Countably compact group topologies on abelian groups from selective ultrafilters

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(joint work with A. H. Tomita)

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About

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In this talk, I will try to give an idea of how their construction can be adapted in order to endow the additive group of the real line with a countably compact Hausdorff group topology (without non-trivial convergent sequences).

Overview

We will identify $\mathbb{R}$ with $\mathbb{Q}^{(\mathfrak{c})}$ and construct a group monomorphism

$$\Phi : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}^{\mathfrak{c}}$$

so that $\Phi[\mathbb{Q}^{(\mathfrak{c})}]$ is countably compact and has no convergent sequences (except the trivial ones) when considered with the subspace topology induced by $\mathbb{T}^{\mathfrak{c}}$.

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How such embedding will be constructed?

We will associate to each $\alpha < \mathfrak{c}$ a group homomorphism $\phi_\alpha : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}$ satisfying two conditions and Φ will be given by

$$\begin{array}{rccc} \Phi : & \mathbb{Q}^{(\mathfrak{c})} & \to & \mathbb{T}^{\mathfrak{c}} \\ & J & \mapsto & \Phi(J) \end{array}$$

where

$$\Phi(J)(\alpha) = \phi_\alpha(J), \text{ for every } \alpha < \mathfrak{c}.$$

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In every inductive step, we will approximate the values of ϕ by non-empty open arcs of $\mathbb{T}$. To make this possible, we will deal only with sequences where there is enough "freedom" to prescribe a pre-determined accumulation point.

Sorting the sequences

If $J \in \mathbb{Q}^{(\mathfrak{c})}$ and $\mu \in \operatorname{supp} J$, then

$$J(\mu) = \frac{p(J,\mu)}{q(J,\mu)}$$

where $q(J,\mu) > 0$ and $\gcd(p(J,\mu), q(J,\mu)) = 1$. Define

$$d(J) = \operatorname{lcm}\{q(J,\mu) : \mu \in \operatorname{supp} J\} \text{ and } a(J,\mu) = d(J) \cdot J(\mu).$$

Define, also,

$$|p(J)| = \max\{|p(J,\mu)| : \mu \in \operatorname{supp} J\}$$

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Definition

We say that $f : \omega \to \mathbb{Q}^{(\mathfrak{c})}$ is *of type i* if f satisfies condition (i) described below:

(1) $\operatorname{supp} f(n) \setminus \bigcup_{m<n} \operatorname{supp} f(m) \neq \emptyset$, for every $n \in \omega$;

(2) $|q(f(n))| > n$, for every $n \in \omega$;

(3) $\{|q(f(n))| : n \in \omega\}$ is finite and $|p(f(n))| > n$, for every $n \in \omega$.

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Proposition

If $f : \omega \to \mathbb{Q}^{(\mathfrak{c})}$, there exists $j : \omega \to \omega$ strictly increasing such that $f \circ j$ is either constant or of type i, for some $i \in \{1, 2, 3\}$.

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Let $\mathcal{F}$ be the set of all sequences in $\mathbb{Q}^{(\mathfrak{c})}$ that are of type 1, 2 or 3. Fix $\{f_\xi : 0 < \xi < \mathfrak{c}\}$ an enumeration of $\mathcal{F}$ such that

$$\bigcup_{n \in \omega} \operatorname{supp} f_\xi(n) \subset \xi, \text{ for every } \xi \in]0, \mathfrak{c}[.$$

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Fix also $\{p_\xi : 0 < \xi < \mathfrak{c}\}$ a family of pairwise incomparable selective ultrafilters and $\{J_\alpha : \alpha < \mathfrak{c}\}$ an enumeration of $\mathbb{Q}^{(\mathfrak{c})}$.

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For every $\alpha < \mathfrak{c}$, there exists $\phi_\alpha : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}$ a group homomorphism such that:

(i) $\phi_\alpha(J_\alpha) \neq 0 + \mathbb{Z}$;

(ii) $\phi_\alpha(\chi_\xi) = p_\xi - \lim\{\phi_\alpha(f_\xi(n)) : n \in \omega\}$, *for every* $\xi \in]0, \mathfrak{c}[$.

A countably compact group topology on the real line

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Proof.

Consider $f : \omega \to \mathbb{Q}^{(\mathfrak{c})}$. If f is non-trivial, then f has two constant and distinct subsequences or f has an injective subsequence. If $f \circ j_1$ is an injective subsequence of f, there exists $j_2 : \omega \to \omega$ strictly increasing such that $n \mapsto f \circ j_1 \circ j_2(2n)$ and $n \mapsto f \circ j_1 \circ j_2(2n+1)$ belongs to $\mathcal{F}$. So, they are given respectively by f_{ξ_0} and f_{ξ_1}, for some $\xi_0, \xi_1 \in]0, \mathfrak{c}[$.

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Since $\phi_\alpha(\chi_{\xi_i}) = p_{\xi_i} - \lim\{\phi_\alpha(f_{\xi_i}(n)) : n \in \omega\}$ for every $\alpha < \mathfrak{c}$, we have $\Phi(\chi_{\xi_i}) = p_{\xi_i} - \lim\{\Phi(f_{\xi_i}(n)) : n \in \omega\}$. Thus, χ_{ξ_0} and χ_{ξ_1} are (distinct) accumulation points of $\{f(n) : n \in \omega\}$.

