# Winter School in Abstract Analysis
# section Set Theory & Topology

25th Jan — 1st Feb 2014

TUTORIAL SPEAKERS

**Antonio Avilés**
**Jaroslav Nešetřil**
**Dilip Raghavan**
**Matteo Viale**

## DEADLINES

SUPPORT
early Dec 2013

REGISTRATION
30th Dec 2013

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# PID and $\omega_1$-towers in $\mathcal{P}(\omega)$

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## Definition (Tower)

We say that $\mathcal{T} = (T_\alpha)_{\alpha\in\omega_1}$ is a *tower* if

- $T_\alpha \subset \omega$ for each $\alpha \in \omega_1$
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*Relations between properties of* $(\mathcal{T}, \subset)$ *and* $\langle \mathcal{T} \rangle$*?*

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### Fact

*Every Hausdorff tower is special.*

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- (OCA) implies that every tower is special.
- $(T_\alpha \cap c)_{\alpha \in \omega_1}$ is Suslin for a Cohen real $c$ and each groundmodel tower $\mathcal{T} = (T_\alpha)_{\alpha \in \omega_1}$.

**Theorem (Todorcevic – oscillation theory)**

*If $\langle \mathcal{T} \rangle$ is a non-meager ideal then $\mathcal{T}$ is Suslin.*

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*Tower $\mathcal{T}$ is Hausdorff iff $(\langle \mathcal{T} \rangle, \subset)$ is Tukey equivalent to $([\omega_1]^{<\omega}, \subset)$ (i.e. Tukey maximal).*

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*Assume PID. TFAE*

- *Every tower is Hausdorff*
- $\mathfrak{b} > \omega_1$

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### Theorem (Raghavan, Todorcevic)

*Assume PID. TFAE*

- *There is no Tukey type between $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$*
- $\min(\mathfrak{b}, \mathrm{cof}(F_\sigma)) > \omega_1$

### Theorem

*Let* $\mathcal{T} = (T_\alpha)_{\alpha \in \omega_1}$ *be a tower, assume* PID *and* $\mathfrak{b} > \omega_1$*.*
*The tower* $\mathcal{T}$ *is Hausdorff.*

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Let $\mathcal{I} \subset [\omega_1]^{\leq\omega}$ be a P-ideal. One of the following holds:

1. There is an uncountable $K \subset \omega_1$ such that $[K]^\omega \subset \mathcal{I}$;
2. $\omega_1 = \bigcup_{n\in\omega} A_n$ and $A_n \cap I$ is finite for each $n \in \omega$ and $I \in \mathcal{I}$.

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Define $\mathcal{I} \subset [\omega_1]^{\leq\omega}$ by $I \in \mathcal{I}$ iff

$$C^n_\alpha(I) = \{\xi \in \alpha \cap I : T_\xi \setminus T_\alpha \subset n\}$$

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We want to show that 1. of PID holds.

$$I \in \mathcal{I} \text{ iff } C^n_\alpha(I) = \{\xi \in \alpha \cap I \colon T_\xi \setminus T_\alpha \subset n\} \text{ is finite}$$

$$I \in \mathcal{I} \text{ iff } C_\alpha^n(I) = \{\xi \in \alpha \cap I \colon T_\xi \setminus T_\alpha \subset n\} \text{ is finite}$$

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**Claim**

*$\mathcal{I}$ is an ideal.*

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**Claim**

*$\mathcal{I}$ is an ideal.*

$I \subset J \Rightarrow C^n_\alpha(I) \subset C^n_\alpha(J)$

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**Claim**

*$\mathcal{I}$ is an ideal.*

$I \subset J \Rightarrow C^n_\alpha(I) \subset C^n_\alpha(J)$, $C^n_\alpha(I \cup J) = C^n_\alpha(I) \cup C^n_\alpha(J)$

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**Lemma**

*$\mathcal{I}$ is a P-ideal.*

We need to show that for each (pairwise disjoint) $\{I_\ell \colon \ell \in \omega\} \subset \mathcal{I}$ there is $I \in \mathcal{I}$ such that $I_\ell \subset^* I$ for each $\ell$.

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Fix enumerations $I_\ell = \left\{\xi^k_\ell \colon k \in \omega\right\}$.
For $\alpha \in \omega_1$ and $n \in \omega$ define $f^n_\alpha \colon \omega \to \omega$

$$f^n_\alpha(\ell) = \max\left\{k \colon \xi^\ell_k \in C^n_\alpha(I_\ell)\right\}$$

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Let $g \colon \omega \to \omega$ be a function $\leq^*$-dominating $\{f^n_\alpha \colon \alpha \in \omega_1, n \in \omega\}$.
Let

$$I = \bigcup_{\ell \in \omega} I_\ell \setminus \left\{\xi^k_n \colon k \leq g(\ell)\right\}.$$

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Alternative 2. of PID:
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**Lemma**

*Alternative 2. of* PID *fails.*

We show that for each uncountable $K \subset \omega_1$ there is $I \in \mathcal{I} \cap [K]^\omega$.

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If $\overline{\{T_\alpha \colon \alpha \in K\}} \subset \langle \mathcal{T} \rangle$, then $\langle \mathcal{T} \rangle$ is generated by this closed set and is a Borel (P-)ideal.
Borel P-ideals cannot be generated by $< \mathfrak{d}$ many sets.

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$I \in \mathcal{I}$

Suppose $C^n_\beta(I)$ is infinite for some $\beta \in \omega_1, n \in \omega$.
I.e. $T_\alpha \subset T_\beta \cup n$ for infinitely many $\alpha \in I$.

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$\{y \in 2^\omega \colon y \subset T_\beta \cup n\}$ is closed, thus

$$x \in \overline{C^n_\beta(I)} \subset \{y \colon y \subset T_\beta \cup n\} \subset \langle T \rangle$$

The limit point of $C^n_\beta(I)$ is a subset of $T_\beta \cup n$ and hence in $\langle \mathcal{T} \rangle$, a contradiction.

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YES if one of:

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- $\diamondsuit(2, =)$

# Reference

Piotr Borodulin-Nadzieja, David Chodounský, *Hausdorff gaps and towers in* $P(\omega)/\mathrm{Fin}$, arXiv:1302.4550 [math.LO]