Compact indestructibility

$\mathcal{O}$ denotes the collection of all open covers of a space X.

The Rothberger game of length α $\mathrm{G}_1^\alpha(\mathcal{O},\mathcal{O})$ is played as follows: at the β-inning 1 chooses an open cover $\mathcal{U}_\beta \in \mathcal{O}$ and 2 responds by taking $U_\beta \in \mathcal{U}_\beta$. At the end, 2 wins if and only if $\{U_\beta : \beta < \alpha\} \in \mathcal{O}$.

The selection principle $\mathrm{S}_1^\alpha(\mathcal{O},\mathcal{O})$ is the assertion "for every sequence $\{\mathcal{U}_\beta : \beta < \alpha\} \subseteq \mathcal{O}$ one may pick $U_\beta \in \mathcal{U}_\beta$ in such a way that $\{U_\beta : \beta < \alpha\} \in \mathcal{O}$".

Proposition. *If 1 has no winning strategy in $\mathrm{G}_1^\alpha(\mathcal{O},\mathcal{O})$, then $\mathrm{S}_1^\alpha(\mathcal{O},\mathcal{O})$.*

A space is Rothberger if $\mathrm{S}_1^\omega(\mathcal{O},\mathcal{O})$ holds.

Theorem. *(Pawlikowski, 1994) $\mathrm{S}_1^\omega(\mathcal{O},\mathcal{O}) \Rightarrow$ 1 does not have a winning strategy in $\mathrm{G}_1^\omega(\mathcal{O},\mathcal{O})$.*

A compact space is indestructible if it remains compact in any countably closed forcing extension.

Proposition 1. *(Scheepers-Tall) A compact space X is indestructibly compact if and only if 1 does not have a winning strategy in $G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$.*

Some facts discovered by Dias and Tall are:

– Every compact hereditarily Lindelöf space is indestructible.

– The one-point compactification of a discrete space is indestructible.

– CH A first countable compact T_2 space is indestructible.

– If a compact space contains a closed copy of 2^{ω_1}, then it is desctructible.

– A compact T_2 indestructible space contains a non-trivial convergent sequence.

Theorem 1. *Every compact indestructible space is sequentially compact.*

Sketch. Let X be a compact non sequentially compact space.

Fix a sequence $\langle a_n : n < \omega \rangle$ with no convergent subsequence and let $A_0 = \{a_n : n < \omega\}$. For each $x \in X$ there is an open set U_x^1 such that $x \in U_x^1$ and $|A_0 \setminus U_x^1| = \aleph_0$. The first move of 1 is the open cover $\mathcal{U}_1 = \{U_x^1 : x \in X\}$.

If 2 responds by choosing $V_1 \in \mathcal{U}_1$, then let $A_1 = A_0 \setminus V_1$. For each $x \in X$ there is an open set U_x^2 such that $x \in U_x^2$ and $|A_1 \setminus U_x^2| = \aleph_0$. The second move of 1 is the open cover $\mathcal{U}_2 = \{U_x^2 : x \in X\}$.

In general, at the α-th inning the moves of the two players have defined a mod finite decreasing family $\{A_\beta : \beta < \alpha\}$ of infinite subsets of A_0. Then 1 fixes an infinite set $B_\alpha \subseteq A_0$ such that $B_\alpha \subseteq^* A_\beta$ for each $\beta < \alpha$ and he plays $\mathcal{U}_\alpha = \{U_x^\alpha : x \in X\}$, where U_x^α is an open set such that $x \in U_x^\alpha$ and $|B_\alpha \setminus U_x^\alpha| = \aleph_0$ and so on.

At the end of the game, the set resulting from the moves of 2 is a collection $\mathcal{V} = \{V_\alpha : 1 \leq \alpha < \omega_1\}$ and for each α there is an infinite set $A_\alpha \subseteq X \setminus V_\alpha$ such that $\{A_\alpha : \alpha < \omega_1\}$ is a mod finite decreasing sequence.

The compactness of X implies that $\mathcal{V}$ cannot be a cover and so 1 wins. $\square$

A space X has the *finite derived set* (briefly FDS) property provided that every infinite set of X contains an infinite subset with at most finitely many accumulation points.

Proposition. *(Alas-Tkachenko-Tkachuk-Wilson, 2005) Every hereditarily Lindelöf* T_2 *space has the FDS property.*

Proposition. *(Tall, 1995) Hereditarily Lindelöf* $\Rightarrow$ *Indestructibily Lindelöf.*

Theorem 2. *A Lindelöf* T_2 *indestructible space has the finite derived set property.*

A space X is pseudoradial provided that for any non-closed set $A \subseteq X$ there exists a well-ordered net $S \subseteq A$ which converges to a point outside A.

