Perfect sets, partitions and chromatic numbers

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Introduction

We study perfect set properties of sets of real numbers or of subsets of some other Polish spaces.
Among other things, we are interested in the way the Axiom of Choice is related to these perfect set properties.

We will examine some applications to the theory of chromatic numbers of infinite graphs.

Work in collaboration with:
J. Bagaria, F. Galindo, J. Henle, J. Llopis, and S. Todorcevic.

The Baire space

The Baire space $\mathbb{N}^\infty$ is the set of infinite sequences of natural numbers with the product topology resulting from the discrete topology on $\mathbb{N}$.

The Baire space is homeomorphic to the irrationals.

The space $[\mathbb{N}]^\infty$ of infinite subsets of $\mathbb{N}$ with the product topology is homeomorphic to the Baire space.

Perfect sets

Definition

A non-empty set $\mathcal{A} \subseteq \mathbb{N}^\infty$ (or $\mathcal{A} \subseteq \mathbb{R}$) is perfect if it is closed and contains no isolated points.

If P is perfect then the cardinality of P is $2^{\aleph_0}$.

Cantor-Bendixon

If C is a closed subset of $\mathbb{R}$, then

$$C = P \cup N$$

where P is perfect (or empty) and N is countable.

Alexandrov and Hausdorff extended this to Borel sets, and later Suslin to analytic sets.

Definition

A subset of $\mathbb{R}$ (or of the Baire space $\mathbb{N}^\infty$, or of $[\mathbb{N}]^\infty$, etc.) has the *Bernstein Property* if it contains a perfect subset or its complement contains a perfect set.

The Baire property and the Bernstein property

It is easy to verify that a set with the Baire property has the Bernstein property.

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Moreover,

Proposition

Given any partition of $\mathbb{N}^\infty$ into countably many sets with the Baire property, then there is a perfect set that lies in one of the pieces of the partition.

The Bernstein property can be extended in both directions:

- higher dimensions, and
- partitions into infinitely many pieces.

The following is a consequence of a result of Galvin.

Proposition

For every Borel partition

$$c : (\mathbb{N}^{\infty})^{\omega} \to \omega$$

there are perfect sets $P_0, P_1, \ldots$ *with* $P_i \subseteq \mathbb{N}^{\infty}$ *such that* c *is constant on* $\Pi_{i \in \mathbb{N}} P_i$

Bagaria-DP, Parameterized partitions on the real numbers. Annals of Math. Logic 2009
presents a study of this kind of partition property and their parametrizations.

Axiom of Choice and the Bernstein property

The axiom of choice implies the existence of sets without the Bernstein property.

Proof: It is enough to well order the collection of perfect subsets of $\mathbb{N}^\infty$ in order type $2^{\aleph_0}$ and diagonalize.

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The axiom of choice implies the existence of sets without the Bernstein property.

Proof: It is enough to well order the collection of perfect subsets of $\mathbb{N}^\infty$ in order type $2^{\aleph_0}$ and diagonalize.

Notice that sets without the Bernstein property are not Lebesgue measurable.

The Perfect Subset Property

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It can be shown that every analytic set has the PSP.

If $\mathcal{A}$ has the PSP, then it has the Bernstein property.

Ultrafilters on $\mathbb{N}$

Definition

A filter on $\mathbb{N}$ is a family $\mathcal{F}$ of subsets of $\mathbb{N}$ such that

- $\emptyset \notin \mathcal{F}$, $\mathbb{N} \in \mathcal{F}$,
- *if* $A \subseteq B$ *and* $A \in \mathcal{F}$, *then* $B \in \mathcal{F}$,
- *if* $A, B \in \mathcal{F}$, *then* $A \cap B \in \mathcal{F}$.

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An ultrafilter is free (or non-principal) if it does not contain finite sets.

The existence of free ultrafilters on $\mathbb{N}$ is a consequence of the Axiom of Choice.

Uncountable sequences of real numbers

Using the Axiom of Choice the set or real numbers can be well ordered, but
in the absence of the Axiom of Choice the set of real numbers might not have cardinality.

The formula

$$\aleph_1 \leq 2^{\aleph_0}$$

expresses that there is an $\aleph_1$- sequence of reals.
In other words, there is an injection of $\aleph_1$ into the set of reals.

Bernstein, in 1908, while working on trigonometric series and their sets of uniqueness, proved:

$\aleph_1 \leq 2^{\aleph_0}$ implies that there is an uncountable set of reals without perfect subsets

(i.e. there is a set without the PSP).

