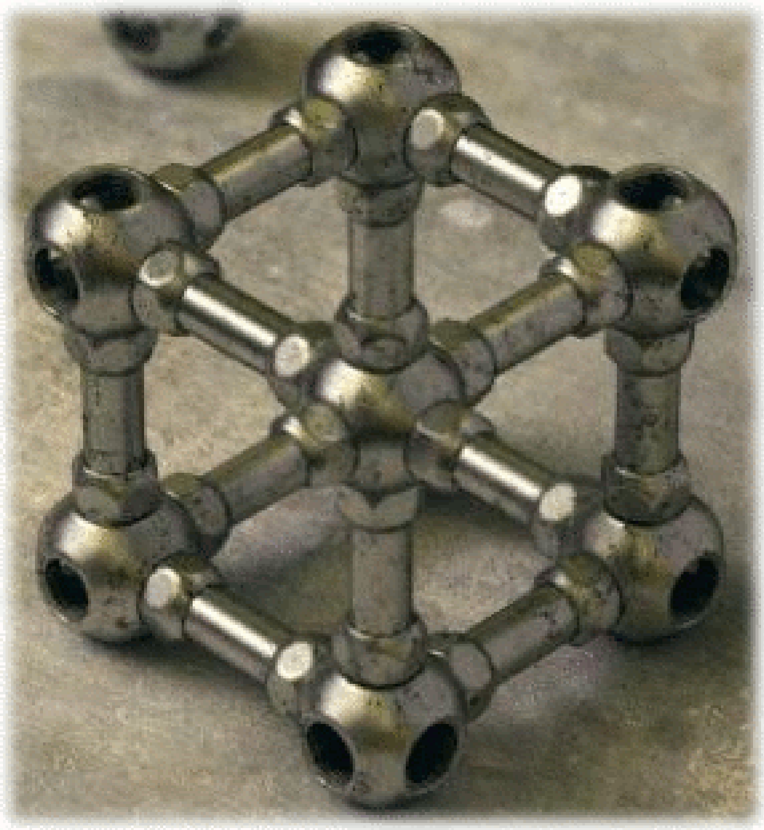

Models for anodic and cathodic multimodalities [1]

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Models for anodic and cathodic multimodalities

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Abstract

A system is classified as *multimodal* if its language has more than one modal operator as primitive, and such operators are not interdefinable. We extend the anodic and cathodic modal systems, introduced in [BS09a] and [BS10], to a class of the so-called basilar multimodal systems generating, in this way, the classes of *anodic and cathodic multimodal logics*. The cathodic multimodal systems are defined as extensions of positive multimodal systems (anodic multimodal systems) by adding degrees of negation plus consistency (and inconsistency) operators. In this way, cathodic multimodal systems are logics of formal inconsistency (the paraconsistent LFIs, as treated in [CCM07]) enriched with multimodal operators. We focus the attention on models for such classes of systems and discuss how modal possible-translations semantics, as well as possible-worlds (or Kripke semantics), can be defined to interpret basilar cathodic multimodal systems. While anodic systems are modeled by Kripke models only, we introduce the modal possible-translations models for cathodic systems. Such models, given by combinations of three-valued modal logics, besides their own interest, explain the role of non-trivializing contradictions in multimodal environment.

Keywords: Multimodal logics; paraconsistent logics; Kripke semantics; possible-translations semantics; completeness.

Introduction

The study of multimodalities has received considerable attention for the last four decades, initiated by the bi-modal involvement of time and modality due to Prior in [Pri57], a line to which Segerberg in [Seg77] and van Benthem in [vB83] also contributed. This paper intends to advance in this direction by following the same lines as of Carnielli and Pizzi in [CP08], so as to define the class of basilar anodic and cathodic multimodal systems.

Monomodal systems treat modal operators one by one, such as the notions expressed by $\Box$ and $\Diamond$, which are typically read as possibility and necessity, or obligation and permission, or even provability and consistency in the familiar systems **K**, **KT**, **KD**, **S4**, **S5**, etc. This approach offers a poor perspective to modal logic, since from a historical perspective already Aristotle and commentators refer to the interest in arguments combining time and modality, and hence, as legitimate multimodalities (see, for example *De Caelo* 1: 11-12). Even if contemporary modal logic in our heritage from the book which marked its birth, [LL32], conspicuously differs from the Aristotelian approach, the notions of logical pluralism are neither foreign to Aristotle, nor to Lewis and Langford. According to Kuhn in [Kuh98], p. 3, Aristotle established four different means for "necessity" in the Metaphysic V, chapter 5, and made various similar distinctions in other points, which means that modal logic was born inherently multimodal. In this sense, it is rather restrictive to fix attention on monomodal systems, as much as it is restrictive to circumscribe modal logics by the ones built on classical basis only. This paper intends to liberate modal logics from classical bounds in the sense of allowing modalities based upon non-classical logics, and also from modal monism in the sense of allowing blends of logics in various aspects, as well as to investigate the kinds of models that interpret them.

We show that certain invariants in modal logics permit that modal versions of possible-translations semantics can be ascribed to cathodic multimodal systems. This kind of semantics is completely novel for the case of modal logics, and is specially suitable for modal logics with non-classical basis. Moreover, possible-translations semantics are in line with requirements of consistency while a component of overall rationality, as in [Rov04], p. 323: if, in a group, one person holds one belief while another person believes its contrary, then we are not bound to regard each person as guilty of a rational failure, just the group.

First of all, we introduce the syntax of the anodic and cathodic multimodal systems and then show how these classes of multimodal systems can be characterized by means of multi-relational (Kripke-style) models. By using this result, it is also possible to obtain, for the basilar cathodic multimodal systems, a second semantical characterization by means of *modal possible-translations semantics*, extending the results for the monomodal cases obtained in [BS10].

1 Multimodal systems

A system is *multimodal* if its language has two or more non-interdefinable modal operators as primitive. In our treatment given below multimodal logics are inherently combined; since we build modal logics upon non-classical basis, we cannot appeal to methods of combining modalities such as [KW91], [Gab96] and [Wol97], inspiring as they can be. On the other hand, most of our systems require specific axioms connecting modal operators (called "bridge principles" in [CCG⁺07]); however, such bridge principles do not arise automatically and have to be carefully chosen in order to warrant completeness. In this sense we cannot either appeal to more general methods of combining logics such as fibring by functions or algebraic fibring as in [CCG⁺07]. This does not mean that our results (or at lest part of them) could not be obtained by more sophisticated methods such as modulated fibring, but this is still object of research.

A *multimodal propositional language* consists of a class of propositional variables (Var), a set of primitive connectives (Σ), and a fixed set of *modal atomic parameters* (Φ_0). From this setting it is possible to obtain the new modal operators from two *formation operators* $\cup$ and $\odot$.

