# Domination by metric spaces

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# Contents

**Definition (Cascales, Orihuela, Tkachuk)**

A family $\mathcal{B}$ is called $M$-ordered for some space $M$ if $\mathcal{B} = \{B_K : K \in \mathcal{K}(M)\}$ while $K \subset L$ implies $B_K \subset B_L$. A space $X$ is dominated by a space $M$ if it has an $M$-ordered compact cover. The space $X$ is strongly $M$-dominated if it has an $M$-ordered fundamental compact cover $\mathcal{C}$.

Given a space $X$ let $M$ be the space obtained by giving $\mathcal{K}(X)$ the discrete topology. It is clear that $X$ is strongly dominated by $M$. The fact that every topological space is dominated by a metric space motivates the following.

### Definition (GS)

For a space $X$ the (strong) metric domination index of $X$ denoted by $dm(X), (sdm(X))$ is the cardinal defined by $dm(X) = min\{w(M) : M$ is a metric space that (strongly) dominates $X\}$.

## Basic facts

*For any space $X$ the following hold:*

- $dm(X) \leq \ell\Sigma(X)$
- *If* $dm(X) \leq \kappa$ *then the metric domination index of any continuous image of* $X$ *is also not greater than* $\kappa$.
- *If* $dm(X) \leq \kappa$ *and* $Y$ *is any closed subset of* $X$ *then* $dm(Y) \leq \kappa$.
- *If* $X = \bigcup_{i \in \omega} X_i$ *and* $dm(X_i) \leq \kappa$ *then* $dm(X) \leq \kappa$.
- $ext(X) \leq dm(X)$.

# *dm* vs other invariants

Tkachuk constructed a space $X$ which is not Lindelöf but $dm(X) \leq \omega$. Also, Cascales, Muñoz and Orihuela attributed to J. Pelant an example of a space for which $\omega = dm(X) < \ell\Sigma(X)$. However, C.O.T. proved that for any space $X$ we have $\ell\Sigma(C_p(X)) \leq \omega$, if and only if $dm(C_p(X)) \leq \omega$, they also showed that the equivalence $dm(Y) \leq \omega$ if and only if $\ell\Sigma(Y) \leq \omega$ is true for any space in which every relatively countably compact subset has compact closure, for instance in $C_p(K)$-spaces. Actually both cardinal invariants $dm$ and $\ell\Sigma$ coincide in the class of $C_p(K)$-spaces.

# $dm$ vs $K$-analycity

M. Talagrand proved that for a compact space $K$ we have $dm(C_p(K)) \leq \omega$ if and only if $C_p(K)$ is $K$-analytic. Later, V. Tkachuk generalized Talagrand's result for any space $C_p(X)$.

# Metrizability

Christensen proved that a second countable space is strongly dominated by the irrationals if and only if it is completely metrizable. This result cannot be extended to metric spaces of arbitrary weight.

## Theorem

*If $M$ is a metric space of weight $\kappa$ then $M$ admits a family of compact sets swallowing compact sets $\{K_\alpha : \alpha \in ([k^\omega]^{<\omega})^\omega\}$.*

***Proof:*** Since $M$ contains at most $\kappa^\omega$ compact subsets.
Therefore we can write $\{C_\beta : \beta \in \xi\}$ the family of all the non void compact subsets of $M$ for some $\xi \leq \kappa^\omega$. Let $L_\beta = C_\beta$ if $\beta, \xi$ and $L_\beta = \varnothing$ if $\xi \leq \beta < \kappa^\omega$.
Let $\alpha \in ([\kappa^\omega]^{<\omega})^\omega$ so $\alpha = (s_0, s_1, ...)$ with $s_n$ a finite subset of $\kappa^\omega$.
Define $K_\alpha = \bigcup\{L_\beta : \beta \in s_0\}$.
Clearly the family $\{K_\alpha : \alpha \in ([\kappa^\omega]^{<\omega})^\omega\}$ is as required.

# Metrizability of compact spaces

Cascales and Orihuela showed that a compact space $K$ is metrizable if $K^2 \setminus \Delta$ is strongly dominated by the irrationals. Later, assuming either $t(K) \leq \omega$ or $MA(\omega_1)$, C.O.T. proved if $K^2 \setminus \Delta)$ is $\omega^\omega$-dominated then $K$ is metrizable.

