Productively countably tight spaces

Leandro F. Aurichi[1]
Joint work with Angelo Bella

ICMC-USP

[1]Sponsored by FAPESP

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If X is productively countably tight at every point x, we simply say that X is productively countably tight.

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$(\Omega_x = \{D \subset X \setminus \{x\} : x \in \overline{D}\})$

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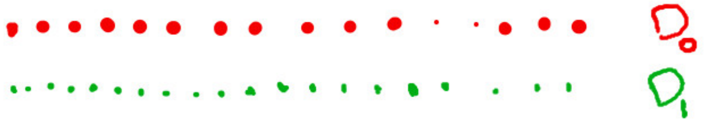

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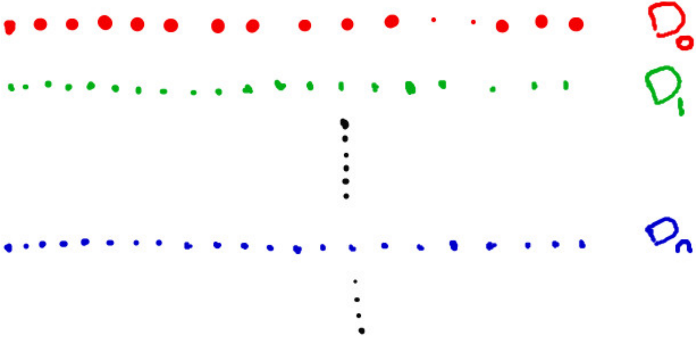

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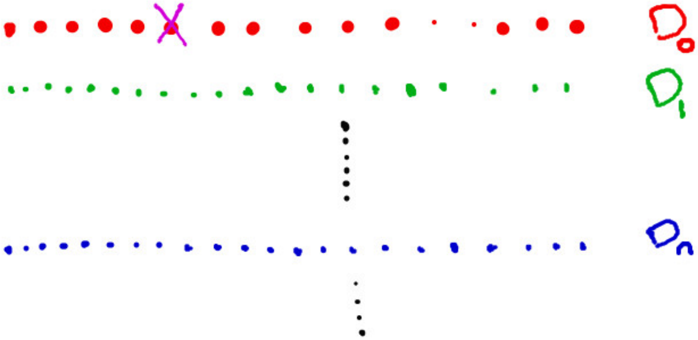

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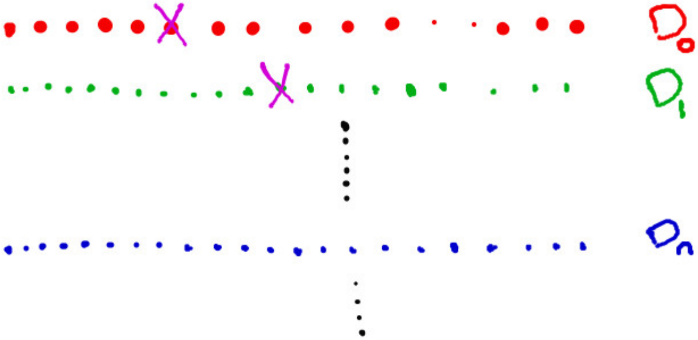

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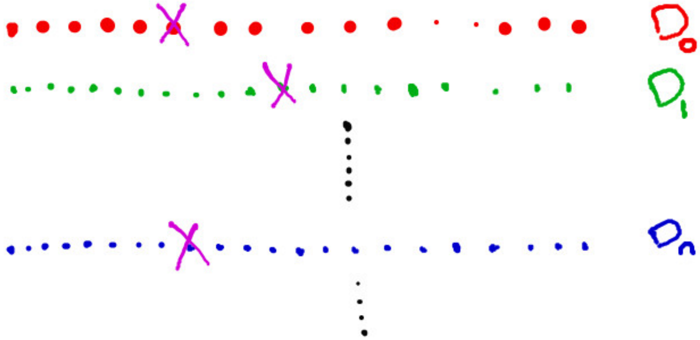