The elements of Φ_0 are denoted by a, b, c etc, and two elements among such parameters are distinguished: the *identity parameter*, denoted by $\mathit{1}$, and the *null parameter*, denoted by $\mathit{0}$. Although we do not treat anodic systems here, it is convenient to introduce cathodic systems as extending the anodic case (see [BS09a]). A *multimodal language* is defined as $MML = \langle Var, \Sigma, \langle \Phi_0, \odot, \cup \rangle \rangle$; a multimodal language is *anodic* if $\Sigma = \{\supset, \wedge\}$, $\mathit{1} \in \Phi_0$, and $\mathit{0} \notin \Phi_0$, and is *cathodic* if $\Sigma = \{\supset, \wedge, \neg\}$ or $\Sigma = \{\supset, \wedge, \neg, \circ\}$, $\mathit{1} \in \Phi_0$

and $0 \in \Phi_0$. From Φ_0 we define the class of *multimodal parameters* as follows.

Definition 1.1

The class Φ of *multimodal parameters* on Φ_0 is defined by the following conditions:
(i) $a \in \Phi_0$ implies $a \in \Phi$;
(ii) $a, b \in \Phi$ implies $a \cup b \in \Phi$;
(iii) $a, b \in \Phi$ implies $a \odot b \in \Phi$.

The class Θ of *indexed modal operators* is the set $\{[a] : a \in \Phi\} \cup \{\langle a \rangle : a \in \Phi\}$. For certain cathodic systems it is possible to define an indexed modal operator $\langle a \rangle$ in terms of $[a]$ and strong (classical) negation, as usual. For convenience, as such an inter-definability between modal operators is not always possible, $\langle a \rangle$ and $[a]$ are taken as primitive.

The set of *formulas* (*For*) of multimodal (anodic and cathodic) systems is defined as usual by adding the following clause:

- If α is a formula and $a \in \Phi$, then $[a]\alpha$ and $\langle a \rangle\alpha$ are formulas.

The anodic multimodal systems are denoted by $\mathbf{K}^{\supset,\wedge,\Phi}$, and the cathodic systems by $\mathbf{PI}^{\Phi}$, $\mathbf{mbC}^{\Phi}$, $\mathbf{bC}^{\Phi}$ and $\mathbf{Ci}^{\Phi}$, where Φ is a collection of multimodal parameters and **PI**, **mbC**, **bC** and **Ci** are paraconsistent systems as treated by Carnielli, Coniglio and Marcos in [CCM07]. $\mathbf{K}^{\supset,\wedge}$ indicates the positive fragment of **K**.

The notation $\alpha(p)$ indicates that p occurs in α, and $\alpha[p/\beta]$ expresses the substitution in α of each occurrence of p by β.

Definition 1.2

A *normal anodic (cathodic) multimodal system* based on Θ is a collection of anodic (cathodic) multimodal formulas governed by the following rules and axiom schemas:

- For normal anodic multimodal systems $\mathbf{K}^{\supset,\wedge,\Phi}$:
  - (**A1**) $p \supset (q \supset p)$
  - (**A2**) $(p \supset q) \supset [(p \supset (q \supset r)) \supset (p \supset r)]$
  - (**A3**) $(p \supset r) \supset [((p \supset q) \supset r) \supset r]$
  - (**A4**) $p \supset (q \supset (p \wedge q))$
  - (**A5**) $(p \wedge q) \supset p$
  - (**A6**) $(p \wedge q) \supset q$
  - ($\mathbf{K}_a$) $[a](p \supset q) \supset ([a]p \supset [a]q)$, for each $a \in \Phi_0$
  - ($\mathbf{K1}_a$) $[a](p \supset q) \supset (\langle a \rangle p \supset \langle a \rangle q)$, for each $a \in \Phi_0$
  - ($\mathbf{K2}_a$) $\langle a \rangle(p \vee q) \supset (\langle a \rangle p \vee \langle a \rangle q)$, for each $a \in \Phi_0$
  - ($\mathbf{K3}_a$) $(\langle a \rangle p \supset [a]q) \supset [a](p \supset q)$, for each $a \in \Phi_0$
  - (**MP**) α, $\alpha \supset \beta$ implies β
  - (**US**) $\vdash \alpha$ implies $\vdash \alpha[p/\beta]$
  - ($\mathbf{Nec}_a$) $\vdash \alpha$ implies $\vdash [a]\alpha$, for each $a \in \Phi_0$
- For normal cathodic multimodal systems, all the previous plus:
  - (**PI**) $p \vee \neg p$
  - (**mbC**) $\circ p \supset [p \supset (\neg p \supset q)]$
  - (**bC**) $\neg\neg p \supset p$
  - (**Ci**) $\neg \circ p \supset (p \wedge \neg p)$

In all the systems given in the Definition 1.2 the disjunction connective is defined as:

$$\alpha \vee \beta \overset{\text{Def}}{=} (\alpha \supset \beta) \supset \beta$$

From such definition one obtains the expected propositional properties of disjunction, such as: *expansion*, *commutativity*, *associativity*, *Dummett's law* and *proof by cases*.

Among the cathodic multimodal systems, $\mathbf{PI}^\Phi$ is the only class that fails in defining a form of classical negation, commonly known as *strong negation*, because its language does not contain the consistency operator $\circ$. In all other cathodic classes treated here a strong negation can be defined as:

$$\sim\alpha \stackrel{\text{Def}}{=} \alpha \supset [p \wedge (\neg p \wedge \circ p)]$$

We write $\alpha \equiv \beta$ as an abbreviation for $(\alpha \supset \beta) \wedge (\beta \supset \alpha)$. In the sequel we define the standard anodic and cathodic multimodal systems.

Definition 1.3
A multimodal system $\mathbf{S}^\Phi$ is classified as *standard* if it is normal (in the sense of Definition 1.2) and its axioms include the following:

(i) $\mathbf{S}^\Phi$ is *standard anodic* if it satisfies, for each multimodal parameter $a, b \in \Phi$ and $p \in Var$:
(**MM1**) $[a \cup b]p \equiv [a]p \wedge [b]p$
(**MM2**) $\langle a \cup b\rangle p \equiv \langle a\rangle p \vee \langle b\rangle p$
(**MM3**) $[a \odot b]p \equiv [a][b]p$
(**MM4**) $\langle a \odot b\rangle p \equiv \langle a\rangle\langle b\rangle p$
(**MM5**) $[1]p \equiv p$
(**MM6**) $\langle 1\rangle p \equiv p$

(ii) $\mathbf{S}^\Phi$ is *standard cathodic* if it also satisfies:
(**MM7**) $[0]p \equiv \top$
(**MM8**) $\langle 0\rangle p \equiv \bot$ (except for $\mathbf{PI}^\Phi$)

We note that in $\mathbf{PI}^\Phi$ formulas of the kind $\langle 0\rangle\alpha$ are not equivalent to $\bot$ (since $\bot$ is not even definable in the language of $\mathbf{PI}^\Phi$). While $\langle 0\rangle\alpha$ cannot be part of a non-trivial set of sentences in most logics, in $\mathbf{PI}^\Phi$ *some* sentence of the form $\langle a\rangle\alpha$ will be excluded from the so-called 'factual' sets of sentences. This proviso is counterbalanced by means of a semantical adjustment (using the same strategy as for anodic systems in [BS09a]) considering factual deductions in the canonical models for completeness (see discussion on page 8).