Questions

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The answer is negative in both cases.

Models of Set theory

Models of the axioms of Set Theory can be built where

- The Axiom of Choice does not hold
- Weak forms of choice hold (like the existence of a non principal ultrafilter or $\aleph_1 \leq 2^{\aleph_0}$)
- all sets of reals have the Bernstein Property.

Inner models of Cohen extensions

Start from a model M of ZFC, and fix an uncountable cardinal κ in M.

Consider the forcing order C_κ which adds κ Cohen reals to M to obtain the forcing extension $M[\{r_\xi : \xi < \kappa\}]$.

Now, construct $L(A)$ in this extension with $A = \{r_\xi : \xi < \kappa\}$

Theorem

Every r_ξ is in $L(A)$,
$A \notin L(A)$.
In $L(A)$ every set of real numbers has the Bernstein property.

A variation of the inner model construction

In the Cohen extension $M[\{r_\xi : \xi < \kappa\}]$ we construct $L(A)$ but now with

$$A = \{\langle r_\xi : \xi < \alpha\rangle : \alpha < \kappa\}$$

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In the resulting model $L(A)$

- The axiom of choice fails.
- For each $\alpha < \kappa$, $\langle r_\xi : \xi < \alpha\rangle \in L(A)$,
- The whole sequence $\langle r_\xi : \xi < \kappa\rangle$ is not in $L(A)$.

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As before, every set of reals in $L(A)$ has the Bernstein property, but (if $\kappa \geq \omega_2$) the sequence $\langle r_\xi : \xi < \omega_1\rangle \in L(A)$, thus

$$\aleph_1 \leq 2^{\aleph_0}$$

holds.

Models of this sort were used by John Truss in articles published in the 1970's.

D. P and F. Galindo, Perfect set properties in models of ZF. Fundamenta Mathematica, 2010.

Models with free ultrafilters on $\mathbb{N}$ where all sets of reals have the Bernstein property or the PSP were constructed in

DP, Partition properties and perfect sets. Notas de Lógica Matemática, No. 38 INMABB-CONICET, Bahia Blanca, Argentina. (1993) 119-127.

DP y S. Todorcevic, Perfect set properties in $L(\mathbb{R})[U]$. Advances in Mathematics 139 (1998) 240-259.

Cubes, The Ramsey Property

Given $A \in [\mathbb{N}]^\infty$, the "cube"

$$A^{[\infty]} = \{B \subseteq A : B \text{ infinite}\}$$

is a perfect subset of $[\mathbb{N}]^\infty$.

The Bernstein property can be strengthened:
Instead of just requiring a perfect set, we require a cube contained in one piece of the partition.

Definition

Ramsey property: A set $\mathcal{A} \subseteq [\mathbb{N}]^\infty$ is Ramsey if it contains or is disjoint from a cube.

- Galvin and Prikry showed that every Borel subset of $[\mathbb{N}]^\infty$ is Ramsey.
- This was extended to analytic sets by Silver using metamathematical methods.
- Ellentuck gave a topological proof os Silver's result.

- Erdös-Rado arrow notation

$$\omega \to (\omega)^\omega$$

expresses that every subset of $[\mathbb{N}]^\infty$ is Ramsey

- *AC* implies there are non-Ramsey sets.
- The property

$$\omega \to (\omega)^\omega$$

is consistent with *ZF* and dependent choice.

Consistency of the Ramsey Property

Solovay (1970) A model of $ZF + DC$ where every set of reals is

- Lebesgue measurable,
- has the property of Baire,
- has the PSP.

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To construct his model Solovay started from a model of $ZFC+$ There is an insccessible cardinal.

The hypothesis of inaccessibility is necessary for

"every set is Lebesgue Measurable."

It is not needed for "every set thas the Baire property."

It is open if it is necessary for

$$\omega \to (\omega)^\omega.$$

Parametrized Partition Properties

Solovay model:
$L(\mathbb{R})$ as computed in the Levy collapse of an inaccessible.

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$L(\mathbb{R})[U]$ satisfies *OCA*, *PSP*, Other perfect set properties.

An intermediate property?