Lemma. *(Dias-Tall) A compact T_2 space which is not first countable at any point is destructible.*

Theorem 3. *Any compact T_2 indestructible space is pseudoradial.*

Sketch. Let X be a compact T_2 indestructible space and let A be a non-closed subset. WLOG we assume $X = \overline{A}$. Let λ be the smallest cardinal such that there exists a non-empty closed G_λ-set $H \subseteq X \setminus A$. As X is indestructible, so is H and by the lemma H is first countable at some point p. Clearly, $\{p\}$ is a G_λ-set in X. Now, the minimality of λ and the compactness of X suffice to find a well-ordered net in A which converges to p. $\square$

Theorem 3 is no longer true for Lindelöf spaces.

Proposition. *(Koszmider-Tall, 2002) There is a model of ZFC+CH with a regular Lindelöf P-space Z of cardinality $\aleph_2$ without Lindelöf subspaces of size $\aleph_1$.*

Z is indestructibily Lindelöf, because a Lindelöf P-space is Rothberger.

Z is not pseudoradial, because any subset of Z of size $\aleph_1$ is radially closed but not closed.

Theorem 3 is far to be invertible:

Proposition. *(Dias-Tall) There exists a desctructible compact LOTS.*

Theorem 4. *CH Every compact sequential T_2 space is indestructible.*

As CH is consistent with the existence of a compact T_2 space of countable tightness which is not sequentially compact, in Theorem 4 "sequential" cannot be weakened to "countable tightness".

Question 1. *CH Is a compact T_2 pseudoradial space of countable tightness indestructible?*

A space X is weakly Whyburn provided that for any non-closed set $A \subseteq X$ there exists a set $B \subseteq A$ such that $|\overline{B} \setminus A| = 1$.

Every sequential T_2 space is weakly Whyburn and every compact T_2 weakly Whyburn space is pseudoradial.

Thanks to a recent result of Alas and Wilson, we can prove:

Theorem 5. *CH Every compact weakly Whyburn space of countable tightness is indestructible.*

Theorem. *(Pawlikowski, 1994)* $S_1^{\omega}(\mathcal{O},\mathcal{O})$ *implies 1 does not have a winning strategy in* $G_1^{\omega}(\mathcal{O},\mathcal{O})$.

Proposition. *(Dias-Tall) [CH] There exists a destructible compact spaces which satisfies the selection principle* $S_1^{\omega_1}(\mathcal{O},\mathcal{O})$.

Question 2. *Let* X *be a compact (or compact* T_2*) space satisfying* $S_1^{\omega_1}(\mathcal{O},\mathcal{O})$. *Is* X *sequentially compact?*

Question 2^+. *Is it true that any compact* T_2 *space satisfying* $S_1^{\omega_1}(\mathcal{O},\mathcal{O})$ *contains a non-trivial convergent sequence?*

Any counterexample to Question 2^+ would be an Efimov space.

Some cardinal inequalities

Theorem A. *(Arhangel'skiĭ, 1969) If X is a first countable Lindelöf T_2 space, then $|X| \leq 2^{\aleph_0}$.*

Theorem B. *(Gorelic, 1993) In a model of ZFC there exists a Lindelöf T_2 space X with points G_δ in which 1 does not have a winning strategy in $G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ and $|X| > 2^{\aleph_0}$.*

Theorem C. *(Scheepers-Tall, 2010) If X is a space with points G_δ and $2 \uparrow G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$, then $|X| \leq 2^{\aleph_0}$.*

The game $\mathrm{G}_{\mathrm{fin}}^{\alpha}(\mathcal{O}, \mathcal{O})$ is played as follows: at the β-inning 1 chooses $\mathcal{U}_\beta \in \mathcal{O}$ and 2 responds by taking a finite set $\mathcal{V}_\beta \subseteq \mathcal{U}_\beta$. 2 wins if and only if $\bigcup\{\mathcal{V}_\beta : \beta < \alpha\} \in \mathcal{O}$.

$$2 \uparrow \mathrm{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O}) \Rightarrow 2 \uparrow \mathrm{G}_{\mathrm{fin}}^{\omega_1}(\mathcal{O}, \mathcal{O})$$

Problem. *Does Theorem C continue to hold for $G_{\mathrm{fin}}^{\omega_1}$?*

Given a space X, a K-cover is an open collection $\mathcal{U}$ such that for every compact set $K \subseteq X$ there is some $U \in \mathcal{U}$ such that $K \subseteq U$.

$\mathcal{K}$ is the collection of all K-covers.

Let $\mathcal{O}^*$ be the collection of all open covers which are closed under finite unions.