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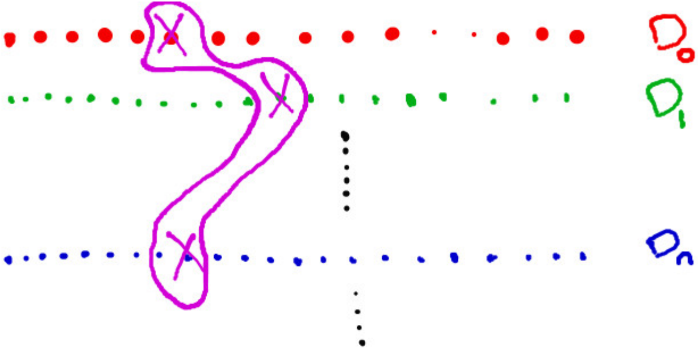

The first theorem

Theorem

If player II has a winning strategy for the game $\mathsf{G}_1(\Omega_x, \Omega_x)$ *played over* X*, then* X *is productively countably tight at* x*.*

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For every collection $\mathcal{P}$ of centered families of countable subsets of X such that, for any neighborhood O_x of x, there exists $\mathcal{B} \in \mathcal{P}$ and $B \in \mathcal{B}$ such that $B \subset O_x$, there is a subfamily $\{\mathcal{B}_n : n < \omega\} \subset \mathcal{P}$ such that for every choice $B_n \in \mathcal{B}_n$, $x \in \overline{\bigcup\{B_n : n < \omega\}}$.

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Actually, we will do a little better. We will find a subfamily $(B_s)_{s \in \omega^{<\omega}}$ of $\mathcal{P}$ such that, for every $f : \omega \longrightarrow \omega$, $(B_s)_{s \subset f}$ is a collection of sets containing the answers from player II using the winning strategy for the game $\mathsf{G}_1(\Omega_x, \Omega_x)$. Thus, $x \in \overline{\bigcup_{s \subset f} B_s}$.

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The key idea is: "how can we make this in such way that all the branches in this tree follow the winning strategy?".

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Lemma

Let X be a space and let F be a strategy for player II in the game $\mathsf{G}_1(\Omega_x, \Omega_x)$ for some $x \in X$. Then, for every sequence $D_0, ..., D_n \in \Omega_x$, there is an open set A such that $x \in A$ and, for every $a \in A \setminus \{x\}$, there is a $D_a \in \Omega_x$ such that $F(D_0, ..., D_n, D_a) = a$.

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Proof.

Let $B = \{y \in X \setminus \{x\} : \text{there is no } D \in \Omega_x \text{ such that } F(D_0, ..., D_n, D) = y\}$. Note that $B \notin \Omega_x$ since, otherwise, $F(D_0, ..., D_n, B) \in B$ which is a contradiction. Thus, there is an open set A such that $x \in A$ and $A \cap D = \emptyset$. □

Playing it

Then we only have to pick each $B_s \subset A_s$ where A_s is the appropriate open set given by the Lemma.

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This is done by proving that, if $X \times S_{\mathfrak{c}}$ has countable tightness at $(x, 0)$, then $\mathsf{S}_1(\Omega_x, \Omega_x)$ holds ($S_{\mathfrak{c}}$ is the sequential fan space of cardinality $\mathfrak{c}$).

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Theorem (Scheepers [4])

Let X be a Tychonoff space. Player II has a winning strategy in $\mathsf{G}_1(\Omega, \Omega)$ played on X if, and only if, player II has a winning strategy in $\mathsf{G}_1(\Omega_{\underline{0}}, \Omega_{\underline{0}})$ played on $C_p(X)$.

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Theorem (Sakai [3])

Let X be a Tychonoff space. The following are equivalent:

1. *$C_p(X)$ satisfies $\mathsf{S}_1(\Omega_{\underline{0}}, \Omega_{\underline{0}})$;*
2. *X satisfies $\mathsf{S}_1(\Omega, \Omega)$;*

A diagram

$$
\begin{array}{ccc}
C_p(X) & & X \\
 & & \\
\mathrm{II} \uparrow \mathsf{G}_1(\Omega_{\underline{0}}, \Omega_{\underline{0}}) & \longleftrightarrow & \mathrm{II} \uparrow \mathsf{G}_1(\Omega, \Omega) \\
\downarrow & & \\
\text{productively countably tight} & \longleftrightarrow & X_\delta \text{ is Lindelöf} \\
\downarrow & & \\
\mathsf{S}_1(\Omega_{\underline{0}}, \Omega_{\underline{0}}) & \longleftrightarrow & \mathsf{S}_1(\Omega, \Omega)
\end{array}
$$

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Trivia

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