How does the distortion of linear embedding of $C_0(K)$ into $C_0(\Gamma, X)$ spaces depend on the height of K?

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Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Banach 1932

$T : c \to c_0$

$$T(x_1, x_2, x_3, ...) = (2 \lim_{n\to\infty} x_n,\ x_1 - \lim_{n\to\infty} x_n,\ x_2 - \lim_{n\to\infty} x_n, ...).$$

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$$\|T\|\ \|T^{-1}\| = 3.$$

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Cambern 1968

$$\mathrm{d}(c, c_0) = 3.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

$$\mathrm{d}(C(K), c_0) = ?$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

$$\mathrm{d}(C(K), c_0) =?$$

$$C(K) \sim c_0 \Longrightarrow K \approx [1, \omega^n k], \text{ com } 1 \leq n, k < \omega.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

$$\mathrm{d}(C(K), c_0) = ?$$

$$C(K) \sim c_0 \Longrightarrow K \approx [1, \omega^n k], \text{ com } 1 \leq n, k < \omega.$$

Question

$$\mathrm{d}(C([1, \omega^n k]), c_0) = ?, \text{ for } 1 \leq n, k < \omega.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Definition

A Banach space $X \neq \{0\}$ is said to have *finite cotype* $2 \leq q < \infty$ if there is a constant $\kappa > 0$ such that no matter how we select finitely many vectors $v_1, v_2 \ldots, v_n$ from X,

$$(\sum_{i=1}^{n} \|v_i\|^q)^{\frac{1}{q}} \leq \kappa \left(\int_0^1 \| \sum_{i=1}^{n} r_i(t) v_i \|^2 dt \right)^{\frac{1}{2}},$$

where $r_i : [0,1] \to \mathbb{R}$ denote the *Rademacher functions*, defined by setting

$$r_i(t) = \mathrm{sign}(\sin 2^i \pi t).$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Recall that the derivative of a topological space K is the space $K^{(1)}$ obtained by deleting from K its isolated points. The α-th derivative $K^{(\alpha)}$ is defined recursively setting $K^{(0)} = K$ and

$$K^{(\alpha)} = \begin{cases} (K^{(\delta)})^{(1)} & \text{if } \alpha = \delta + 1, \\ \bigcap_{\beta<\alpha} K^{(\beta)} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

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Definition

A topological space K is said to be scattered if $K^{(\alpha)} = \varnothing$ for some ordinal α. In this case, the minimal α such that $K^{(\alpha)} = \varnothing$ is called the height of K (in short, $ht(K)$).

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Theorem

Let K be a locally compact Hausdorff space, Γ an infinite set with the discrete topology and X a Banach space with finite cotype. Then for every integer $n \geq 1$ and for every linear embedding T from $C_0(K)$ into $C_0(\Gamma, X)$ we have

$$K^{(n)} \neq \varnothing \Longrightarrow \|T\| \, \|T^{-1}\| \geq 2n + 1.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

- For a non-empty closed subset $K_1 \subseteq K$ we denote

$$\|f\|_{K_1} = \sup_{x \in K_1} \{|f(x)|\}.$$

- For every function $f \in C_0(K, X)$ and $\epsilon > 0$ we denote

$$\mathcal{K}(f, \epsilon) = \{x \in K : \|f(x)\| \geq \epsilon\}.$$

- For $n+1$ functions $g_0, g_1, \ldots, g_n$ in $C_0(K)$ satisfying

$$0 \leq g_0(x) \leq g_1(x) \leq \ldots \leq g_n(x) \leq 1, \forall x \in K,$$

we denote by $\mathcal{F}_{g_0, \ldots, g_n}$ the set of all $(f_1, \ldots, f_n) \in C_0(K)^n$ such that

$$0 \leq g_0(x) \leq f_1(x) \leq g_1(x) \leq \ldots \leq f_n(x) \leq g_n(x), \forall x \in K.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Proposition

Let J and K be locally compact Hausdorff spaces, X a Banach space with finite cotype and suppose that T is a linear embedding of $C_0(K)$ into $C_0(J, X)$ with $\|T^{-1}\| = 1$ and $\|T\| < 2n+1$ for some integer $n \geq 1$. Take $\delta > 0$ and $\theta < 1$ such that $\|T\| + 2\delta \leq (2n+1)\theta$, and $g_0, g_1, \dots, g_n$ in $C_0(K)$ satisfying $0 \leq g_0(x) \leq g_1(x) \leq \ldots \leq g_n(x) \leq 1, \forall x \in K$. Assume that for each $1 \leq i < j \leq n$

$$\mathcal{K}(Tg_i, \frac{\delta}{2n}) \cap \mathcal{K}(Tg_j, \frac{\delta}{2n}) = \varnothing.$$

Then

$$\|g_0\|_{K^{(1)}} > \theta \Longrightarrow \bigcap_{\mathcal{F}_{g_0,\dots,g_n}} \mathcal{K}(T(\sum_{i=1}^{n} f_i), \delta) \cap J^{(1)} \neq \varnothing.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Theorem

Let K be a locally compact Hausdorff space, Γ an infinite set with the discrete topology and X a Banach space with finite cotype. Suppose that there exists a linear embedding T from from $C_0(K)$ into $C_0(\Gamma, X)$. Then K has finite height and

$$\|T\| \, \|T^{-1}\| \geq 2 \; ht(K) - 1.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

$$C([0,1]) \hookrightarrow C_0(\mathbb{N}, C([0,1])).$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

$$C([0,1]) \hookrightarrow C_0(\mathbb{N}, C([0,1])).$$

$$[0,1]^{(\omega)} = [0,1].$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Corollary