**Definition (Kakol)**

Given a topological group $G$ a $\mathcal{G}$-base of the identity $e \in E$ is a topological local base $\mathcal{U}$ of $e$ such that $\mathcal{U} = \{U_K : K \in \mathcal{K}(\omega^\omega)\}$ and $K \subset L$ implies $U_K \subset U_L$.

# Metrizability of compact spaces

## Theorem

*Assume that $K$ is a compact space in a topological group $E$ that has a $\mathcal{G}$-base $\mathcal{U} = \{U_C : C \in \mathcal{K}(\omega^\omega)\}$ of the identity, then $K$ is metrizable.*

***Proof:*** Let $X = (K \times K) \setminus \Delta$. For each $C \in \mathcal{K}(\omega^\omega)$ the set $W_C = \{(x, y) \in X : xy^{-1} \notin U_C\}$ is closed in $X$, so it is compact. Moreover, the cover $\mathcal{W} = \{W_C : C \in \mathcal{K}(\omega^\omega)\}$ is $\omega^\omega$-ordered. For each $C \in \mathcal{K}(\omega^\omega)$ the set $V_C := \{(x, y) \in X : xy^{-1} \notin \overline{U_C}\}$ is open and $V_C \subset W_C$. Take $T \in \mathcal{K}(X)$ compact. For every $p = (x, y) \in T$ there is $C(p) \in \mathcal{K}(\omega^\omega)$ such that $xy^{-1} \notin \overline{U_{C(p)}}$ therefore $p \in V_{C(p)}$. There is a finite set $F = \{p_0, ..., p_n\}$ such that $T \subset \bigcup\{V_{C(p)} : p \in F\}$. Find $L \in \mathcal{K}(\omega^\omega)$ such that $C(p) \subset L$ for every $p \in F$. It follows from the definition of the $V_C$'s that $T \subset V_L \subset W_L$. So the family $\{W_C : C \in \mathcal{K}(\omega^\omega)\}$ swallows all compact sets in $X$ and hence $K$ is metrtizable.

# Metrizability of function spaces

C.O.T. proved under CH in that if $K$ is compact and $C_p(K)$ is strongly dominated by a second countable space then $K$ is countable and hence $C_p(K)$ is metrizable.

**Lemma (GS)**

*Given an infinite cardinal $\kappa$, if $X$ is a space of cardinality $\kappa$ with a unique non-isolated point and $C_p(X)$ is strongly dominated by some space $M$ then $\mathbb{R}^\kappa$ is strongly by dominated by $M$.*

# Corollary (GS)

Define the cardinal $\mathfrak{l} = min\{\gamma : \mathbb{R}^\gamma$ does not contain a closed discrete set of cardinality $\gamma\}$, observe that $\omega_1 < \mathfrak{l}$.
Suppose $\kappa < \mathfrak{l}$. If $X$ is a space of cardinality $\kappa$ with a unique non-isolated point and $C_p(X)$ is strongly dominated by some metric space $M$ then $\kappa \leq w(M)$.
So, if $\kappa < \mathfrak{l}$ and $X$ is a space with a unique non-isolated point such that $sdm(C_p(X)) \leq \kappa$, then $|X| \leq \kappa$.

# Theorem (GS)

*For a space $X$ such that $C_p(X)$ is strongly dominated by a second countable space the following hold:*

- *If $X$ is separable then it is countable.*
- *If $X$ is scattered then it is countable.*
- *Every second countable continuous image of $X$ is countable.*
- *If $X$ is compact then it is countable.*
- *If $X$ is pseudocompact then it is countable.*
- *If $K \subset X$ is compact then $K$ is scattered.*
- *If $X$ is Lindelöf-p then $X$ is the union of countably many compact scattered subsets.*

# Open problems

## Problem (GS)

Is it true that $dm(C_p(X) = \ell\Sigma(C_p(X))$ for every Tychonoff space $X$?

## Problem (Kakol)

Is it true that every Frechet-Urysohn topological group which is Frechet-Urysohn and has a G-base is metrizable?

## Problem

Is it possible to give a direct prove in ZFC that if $S_\kappa = \{x \in \{0,1\}^\kappa : x{-}1(1) \text{ is finite}\}$ is strongly dominated by a second countable space then $\kappa \leq \omega$?

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