To avoid redundancies on multimodal operators we work with equivalent classes of modal parameters, denoted by $\Phi/_\approx$. The class is reduced by the following equivalent relation, for each $p \in Var$:

$$a \approx b \quad \text{iff} \quad \vdash [a]p \equiv [b]p \quad \text{and} \quad \vdash \langle a\rangle p \equiv \langle b\rangle p$$

For simplicity, the equivalence relation above is denoted by Φ only.

It is to be remarked that the Deduction Metatheorem holds for all anodic and cathodic multimodal systems; the proof is virtually the same as for the classical modal logics, because anodic and cathodic multimodal systems do not require any new rules other than (**MP**), (**US**) and ($\mathbf{Nec}_a$), where $a \in \Phi_0$.

The following result shows that the generalized versions of the axiom (**K**) and rule (**Nec**) hold, in the anodic and cathodic multimodal systems, for any multimodal parameter $a \in \Phi$ (and not just for the atomic parameters):

THEOREM 1.4
For any anodic (cathodic) multimodal system $\mathbf{S}^\Phi$, and for each parameter $c \in \Phi$, the following holds in $\mathbf{S}^\Phi$:
(i) $\vdash \alpha$ implies $\vdash [c]\alpha$;
(ii) $[c](p \supset q) \supset ([c]p \supset [c]q)$;
(iii) $[c](p \supset q) \supset (\langle c\rangle p \supset \langle c\rangle q)$;
(iv) $\langle c\rangle(p \vee q) \supset (\langle c\rangle p \vee \langle c\rangle q)$;
(v) $(\langle c\rangle p \supset [c]q) \supset [c](p \supset q)$.

PROOF. By induction on the complexity on the multimodal parameters c. ∎

2 Multi-relational models

Cathodic multimodal systems can be characterized by two kinds of models: the *relational models*, which extend the familiar possible-worlds or Kripke models, and the *modal possible-translations models*, as mentioned below, which are based on translating complex modal systems into many-valued modal systems. This section concentrates on the former: relational models will have a relation associated to each indexed modal operator taken as primitive. For this reason we need to take into account some primitive relations: the *empty relation*, denoted by $\mathbf{0} = \emptyset$; the *universal relation*, denoted by $\mathbf{1} = W \times W$; and the *identity relation* denoted by $\mathbf{Id} = \{\langle w, w\rangle : w \in W\}$, plus *operations over relations* defined as follows:

1. $R \cup S = \{\langle w, w'\rangle : wRw' \text{ or } wSw'\}$ (Union)
2. $R \cap S = \{\langle w, w'\rangle : wRw' \text{ and } wSw'\}$ (Intersection)
3. $R \odot S = \{\langle w, w'\rangle : \exists w''(wRw'' \text{ and } w''Sw')\}$ (Relative Product)
4. $R^{-1} = \{\langle w, w'\rangle : \langle w', w\rangle \in R\}$ (Inverse)
5. $R \Rightarrow S = \{\langle w, w'\rangle : \forall w''(wRw'' \text{ implies } w''Sw')\}$ (Relative Implication)

A *multi-relational frame* $\mathfrak{F}^\Phi$ is a pair $\langle W, \Omega\rangle$ where W is a set of worlds and Ω is a set of binary relations over W.

DEFINITION 2.1
Let $\mathbf{S}^\Phi$ be an anodic (cathodic) multimodal system; $\mathfrak{F}^\Phi$ is a multi-relational frame for $\mathbf{S}^\Phi$ if there exists a function $\rho : \Phi \longrightarrow \Omega$ associating a relation R_a to every multimodal parameter a in Φ satisfying the following conditions:

- For $\mathbf{S}^\Phi$ an anodic multimodal system:
  (i) $\rho(1) = \mathbf{Id}$ (i.e. $R_1 = \mathbf{Id}$)
  (ii) $\rho(a \cup b) = \rho(a) \cup \rho(b)$ (i.e. $R_{a\cup b} = R_a \cup R_b$)
  (iii) $\rho(a \odot b) = \rho(a) \odot \rho(b)$ (i.e. $R_{a\odot b} = R_a \odot R_b$)
- For $\mathbf{S}^\Phi$ a cathodic multimodal system, add:
  (iv) $\rho(0) = \mathbf{0}$ (i.e. $R_0 = \emptyset$)

In the sequel we define *multi-relational models* $\mathfrak{M}^{\mathcal{L}^\Phi}_{\mathrm{Biv}}$ for anodic and cathodic multimodal systems in the standard way. The subindex `Biv` aims to indicate that the valuation is a bi-valuation.

DEFINITION 2.2
A *multi-relational model* $\mathfrak{M}^{\mathcal{L}^\Phi}_{\mathrm{Biv}}$ for a multimodal system $\mathcal{L}^\Phi$ is a pair $\langle \mathfrak{F}^\Phi, v \rangle$, where $\mathfrak{F}^\Phi$ is a multi-relational frame for $\mathcal{L}^\Phi$ and $v : For \times W \longrightarrow \{0, 1\}$ is a function satisfying the following conditions:

- For $\mathcal{L}^\Phi = \mathbf{K}^{\supset,\wedge,\Phi}$:
  (i) $v(p, w) = 1$ or $v(p, w) = 0$;
  (ii) $v(\alpha \supset \beta, w) = 1$ iff $v(\alpha, w) = 0$ or $v(\beta, w) = 1$;
  (iii) $v(\alpha \wedge \beta, w) = 1$ iff $v(\alpha, w) = 1$ and $v(\beta, w) = 1$;
  (iv) $v([a]\alpha, w) = 1$ iff $v(\alpha, w') = 1$, for all $w' \in W$ such that $wR_a w'$, for each $a \in \Phi$;
  (v) $v(\langle a \rangle \alpha, w) = 1$ iff $v(\alpha, w') = 1$, for some $w' \in W$ such that $wR_a w'$, for each $a \in \Phi$.
- For $\mathcal{L}^\Phi = \mathbf{PI}^\Phi$, we add:
  (vi) $v(\alpha, w) = 0$ implies $v(\neg\alpha, w) = 1$.
- For $\mathcal{L}^\Phi = \mathbf{mbC}^\Phi$, we add:
  (vii) $v(\circ\alpha, w) = 1$ implies $v(\alpha, w) = 0$ or $v(\neg\alpha, w) = 0$.
- For $\mathcal{L}^\Phi = \mathbf{bC}^\Phi$, we add:
  (viii) $v(\neg\neg\alpha, w) = 1$ implies $v(\alpha, w) = 1$.
- For $\mathcal{L}^\Phi = \mathbf{Ci}^\Phi$, we add:
  (ix) $v(\neg \circ \alpha, w) = 1$ implies $v(\alpha, w) = 1$ and $v(\neg\alpha, w) = 1$.