Let X a Banach space with finite cotype and $1 \leq n, k < \omega$. Then

$$\mathrm{d}(C([1, \omega^n k], X), C_0(\mathbb{N}, X)) \geq 2n + 1.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Recall that every ordinal number $1 \leq \xi < \omega^\omega$ has an unique representation in the *Cantor normal form*,

$$\xi = \omega^{n_k} m_k + \ldots + \omega^{n_2} m_2 + \omega^{n_1} m_1$$

where $0 \leq n_1 < n_2 < \ldots < n_k < \omega$ and $1 \leq m_1 < \omega$, $1 \leq m_2 < \omega, \ldots, 1 \leq m_k < \omega$ and $1 \leq k < \omega$.

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

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Definition

For an ordinal number $1 \leq \xi < \omega^\omega$, represented in the Cantor normal form as above, we set $\xi^{[0]} = \xi$ and by induction

$$\xi^{[r]} = \begin{cases} \omega^{n_k} m_k + \ldots + \omega^{n_2} m_2 + \omega^{n_1+1} & \text{if } r = 1, \\ \left(\xi^{[r-1]}\right)^{[1]} & \text{if } 1 \leq r < \omega. \end{cases}$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Let Γ_n be the ordinal space $[1, \omega^n]$ provided with the discrete topology and replace the space $C_0(\mathbb{N}, X)$ by $C_0(\Gamma_n, X)$.

Embeddings of $C_0(K)$ *into* $C_0(\Gamma, X)$ *spaces*

Let Γ_n be the ordinal space $[1, \omega^n]$ provided with the discrete topology and replace the space $C_0(\mathbb{N}, X)$ by $C_0(\Gamma_n, X)$.

For each function $f \in C([1, \omega^n], X)$ set $T(f) : \Gamma_n \to X$ by

$$T(f)(\xi) = \begin{cases} 2f(\omega^n) & \text{if } \xi = \omega^n, \\ f(\xi) - f(\xi^{[1]}) & \text{if } 1 \leq \xi < \omega^n. \end{cases}$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

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T defines a bounded linear operator from $C([1, \omega^n], X)$ to $C_0(\Gamma_n, X)$ with

$$\|T\| = 2.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Remark

By using the Cantor normal form we an check that each ordinal number $1 \leq \xi < \omega^n$ admits an unique representation in the form

$$\xi = \omega^{n-1} i_1 + \omega^{n-2} i_2 + \omega^{n-3} i_3 + \ldots + \omega^{n-j} i_j \quad (1)$$

where $1 \leq j \leq n$, $0 \leq i_k < \omega$ for $1 \leq k \leq j-1$ and $1 \leq i_j < \omega$.

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

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where $1 \leq j \leq n$, $0 \leq i_k < \omega$ for $1 \leq k \leq j-1$ and $1 \leq i_j < \omega$.

$$\begin{aligned}
\xi^{[1]} &= \omega^{n-1} i_1 + \omega^{n-2} i_2 + \ldots + \omega^{n-j+1}(i_{j-1} + 1) \\
\xi^{[2]} &= \omega^{n-1} i_1 + \omega^{n-2} i_2 + \ldots + \omega^{n-j+2}(i_{j-2} + 1) \\
&\vdots \\
\xi^{[j-1]} &= \omega^{n-1}(i_1 + 1) \\
\xi^{[j]} &= \omega^n
\end{aligned}$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Next, for each function $g \in C_0(\Gamma_n, X)$, set $S(g) : [1, \omega^n] \to X$ by

$$S(g)(\xi) = \begin{cases} \frac{1}{2} g(\omega^n) & \text{if } \xi = \omega^n, \\ \sum_{r=0}^{j-1} g(\xi^{[r]}) + \frac{1}{2} g(\omega^n) & \text{if } 1 \leq \xi < \omega^n \text{ as in (1).} \end{cases}$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

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S defines a bounded linear operator from $C([1, \omega^n], X)$ to $C_0(\Gamma_n, X)$ with

$$\|S\| = \frac{2n+1}{2}$$

Embeddings of $C_0(K)$ *into* $C_0(\Gamma, X)$ *spaces*

Moreover

$$T \circ S = I_{C_0(\Gamma_n, X)} \quad \text{and} \quad S \circ T = I_{C([1,\omega^n],X)}.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Moreover

$$T \circ S = I_{C_0(\Gamma_n, X)} \quad \text{and} \quad S \circ T = I_{C([1,\omega^n], X)}.$$

$$C([1,\omega^n k], X) = \underbrace{C([1,\omega^n], X) \oplus \ldots \oplus C([1,\omega^n], X)}_{k},$$

$$C_0(\mathbb{N}, X) = \underbrace{C_0(\mathbb{N}, X) \oplus \ldots \oplus C_0(\mathbb{N}, X)}_{k}.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Corollary

Let X a Banach space with finite cotype and $1 \leq n, k < \omega$. Then

$$\mathrm{d}(C([1, \omega^n k], X), C_0(\mathbb{N}, X)) = 2n + 1.$$

Embeddings of $C_0(K)$ into $C_0(\Gamma, X)$ spaces

Thank you for your attention!

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