Given a set K

$$\omega \to [\omega]^{\omega}_{K}$$

means that for every $c : [\mathbb{N}]^\infty \to \omega$ there is $A \in [\mathbb{N}]^\infty$ such that $c''[A]^\infty \subsetneq K$

The following implications follow from the definitions.

$$\omega \to (\omega)^\omega \Rightarrow \omega \to [\omega]^\omega_\omega \Rightarrow \textit{Bernstein Property}$$

Square bracket partitions

Kleinberg proved that

$$\omega \to [\omega]^{\omega}_{\omega}$$

implies that there is $n \in \mathbb{N}$ such that

$$\omega \to [\omega]^{\omega}_{n}$$

It is not known which is the smallest such n.
For $n = 2$ we just have

$$\omega \to (\omega)^{\omega}$$

Non-Ramsey sets

A well known result of Mathias (1974) if there is a non-meager filter on $\mathbb{N}$, then there is a non-Ramsey set.
In particular, from a non-principal ultrafilter on $\mathbb{N}$ we can get a non-Ramsey set.

Open Question: Is $\aleph_1 \leq 2^{\aleph_0}$ enough to produce a non-Ramsey set?

The shift graph

We will see that if there is a finite coloring of the shift graph then there is a non-Ramsey set.

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The shift $S : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ is defined by $S(A) = A \setminus \{\min(A)\}$

$\mathcal{G}_S = ([\mathbb{N}]^\infty, S)$:
two elements of $[\mathbb{N}]^\infty$ form an edge if one is the shift of the other.

There are no cycles in this graph. AC implies that its chromatic number is 2.

DP-Todorcevic (2013) :

$\omega \to [\omega]^{\omega}_{\omega}$ implies that the chromatic number of $\mathcal{G}_S$ is $\aleph_0$.

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$\omega \to [\omega]^\omega_\omega$ implies that the chromatic number of $\mathcal{G}_S$ is $\aleph_0$.

The proof uses Kleinberg's result and van der Waerden's theorem on arithmetic progressions.

description of the proof

Using results of Kechris, Solecki and Todorcevic (2000), the only possible values of the chromatic number of $\mathcal{G}_S$ are $2, 3$, and $\aleph_0$.

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So, it is enough to show that if $\omega \to [\omega]^\omega_\omega$ holds, there is no 3-coloring of $\mathcal{G}_S$

Sketch of proof

If there is a 3 coloring of $\mathcal{G}_S$, then, for every k, $\omega \not\to [\omega]^{\omega}_{k}$.

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If there is a 3 coloring of $\mathcal{G}_S$, then, for every k, $\omega \not\to [\omega]^\omega_k$.

Fix $k \in \omega$ and let $n_k = W(k,k)$ be the corresponding van der Waerden number. Put $p_k = n_k!$

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If there is a 3 coloring of $\mathcal{G}_S$, then, for every k, $\omega \not\to [\omega]^\omega_k$.

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Fix $c : [\omega]^\omega \to 3$ a coloring of the graph $([\omega]^\omega, S)$, and define $\vec{c} : [\omega]^\omega \to 3^\omega$ by

$$\vec{c}(A) = \langle c(A), c(S(A)), c(S^2(A)), \ldots \rangle.$$

Let $A = \{a_0, a_1, \dots\}$ be the increasing enumeration of an infinite set $A \subseteq \omega$, for each $1 \leq i \leq k$, let

$$A_i = \{a_n : p_k^i | n \quad \& \quad p_k^{i+1} \not| n\}.$$

Define $d : [\omega]^\omega \to (3^{n_k})^k$ by

$$d(A) = \langle \vec{c}(A_1) \restriction n_k, \dots, \vec{c}(A_k) \restriction n_k \rangle.$$

Notice that $d(A)$ takes values in a finite set, namely, the set $(3^{n_k})^k$. Therefore d is a finite coloring of $[\omega]^\omega$.

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For every $B \in [\omega]^\omega$, $|\{d(A) : A \in [B]^\omega\}| \geq k$ (and therefore, $\omega \not\to [\omega]^\omega_k$).

Let $B \in [\omega]^\omega$, we find k infinite subsets of B, $A^1, \ldots, A^k$, such that $d(A^1), \ldots, d(A^k)$ are all different.

Reference

Di Prisco, C. A. and S. Todorcevic, The shift graph and the Ramsey degree of $[\mathbb{N}]^\omega$. Acta Mathematica Hungarica (to appear).

Open problems

- (very old) Are inaccessibles needed for the consistency of $ZF + DC +$" "all sets of reals have the Ramsey property"?
- Are the following partition relations equivalent? $\omega \to (\omega)^\omega$ and $\omega \to [\omega]^\omega_\omega$
- Can we get a non-Ramsey set from an $\aleph_1$-sequence of real numbers?

GRACIAS