THEOREM 2.3
Let $\mathcal{L}^\Phi$ be any system among $\mathbf{K}^{\supset,\wedge,\Phi}$, $\mathbf{PI}^\Phi$, $\mathbf{mbC}^\Phi$, $\mathbf{bC}^\Phi$ and $\mathbf{Ci}^\Phi$. Each theorem of $\mathcal{L}^\Phi$ is valid in the class $\mathcal{F}^\Phi$ of multi-relational frames $\mathfrak{F}^\Phi$, where the relations in Ω are arbitrary.

PROOF. It is sufficient to check that all multimodal axiom are sound, and that rules preserve validity. For propositional axioms see [BS09a] and [BS10]. For the axioms ($\mathbf{K}_a$), ($\mathbf{K1}_a$), ($\mathbf{K2}_a$), ($\mathbf{K3}_a$) and the rule ($\mathbf{Nec}_a$), the argument is routine. Observe that the modal parameter a does not modify the classical argument. It remains to be shown the result for the multimodal cases (**MM1**)–(**MM8**). We will treat one case only (the others are analogous).

- Axiom (**MM1**)
  ($\Longrightarrow$) Suppose, by *Reductio*, that there exists a multi-relational model $\mathfrak{M}^\Phi$ based on $\mathfrak{F}^\Phi$ such that $\mathfrak{M}^\Phi \nvDash [a \cup b]p \supset [a]p \wedge [b]p$. By Definition 2.2 (iv) and (ii), we have that there exists $w \in W$ such that $v([a \cup b]p), w) = 1$ and $v([a]p \wedge [b]p, w) = 0$. By analyzing each case we derive a contradiction.
  1. $v([a \cup b]p), w) = 1$ iff $v(p, w') = 1$ *for all* $w' \in W$ such that $wR_{a\cup b}w'$.
  2. $v([a]p \wedge [b]p, w) = 0$ iff $v([a]p, w) = 0$ or $v([b]p, w) = 0$. Consider $v([a]p, w) = 0$, the other case is analogous:
     $v([a]p, w) = 0$ iff $v(p, w') = 0$ for *some* $w' \in W$ such that $wR_a w'$. Since $R_a \subseteq R_a \cup R_b$, and as $R_a \cup R_b = R_{a\cup b}$ (Definition 2.1 (ii)), then $wR_{a\cup b}w'$, which contradicts item 1.

  ($\Longleftarrow$) The argument is analogous.
  Therefore, $\mathfrak{M}^\Phi \vDash [a \cup b]p \equiv [a]p \wedge [b]p$ for all $\mathfrak{M}^\Phi$ based on $\mathfrak{F}^\Phi \in \mathcal{F}^\Phi$.

∎

The proof of completeness for the systems $\mathbf{K}^{\supset,\wedge,\Phi}$, $\mathbf{PI}^{\Phi}$, $\mathbf{mbC}^{\Phi}$, $\mathbf{bC}^{\Phi}$ e $\mathbf{Ci}^{\Phi}$ can be obtained as a particular case of Theorem 2.12. Our interest is focused in the basilar multimodal systems, which will be defined in the sequel.

We call *basilar* a system $\mathbf{S}^{\Phi}(a, b, c, d)$, for $a, b, c, d \in \Phi$, obtained from a multimodal system $\mathbf{S}^{\Phi}$ by adding the following axioms:

$$\mathbf{G}(a,b,c,d) \quad \langle a\rangle[b]p \supset [c]\langle d\rangle p$$
$$\mathbf{G}(c,d,a,b) \quad \langle c\rangle[d]p \supset [a]\langle b\rangle p$$

We denote by $\mathbf{K}^{\supset,\wedge,\Phi}+\mathbf{G}(a,b,c,d)+\mathbf{G}(c,d,a,b)$ the class of *basilar anodic multimodal* systems, where Φ represents the set of multimodal parameters.

The *basilar cathodic multimodal system* considered here are the following classes of systems:

- $\mathbf{PI}^{\Phi}(a,b,c,d)$, defined as $\mathbf{K}^{\supset,\wedge,\Phi}+\mathbf{G}(a,b,c,d)+\mathbf{G}(c,d,a,b)$ plus (**PI**);
- $\mathbf{mbC}^{\Phi}(a,b,c,d)$, defined as $\mathbf{PI}^{\Phi}(a,b,c,d)$ plus (**mbC**);
- $\mathbf{bC}^{\Phi}(a,b,c,d)$, defined as $\mathbf{mbC}^{\Phi}(a,b,c,d)$ plus (**bC**);
- $\mathbf{Ci}^{\Phi}(a,b,c,d)$, defined as $\mathbf{bC}^{\Phi}(a,b,c,d)$ plus (**Ci**).

The term 'multimodal' is often omitted, and we refer to *basilar anodic (cathodic) systems* instead of basilar anodic (cathodic) multimodal systems.

The property of (a, b, c, d)-interaction, related with the basilar axiom, is described as:

$$\mathbf{P}(a,b,c,d) \quad \rho(a)^{-1} \odot \rho(c) \subseteq \rho(b) \odot \rho(d)^{-1}$$

It is not difficult to prove that every instance of $\mathbf{G}(a,b,c,d)$ is valid in any multi-relational frame $\mathfrak{F}^{\Phi}$ where the relations in Ω satisfy the (a,b,c,d)-interaction property.

Theorem 2.4

Every instance of $\mathbf{G}(a,b,c,d)$ is valid in all class of multi-relational frames $\mathfrak{F}^{\Phi}$, where the relations in Ω satisfy the (a,b,c,d)-interaction property.

Proof. Let $R_a, R_b, R_c, R_d \in \Omega$ be relations that satisfy the (a,b,c,d)-interaction property. Suppose, by *Reductio*, that some instance of $\mathbf{G}(a,b,c,d)$ is invalid in some multi-relational model based on $\mathfrak{F}^{\Phi}$. So, there exists w_1 such that:

(*a*) $v(\langle a\rangle[b]\alpha, w_1) = 1$

(*b*) $v([c]\langle d\rangle\alpha, w_1) = 0$

From (a) and Definition 2.2, it follows that there exists w_2 such that $w_1 R_a w_2$ and $v([b]\alpha, w_2) = 1$ and then, it follows that $v(\alpha, w') = 1$ in *all* w' such that $w_2 R_b w'$.

From (b) and Definition 2.2, it follows that there exists w_3 such that $w_1 R_v w_3$ and $v(\langle d\rangle\alpha, w_3) = 0$, and consequently $v(\alpha, w'') = 0$ in *all* w'' such that $w_3 R_d w''$.