Since $\mathcal{O}^* \subseteq \mathcal{K} \subseteq \mathcal{O}$, we have:

$$2 \uparrow \mathrm{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O}) \Rightarrow 2 \uparrow \mathrm{G}_1^{\omega_1}(\mathcal{K}, \mathcal{O}) \Rightarrow 2 \uparrow \mathrm{G}_1^{\omega_1}(\mathcal{O}^*, \mathcal{O}) \equiv 2 \uparrow \mathrm{G}_{\mathrm{fin}}^{\omega_1}(\mathcal{O}, \mathcal{O})$$

2^{ω_1} witnesses $2 \uparrow \mathrm{G}_1^{\omega_1}(\mathcal{K}, \mathcal{O}) \not\Rightarrow 2 \uparrow \mathrm{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$.

Lemma 1. *(Gryzlov) Let* X *be a space with points* G_δ*. If* K *is a compact subset of* X*, then there is a collection of open sets* $\mathcal{U}$ *such that* $K = \bigcap \mathcal{U}$ *and* $|\mathcal{U}| \leq 2^{\aleph_0}$*.*

Lemma 2. *If* σ *is a winning strategy for 2 in* $G_1^{\omega_1}(\mathcal{K}, \mathcal{O})$ *played on* X*, then for any sequence* $\{U_\beta : \beta < \alpha\}$ *of K-covers there exists a compact set* $K \subseteq X$ *such that for any open set* $U \supseteq K$ *there exists a K-cover* $\mathcal{V}$ *satisfying* $\sigma(\{U_\beta : \beta < \alpha\} \frown \mathcal{V}) = V$*.*

Theorem 1. *Let* X *be a space with points* G_δ*. If* $2 \uparrow G_1^{\omega_1}(\mathcal{K}, \mathcal{O})$*, then* $|X| \leq 2^{\aleph_0}$*.*

Theorem 2. *Let* X *be a Tychonoff space.* $2 \uparrow G_{\text{fin}}^{\omega_1}(\mathcal{O}, \mathcal{O}) \Rightarrow 2 \uparrow G_1^{\mathfrak{c}}(\mathcal{K}, \mathcal{O})$*.*

Corollary. *CH If* X *is a Tychonoff space with points* G_δ *and* $2 \uparrow G_{\text{fin}}^{\omega_1}(\mathcal{O}, \mathcal{O})$*, then* $|X| \leq 2^{\aleph_0}$*.*

A space X is weakly Lindelöf if every open cover $\mathcal{U}$ has a countable subfamily $\mathcal{V}$ such that $\bigcup \mathcal{V}$ is dense in X.

Question 3. *(Bell-Ginsburg-Woods, 1978) Let X be a first countable regular weakly Lindelöf space. Is $|X| \leq 2^{\aleph_0}$?*

For a given space X, $\mathcal{D}$ denotes the collection of all open families $\mathcal{U}$ such that $\bigcup \mathcal{U}$ is dense in X.

$$2 \uparrow \mathrm{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O}) \Rightarrow 2 \uparrow \mathrm{G}_1^{\omega_1}(\mathcal{O}, \mathcal{D})$$

Fact 1. *ccc $\Rightarrow 2 \uparrow G_1^{\omega_1}(\mathcal{O}, \mathcal{D}) \Rightarrow wL(X) \leq \aleph_1$.*

Fact 2. *MA+¬CH $1 \uparrow G_1^{\omega_1}(\mathcal{O}, \mathcal{D})$ in ω^*.*

Fact 3. *CH $2 \uparrow G_1^{\omega_1}(\mathcal{O}, \mathcal{D})$ in ω^*.*

Theorem 3. *Let X be a first countable regular space. If $2 \uparrow G_1^{\omega_1}(\mathcal{O}, \mathcal{D})$, then $|X| \leq 2^{\aleph_0}$.*

Question 4. *Let X be a first countable regular weakly Lindelöf space. Is $2 \uparrow G_1^{\omega_1}(\mathcal{O}, \mathcal{D})$ true?*

A space is homogeneous if for any pair of points there is a homeomorphism sending one point to the other. A space is power homogeneous if some power of it is homogeneous.

Theorem. *(Ridderbos, 2006 - van Mill, 2005 for compact spaces) If X is a regular power homogeneous space, then $|X| \leq 2^{c(X)\pi\chi(X)}$.*

$$ccc \Rightarrow 2 \uparrow \mathrm{G}_1^{\omega_1}(\mathcal{D}, \mathcal{D}) \Rightarrow c(X) \leq \aleph_1$$

Theorem 4. *Let X be a regular power homogeneous space of countable π-character. If $2 \uparrow G_1^{\omega_1}(\mathcal{D}, \mathcal{D})$, then $|X| \leq 2^{\aleph_0}$.*

Power homogeneous is necessary: $\beta\omega$!