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Since $\phi_\alpha(\chi_{\xi_i}) = p_{\xi_i} - \lim\{\phi_\alpha(f_{\xi_i}(n)) : n \in \omega\}$ for every $\alpha < \mathfrak{c}$, we have $\Phi(\chi_{\xi_i}) = p_{\xi_i} - \lim\{\Phi(f_{\xi_i}(n)) : n \in \omega\}$. Thus, χ_{ξ_0} and χ_{ξ_1} are (distinct) accumulation points of $\{f(n) : n \in \omega\}$. It implies that f is not convergent and that $\tau = \{\Phi^{-1}(U \cap \Phi[\mathbb{Q}^{(\mathfrak{c})}]) : U \text{ is an open set of } \mathbb{T}^{\mathfrak{c}}\}$ is a countably compact group topology on $\mathbb{Q}^{(\mathfrak{c})}$. □

Constructing group homomorphisms

Fix $\alpha < \mathfrak{c}$ and let $E \in [\mathfrak{c}]^\omega$ be such that:

- $\operatorname{supp} J_\alpha \subset E$;
- $\bigcup_{n\in\omega} f_\xi(n) \subset E$, for every $\xi \in E \setminus \{0\}$.

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We obtain $\{E_k : k \in \omega\} \subset [E]^{<\omega}$, $\{b_k : k \in \omega\}$ strictly increasing, $\{r_k : k \in \omega\}$ and $i : \omega \to E \setminus \{0\}$ such that:

- $\operatorname{supp} J \subset E_0$ and $E = \bigcup_{k\in\omega} E_k$;
- $i(k) \in E_k$ and $E_{k+1} \supset E_k \cup \operatorname{supp} f_{i(k)}(b_k)$;
- $\{b_k : k \in i^{-1}(\{\xi\})\} \in p_\xi$;
- $f_{i(k)}$ of type 1: $\operatorname{supp} f_{i(k)}(b_k) \setminus E_k \neq \emptyset$;
- $f_{i(k)}$ of type 2: $|q(f_{i(k)}(b_k))| \cdot r_k > d(J) \cdot \prod_{m<k} d(f_{i(m)}(b_m)) =: c_k$;
- $f_{i(k)}$ of type 3: $|a(f_{i(k)}(b_k))| \cdot r_k > 4 \cdot d(f_{i(k)}(b_k))$;
- $r_0 = \frac{1}{4\cdot\sum_{\mu\in\operatorname{supp} J} |a(J,\mu)|}$;
- $r_{k+1} = \frac{r_k}{2\cdot\sum_{\mu\in\operatorname{supp} f_{i(k)}(b_k)} |a(f_{i(k)}(b_k),\mu)|}$, for every $k \in \omega$.

Constructing group homomorphisms

Lemma

Let $\{p_j : j \in \omega\}$ *be a family of pairwise incomparable selective ultrafilters. For each* $j \in \omega$, *let* $\{a^j_k : k \in \omega\} \in p_j$ *be an increasing sequence such that* $k < a^j_k$ *for every* $k \in \omega$. *Then, there exists a family* $\{I_j : j \in \omega\}$ *of pairwise disjoint subsets of* ω *such that:*

(i) $\{a^j_k : k \in I_j\} \in p_j$, *for every* $j \in \omega$;

(ii) $\{[k, a^j_k] : k \in I_j, j \in \omega\}$ *is a family of pairwise disjoint intervals of* ω.

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How to construct $\phi : \mathbb{Q}^{(E)} \to \mathbb{T}$?

First, we guarantee that $\phi(J_\alpha) \neq 0$ associating to each $\xi \in E_0$ an open arc $\psi_0(\xi)$ with diameter $r_0/d(J)$ such that $0 + \mathbb{Z} \notin \sum_{\xi \in \text{supp } J} a(J, \xi) \cdot \psi_0(\xi)$.

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In a successor step $k+1$, we will associate to each $\xi \in E_{k+1}$ an open arc $\psi_{k+1}(\xi)$ such that:

(1) $d(f_{i(k)}(b_k)) \cdot \overline{\psi_{k+1}(\xi)} \subset \psi_k(\xi)$, if $\xi \in E_k$;

(2) $c_k \cdot d(f_{i(k)}(b_k)) \cdot \delta(\psi_{k+1}(\xi)) = r_{k+1} \ \forall \xi \in E_{k+1}$;

(3) $\delta(\sum_{\xi \in \text{supp } f_{i(k)}(b_k)} a(f_{i(k)}(b_k), \xi) \cdot \psi_{k+1}(\xi)) \leq r_k / c_k$;

(4) $\psi_k(i(k)) \cap \sum_{\xi \in \text{supp } f_{i(k)}(b_k)} a(f_{i(k)}(b_k), \xi) \cdot \psi_{k+1}(\xi) \neq \emptyset$;

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If we define $\phi(\chi_\xi) = \bigcap_{k \geq N_\xi} c_k \cdot \psi_k(\xi)$ where $N_\xi = \min\{n \in \omega : \xi \in E_n\}$ we conclude that

$$\{\phi(f_{i(k)}(b_k)) : k \in i^{-1}(\{\xi\})\} \to \phi(\chi_\xi)$$

and, therefore, that $\phi(\chi_\xi) = p_\xi - \lim\{\phi(f_\xi(n)) : n \in \omega\}$.

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Proof.

We have $\phi : \mathbb{Q}^{(E)} \to \mathbb{T}$ satisfying (i) and (ii) for each $\xi \in E \setminus \{0\}$.

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Theorem (B.-Tomita, Houston J. Math. (2013))

Under the existence of $\mathfrak{c}$ many pairwise incomparable selective ultrafilters, if G is an abelian almost torsion-free group such that $|G| = \mathfrak{c}$, then G can be endowed with a countably compact group topology without non-trivial convergent sequences.