Since $w_1 R_a w_2$ and $w_1 R_c w_3$, the relative product of the inverse of R_a and R_c implies that $\langle w_2, w_3\rangle \in R_a^{-1} \odot R_c$. Since the multi-relational model satisfies the (a,b,c,d)-interaction property then, it follows that $\langle w_2, w_3\rangle \in R_b \odot R_d^{-1}$. By the definition of relative product we have that there exists w_4 such that $\langle w_2, w_4\rangle \in R_b$ and $\langle w_4, w_3\rangle \in R_d^{-1}$. Thus $\langle w_3, w_4\rangle \in R_d$, for some w_4. But, from (a) and (b) it follows, respectively, that $v(\alpha, w_4) = 1$ and that $v(\alpha, w_4) = 0$. A contradiction. ∎

Corollary 2.5
Let $\mathcal{L}^\Phi(a,b,c,d)$ be any system in the collection $\mathbf{K}^{\supset,\wedge,\Phi}+\mathbf{G}(a,b,c,d)+\mathbf{G}(c,d,a,b)$, $\mathbf{PI}^\Phi(a,b,c,d)$, $\mathbf{mbC}^\Phi(a,b,c,d)$, $\mathbf{bC}^\Phi(a,b,c,d)$ and $\mathbf{Ci}^\Phi(a,b,c,d)$. Each theorem of $\mathcal{L}^\Phi(a,b,c,d)$ is valid in all multi-relational frames where the relations in Ω satisfy the (a,b,c,d)-interaction property.

Proof. Immediate from Theorem 2.3 and Theorem 2.4. ■

As usual, we say that a set Δ of sentences is *non-trivial* if $\Delta \nvdash \alpha$ for some sentence α; otherwise, Δ is *trivial*.

As the classes $\mathbf{K}^{\supset,\wedge,\Phi}+\mathbf{G}(a,b,c,d)+\mathbf{G}(c,d,a,b)$ and $\mathbf{PI}^\Phi(a,b,c,d)$ are required to handle prime theories (defined below), then the usual notion of a saturated set (or maximal non-trivial set with respect to a given sentence) is defined with respect to collections of sentences instead of a single sentence, as seen in the next definition.

Definition 2.6
Let $\mathbf{S}$ be a system and Δ and Λ be non-trivial subsets of *For* such that $\Delta \cap \Lambda = \emptyset$. Δ is *non-trivial* Λ*-maximal* if:
(i) $\Delta \nvdash \lambda$, for all $\lambda \in \Lambda$;
(ii) For each $\alpha \in For$ such that $\alpha \notin \Delta$, $\Delta \cup \{\alpha\} \vdash \lambda$, for some $\lambda \in \Lambda$.

If the set Λ is not specified, the non-trivial Λ-maximal set Δ will be referred to as a *non-trivial maximal set*, for short.

Let Δ be a set of $\mathbf{S}$-sentences; Δ is called an $\mathbf{S}$-*theory* if it satisfies: $\Delta \vdash \delta$ implies $\delta \in \Delta$; an $\mathbf{S}$-*theory* Δ is called *prime* if it is non-trivial and satisfies $\Delta \vdash \alpha \vee \beta$ implies $\Delta \vdash \alpha$ or $\Delta \vdash \beta$.

Lemma 2.7
Let Δ be a non-trivial Λ-maximal set. If Λ is a singleton, then Δ is a prime set.

Proof. A simple argument by *Reductio*. ■

Now, consider the definition of the following particular sets: $Den_a(\Delta)$ (*a-denecessitation set* of Δ) and $Pos_a(\Delta)$ (*a-possibilitation set* of Δ) defined as:

$$Den_a(\Delta) = \{\alpha : [a]\alpha \in \Delta \text{ and } a \in \Phi\} \quad \text{and} \quad Pos_a(\Delta) = \{\langle a\rangle\alpha : \alpha \in \Delta \text{ and } a \in \Phi\}$$

Lemma 2.8
If Δ is a $\mathcal{L}^\Phi$-theory, then $Den_a(\Delta)$ is also an $\mathcal{L}^\Phi$-theory.

Proof. The analogous argument used in fact 4.5. of [BS10] using, respectively, $Den_a(\Delta)$, ($\mathbf{Nec}_a$) and ($\mathbf{K}_a$) instead of $Den(\Delta)$, ($\mathbf{Nec}$) and ($\mathbf{K}$) in the argument. ■

In order to gain absolute positiveness in the anodic multimodal systems, the concept of *factual sets* (sets that are $\langle a\rangle$-non-trivial in the sense of not containing all sentences of the kind $\langle a\rangle\alpha$, for some $a \in \Phi$) is taken into account as in [BS09a]. From factual sets one can define the *factual deductions* (logical consequences of factual sets of premises). Factual deductions express deductions in the actual world, where not everything is possible. These notions are innocuous in the presence of strong negation, as in such cases the notions of 'consistency' and 'non-triviality' coincide[1]. Completeness for

[1] It is noteworthy to recall that the distinction between 'consistency' and 'non-triviality' is a hallmark of the LFIs (cf. [CCM07]) that we inherit here.

anodic multimodal systems w.r.t. multi-relational Kripke models will be granted just for factual deductions.

For the proof of completeness we need to construct the canonical multi-relational models that satisfy the conditions of bi-valued multi-relational models $\mathfrak{M}^{\mathcal{L}^\Phi}_{\mathrm{Biv}}$. Canonical models are based on *maximal prime sets* (non-trivial maximal sets satisfying the condition $\Delta \vdash \alpha \vee \beta$ implies $\Delta \vdash \alpha$ or $\Delta \vdash \beta$) by means of Lindenbaum-type constructions.

Definition 2.9
The *multi-relational canonical model* $\widehat{\mathfrak{M}^\Phi}$ for an anodic (cathodic) multimodal system $\mathcal{L}^\Phi$ is a triple $\langle \widehat{W}, \widehat{\Omega}, \widehat{V} \rangle$ where:
(i) $\widehat{W}$ is a class of maximal non-trivial extension of $\mathcal{L}^\Phi$;
(ii) For each $\widehat{R}_a \in \widehat{\Omega}$ and all $a \in \Phi$, in each case, we have that:

$$\Delta \widehat{R}_a \Delta' \text{ iff } \begin{cases} Den_a(\Delta) \subseteq \Delta' \subseteq Dep_a(\Delta) & \text{for } \mathcal{L}^\Phi \text{ an anodic system.} \\ Den_a(\Delta) \subseteq \Delta' & \text{for } \mathcal{L}^\Phi \text{ a cathodic system.} \end{cases}$$

(iii) Each $\widehat{v}_\Delta \in \widehat{V}$ is a multimodal valuation of $\mathcal{L}^\Phi$, defined from some $\Delta \in \widehat{W}$, as:

$$\widehat{v}_\Delta(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \Delta \\ 0 & \text{if } \alpha \notin \Delta \end{cases}$$

To obtain completeness, the next lemma shows that the canonical multi-relational frame for an anodic (cathodic) system satisfies the conditions required in Definition 2.1.

