ is the sequence defined by $t_{s,k}(i) = U(W_{s\restriction i}, s(i))$ for all $i < n$ and $t_{s,k}(n) = U(W_s, k)$.

Since the strategy σ is winning, it follows that (...) $\{W_s : s \in {}^{<\omega}\omega\} \subseteq \mathcal{W}$ is a cover of X. □

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