Lemma 2.10
Let $\widehat{\mathfrak{F}}^\Phi = \langle \widehat{W}, \widehat{\Omega} \rangle$ be a canonical multi-relational frame for an anodic (cathodic) multimodal system $\mathbf{S}^\Phi$ and $\rho : \Phi \longrightarrow \widehat{\Omega}$ defined as $\rho(a) = \widehat{R}_a$. Then ρ satisfies the following conditions:

- For $\mathbf{S}^\Phi$ an anodic system:
  (i) $\rho(1) = \mathbf{Id}$;
  (ii) $\rho(a \cup b) = \rho(a) \cup \rho(b)$;
  (iii) $\rho(a \odot b) = \rho(a) \odot \rho(b)$;
- For $\mathbf{S}^\Phi$ a cathodic system, add:
  (iv) $\rho(0) = \mathbf{0}$.

Proof. From Definition 2.1 we need to prove that:

(i) $\widehat{R}_1 = \mathbf{Id}$.
($\Longrightarrow$) If $\langle \Delta, \Delta' \rangle \in \widehat{R}_1$ then $Den_1(\Delta) \subseteq \Delta' \subseteq Dep_1(\Delta)$, i.e., $\{\alpha : [1]\alpha \in \Delta\} \subseteq \Delta' \subseteq \{\alpha : \langle 1 \rangle \alpha \in \Delta\}$. Therefore, from (**MM5**) and (**MM6**), it follows that $\langle \Delta, \Delta' \rangle \in \mathbf{Id}$.
($\Longleftarrow$) Immediate from (**MM5**) and (**MM6**).

(ii) $\widehat{R}_{a\cup b} = \widehat{R}_a \cup \widehat{R}_b$.
($\Longrightarrow$) We will show that $\langle \Delta, \Delta' \rangle \in \widehat{R}_{a\cup b}$ implies $\Delta \widehat{R}_a \Delta'$ or $\Delta \widehat{R}_b \Delta'$.
If $\langle \Delta, \Delta' \rangle \in \widehat{R}_{a\cup b}$ then, from Definition 2.9, $\langle \Delta, \Delta' \rangle$ satisfies both conditions:
(a1) If $[a \cup b]\alpha \in \Delta$ then $\alpha \in \Delta'$ for all α;
(a2) If $\delta \in \Delta'$ then $\langle a \cup b \rangle \delta \in \Delta$ for all δ.

Suppose, by *Reductio*, that $\langle \Delta, \Delta' \rangle \notin \widehat{R}_a$ and $\langle \Delta, \Delta' \rangle \notin \widehat{R}_b$. This means that:

$$\text{For } \langle \Delta, \Delta' \rangle \notin \widehat{R}_a \begin{cases} \textbf{(b1)} \quad [a]\beta \in \Delta \;\text{ and }\; \beta \notin \Delta', \text{ for some } \beta \\ \qquad\qquad \text{or} \\ \textbf{(b2)} \quad \delta' \in \Delta' \;\text{ and }\; \langle a \rangle \delta' \notin \Delta, \text{ for some } \delta' \end{cases}$$

$$\text{For } \langle \Delta, \Delta' \rangle \notin \widehat{R}_b \begin{cases} \textbf{(c1)} \quad [b]\gamma \in \Delta \;\text{ and }\; \gamma \notin \Delta', \text{ for some } \gamma \\ \qquad\qquad \text{or} \\ \textbf{(c2)} \quad \delta'' \in \Delta' \;\text{ and }\; \langle b \rangle \delta'' \notin \Delta, \text{ for some } \delta'' \end{cases}$$

In this case, there are four possibilities to be analyzed: (b1) and (c1); (b1) and (c2); (b2) and (c1); (b2) and (c2). We will show that, in each case, a contradiction can be obtained.

Case 1: Suppose (b1) and (c1).
From the hypothesis $[a]\beta \in \Delta$ it follows that $[a]\beta \vee [a]\gamma \in \Delta$. By a propositional modal reasoning we have that $[a]\beta \vee [a]\gamma \supset [a](\beta \vee \gamma)$ so, by (**MP**), $[a](\beta \vee \gamma) \in \Delta$. Analogously, from the hypothesis $[b]\gamma \in \Delta$, we have that $[b](\beta \vee \gamma) \in \Delta$. From (**A4**) and (**MM1**) it follows that $[a \cup b](\beta \vee \gamma) \in \Delta$. Therefore, from (a1), $(\beta \vee \gamma) \in \Delta'$. As Δ' is non-trivial Λ-maximal then $\beta \in \Delta'$ or $\gamma \in \Delta'$. In both cases we have a contradiction.

Case 2: Suppose (b1) and (c2):
From the fact that Δ is a prime set and that $\langle b \rangle \delta'' \notin \Delta$, for some δ'', it follows that $\langle b \rangle \delta'' \supset [b]\beta \in \Delta$. Hence, by (**K3**$_b$) and (**MP**), we have that $[b](\delta'' \supset \beta) \in \Delta$. On the other hand, from $[a]\beta \in \Delta$ and (**A1**) it follows that $\langle a \rangle \delta'' \supset [a]\beta \in \Delta$. Hence, by (**K3**$_a$) and (**MP**), we have that $[a](\delta'' \supset \beta) \in \Delta$. From (**A4**) and (**MM1**) it follows that $[a \cup b](\delta'' \supset \beta) \in \Delta$. Therefore, from (a1), $(\delta'' \supset \beta) \in \Delta'$. As $\delta'' \in \Delta'$, then $\beta \in \Delta'$. Contradiction.

Case 3: Analogous to the previous case.

Caso 4: Suppose (b2) and (c2):
From $\langle a \rangle \delta' \notin \Delta$ and (**A5**) it follows that $\langle a \rangle \delta' \wedge \langle a \rangle \delta'' \notin \Delta$. By propositional reasoning we have that $\langle a \rangle (\delta' \wedge \delta'') \notin \Delta$. Analogously, from $\langle b \rangle \delta'' \notin \Delta$ and (**A6**) it follows that $\langle b \rangle (\delta' \wedge \delta'') \notin \Delta$. As Δ is a prime set it follows that $\langle a \rangle (\delta' \wedge \delta'') \vee \langle b \rangle (\delta' \wedge \delta'') \notin \Delta$. Therefore, from (**MM2**), we have that $\langle a \cup b \rangle (\delta' \wedge \delta'') \notin \Delta$. On the other hand, as both $\delta' \in \Delta'$ and $\delta'' \in \Delta'$ then, from (**A4**), it follows that $\delta' \wedge \delta'' \in \Delta'$ so, from (a2), $\langle a \cup b \rangle (\delta' \wedge \delta'') \in \Delta$. Absurd.

($\Longleftarrow$) Now, rest to be shown that $\langle \Delta, \Delta' \rangle \in \widehat{R}_a \cup \widehat{R}_b$ implies $\langle \Delta, \Delta' \rangle \in \widehat{R}_{a \cup b}$.

From the supposition $\langle \Delta, \Delta' \rangle \in \widehat{R}_a \cup \widehat{R}_b$ it follows that:

$$Den_a(\Delta) \subseteq \Delta' \subseteq Dep_a(\Delta) \quad \text{or} \quad Den_b(\Delta) \subseteq \Delta' \subseteq Dep_b(\Delta)$$

1. If $Den_a(\Delta) \subseteq \Delta'$ or $Den_b(\Delta) \subseteq \Delta'$ then $Den_a(\Delta) \cap Den_b(\Delta) \subseteq \Delta'$, which means that $\{\alpha : [a]\alpha \wedge [b]\alpha \in \Delta\} \subseteq \Delta'$. From (**MM1**), it follows that $Den_{a \cup b}(\Delta) \subseteq \Delta'$.
2. If $\Delta' \subseteq Dep_a(\Delta)$ or $\Delta' \subseteq Dep_b(\Delta)$ then $\Delta' \subseteq Dep_a(\Delta) \cup Dep_b(\Delta)$, which means that $\Delta' \subseteq \{\alpha : \langle a \rangle \alpha \vee \langle b \rangle \alpha \in \Delta\}$. From (**MM2**), it follows that $\Delta' \subseteq Dep_{a \cup b}(\Delta)$.

Therefore, from (1) and (2), $Den_{a \cup b}(\Delta) \subseteq \Delta' \subseteq Dep_{a \cup b}(\Delta)$, i.e., $\langle \Delta, \Delta' \rangle \in \widehat{R}_{a \cup b}$.

(iii) $\widehat{R}_{a \odot b} = \widehat{R}_a \odot \widehat{R}_b$.

($\Longrightarrow$) We will show that $\langle \Delta, \Delta' \rangle \in \widehat{R}_{a \odot b}$ implies $\langle \Delta, \Delta' \rangle \in \widehat{R}_a \odot \widehat{R}_b$.

From Definition 2.9, we have that $\langle \Delta, \Delta' \rangle \in \widehat{R}_{a \odot b}$ means the following:

(Hyp) $\quad Den_{a\odot b}(\Delta) \subseteq \Delta' \subseteq Dep_{a\odot b}(\Delta)$

As $Den_{a\odot b}(\Delta)$ means that $[a \odot b]\alpha \in \Delta$, for some α then, from (**MM3**), we have that $[a][b]\alpha \in \Delta$. From Definition 2.2 we know that for all Δ'' such that $\Delta\widehat{R}_a\Delta''$, the formula $[b]\alpha \in \Delta''$. It remains to be shown that $\Delta''\widehat{R}_b\Delta'$; the argument will be divided into two sub-arguments:

$$\Delta' \subseteq Dep_b(\Delta'') \tag{2.1}$$

If $\alpha \in \Delta'$ then, by (Hyp), it follows that $\alpha \in Dep_{a\odot b}(\Delta)$; this means that $\langle a \odot b\rangle\alpha \in \Delta$. From (**MM4**), it follows that $\langle a\rangle\langle b\rangle\alpha \in \Delta$. As $\Delta\widehat{R}_a\Delta''$ then, for some Δ'', we have that $\langle b\rangle\alpha \in \Delta''$, i.e., $\alpha \in Dep_b(\Delta'')$.

$$Den_b(\Delta'') \subseteq \Delta' \tag{2.2}$$

Suppose, by *Reductio*, that $\alpha \in Den_b(\Delta'')$ and $\alpha \notin \Delta'$.

a. If $\alpha \in Den_b(\Delta'')$ then $[b]\alpha \in \Delta''$. Given that $\Delta\widehat{R}_a\Delta''$ then $[a][b]\alpha \in \Delta$ so, from (**MM3**), it follows that $[a \odot b]\alpha \in \Delta$.
b. If $\alpha \notin \Delta'$ then, by (Hyp), $\alpha \notin Den_{a\odot b}(\Delta)$ so, $[a \odot b]\alpha \notin \Delta$. Contradiction with (a).

From (2.1) and (2.2) it follows that $Den_b(\Delta'') \subseteq \Delta' \subseteq Dep_b(\Delta'')$, for some Δ'' such that $\Delta\widehat{R}_a\Delta''$. Therefore, given that $\Delta\widehat{R}_a\Delta''$ and $\Delta''\widehat{R}_b\Delta'$, for some Δ'', by composition of $\widehat{R}_a$ and $\widehat{R}_b$, we have that $\langle\Delta, \Delta'\rangle \in \widehat{R}_a \odot \widehat{R}_b$.

($\Longleftarrow$) Now, it remains to be shown that $\langle\Delta, \Delta'\rangle \in \widehat{R}_a \odot \widehat{R}_b$ implies $\langle\Delta, \Delta'\rangle \in \widehat{R}_{a\odot b}$.
If $\langle\Delta, \Delta'\rangle \in \widehat{R}_a \odot \widehat{R}_b$ then, by definition of composition relation, we have that there exists Δ'' such that $\langle\Delta, \Delta''\rangle \in \widehat{R}_a$ and $\langle\Delta'', \Delta'\rangle \in \widehat{R}_b$ so, by Definition 2.9, we have that:

(Hyp1) $\quad Den_a(\Delta) \subseteq \Delta'' \subseteq Dep_a(\Delta)$
(Hyp2) $\quad Den_b(\Delta'') \subseteq \Delta' \subseteq Dep_b(\Delta'')$.

The result is obtained by means of the following two sub-arguments:

$$\Delta' \subseteq Dep_{a\odot b}(\Delta) \tag{2.3}$$

If $\alpha \in \Delta'$ then, by (Hyp2), $\alpha \in Dep_b(\Delta'')$, i.e., $\langle b\rangle\alpha \in \Delta''$. From (Hyp1) and (**MM4**) we have that $\langle a \odot b\rangle\alpha \in \Delta$. Therefore, $\alpha \in Dep_{a\odot b}(\Delta)$.

$$Den_{a\odot b}(\Delta) \subseteq \Delta' \tag{2.4}$$

Suppose, by *Reductio*, that $\alpha \in Den_{a\odot b}(\Delta)$ and $\alpha \notin \Delta'$. On the one hand, if $\alpha \in Den_{a\odot b}(\Delta)$, then $[a \odot b]\alpha \in \Delta$. From (**MM3**) it follows that $[a][b]\alpha \in \Delta$, i.e., $[b]\alpha \in Den_a(\Delta)$. From (Hyp1) we have that $[b]\alpha \in \Delta''$. On the other hand, if $\alpha \notin \Delta'$ then, from (Hyp2), $\alpha \notin Den_b(\Delta'')$ so, $[b]\alpha \notin \Delta''$. Absurd.

Therefore, from (2.4) and (2.3) we have that $Den_{a\odot b}(\Delta) \subseteq \Delta' \subseteq Dep_{a\odot b}(\Delta)$, i.e., $\langle\Delta, \Delta'\rangle \in \widehat{R}_{a\odot b}$.

(iv) $\widehat{R}_{\mathit{0}} = \emptyset$.
Suppose, by *Reductio*, that there exist $\Delta, \Delta' \in \widehat{W}$ such that $\langle\Delta, \Delta'\rangle \in \widehat{R}_{\mathit{0}}$. This means that $Den_{\mathit{0}}(\Delta) \subseteq \Delta'$, i.e., $\{\alpha : [\mathit{0}]\alpha \in \Delta\} \subseteq \Delta'$. From (**MM7**) it follows that $\alpha \in \Delta'$, for all α, hence Δ' is trivial. Absurd.

■

THEOREM 2.11
Let $\mathcal{L}^\Phi$ be a basilar anodic (cathodic) system and $\langle \widehat{W}, \widehat{\Omega}, \widehat{V} \rangle$ a canonical multi-relational model for $\mathcal{L}^\Phi$. Then, for any multimodal formula α and any $\Delta \in \widehat{W}$, we have that:

$$v_\Delta(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \Delta \\ 0 & \text{if } \alpha \notin \Delta \end{cases}$$

PROOF. By induction on the complexity of formulas, and sub-induction on the complexity of multimodal parameters. ■

THEOREM 2.12
Let $\Gamma \cup \{\alpha\}$ be a set of sentences of the multimodal language of $\mathcal{L}^\Phi$, where $\mathcal{L}^\Phi$ is among $\mathbf{K}^{\supset,\wedge,\Phi}+\mathbf{G}(a,b,c,d)+\mathbf{G}(a,b,c,d)$, $\mathbf{PI}^\Phi(a,b,c,d)$, $\mathbf{mbC}^\Phi(a,b,c,d)$, $\mathbf{bC}^\Phi(a,b,c,d)$ and $\mathbf{Ci}^\Phi(a,b,c,d)$. In these conditions, respectively, if $\Gamma \vDash_{\mathcal{L}^\Phi} \alpha$, then $\Gamma \vdash_{\mathcal{L}^\Phi} \alpha$.

PROOF. Suppose that $\Gamma \nvdash_{\mathcal{L}^\Phi} \alpha$ for some α then, by a Lindenbaum-style construction, we can extend Γ to a non-trivial maximal $\mathcal{L}^\Phi$-theory Δ such that $\Delta \nvdash_{\mathcal{L}^\Phi} \alpha$ so, $\alpha \notin \Delta$. From Theorem 2.11 we have that $v_\Delta(\alpha) = 0$. As $\Gamma \subseteq \Delta$ we also have that $v_\Delta(\gamma) = 1$, for all $\gamma \in \Gamma$. Therefore, there exists a multi-relational model that validates all formulas in Γ and falsifies α, i.e, $\Gamma \nvDash_{\mathcal{L}^\Phi} \alpha$. ■

We have proved that the basilar anodic and cathodic systems are characterized w.r.t. multi-relational Kripke models. For cathodic systems, however, this kind of characterization does not explain the role of basilar cathodic systems in supporting contradictions without falling into deductive triviality. This will be the subject of the next section.

3 Modal possible-translations semantics

Paraconsistent systems are known for their capability in supporting contradictions, in sense that a contradiction by itself does not entail triviality. A suitable tool to explain this property involving contradictions are the *possible-translations semantics* (a.k.a. PTS). Paraconsistent systems can also be characterized with respect to *bi-valued semantics*, but this kind of semantics does not give a satisfactory explanation about the rationality of maintaining contradictory stands about some topic without committing necessarily to deductive explosion. The idea behind PTS is that the contradiction occurs in terms of a group (e.g., of agents, machines or logics) and not in individual terms. In this process the translations will be fundamental because they will permit us to check the validity of paraconsistent formulas by mapping these formulas within the language of a group of other logics with lower semantical complexity, as explained in [CCM07] (in this section we assume the reader to have some familiarity with that paper, as well as with [BS10]). The question of validity of a given paraconsistent formula will depend on the "answer" of this same question given by each member of the group.

It is important to note that basilar cathodic systems have a paraconsistent system as non-modal (propositional) constituent. This will be crucial to define a *modal possible-translations structure* MPTS for those systems. In this paper we will extend the concept of MPTS, introduced in [BS10], to the classes of basilar cathodic systems.

A MPTS for a basilar cathodic system is obtained in similar way of a PTS for a propositional paraconsistent system. The method consists of constructing a three-valued multi-relational model $\mathfrak{M}^{\mathcal{L}^{\Phi}}_{\mathrm{Thv}}$ (or three-valued Kripke model) based in three-valued matrices $\mathscr{M}$ (or three-valued logics) described in the language that contains the connectives $\sqcap, \sqcup, \sqsupset, \neg_1, \neg_2, \circ_1, \circ_2, \circ_3$ whose truth-values are $\{F, t, T\}$, where T and t are distinguished values. The three-valued matrices we will reproduced in Note 3.1.

Note 3.1 (Tables of $\mathscr{M}$)

$\sqcap$	T	t	F
T	t	t	F
t	t	t	F
F	F	F	F

$\sqcup$	T	t	F
T	t	t	t
t	t	t	t
F	t	t	F

$\sqsupset$	T	t	F
T	t	t	F
t	t	t	F
F	t	t	t

	$\neg_1$	$\neg_2$	$\neg_3$
T	F	F	F
t	F	t	t
F	T	t	T

	$\circ_1$	$\circ_2$	$\circ_3$
T	T	t	F
t	F	F	F
F	T	t	F

In order to obtain a characterization of a basilar cathodic system $\mathcal{L}^{\Phi}(a, b, c, d)$ w.r.t. MPTS we consider the translations from $\mathcal{L}^{\Phi}(a, b, c, d)$ into $\mathscr{M}$ consisting of all functions in the set $\mathtt{Tr}_{\mathcal{L}^{\Phi}(a,b,c,d)}$ of mappings $\mathtt{t} : For_{\mathcal{L}^{\Phi}(a,b,c,d)} \longrightarrow For_{\mathscr{M}}$ subject to the following clauses given in Note 3.2:

Note 3.2 (Clauses on translations)

$(Tr.1)$	$\mathtt{t}(p)$	$=$	p
$(Tr.2)$	$\mathtt{t}(\alpha \supset \beta)$	$=$	$\mathtt{t}(\alpha) \sqsupset \mathtt{t}(\beta)$
$(Tr.3)$	$\mathtt{t}(\alpha \wedge \beta)$	$=$	$\mathtt{t}(\alpha) \sqcap \mathtt{t}(\beta)$
$(Tr.4)$	$\mathtt{t}(\alpha \vee \beta)$	$=$	$\mathtt{t}(\alpha) \sqcup \mathtt{t}(\beta)$
$(Tr.5)$	$\mathtt{t}(\neg\alpha)$	$\in$	$\{\neg_1\mathtt{t}(\alpha),\ \neg_2\mathtt{t}(\alpha)\}$
$(Tr.6)$	$\mathtt{t}(\neg\alpha)$	$\in$	$\{\neg_1\mathtt{t}(\alpha),\ \neg_3\mathtt{t}(\alpha)\}$
$(Tr.7)$	$\mathtt{t}(\circ\alpha)$	$\in$	$\{\circ_2\mathtt{t}(\alpha),\ \circ_3\mathtt{t}(\alpha),\ \circ_2\mathtt{t}(\neg\alpha),\ \circ_3\mathtt{t}(\neg\alpha)\}$
$(Tr.8)$	$\mathtt{t}(\circ\alpha)$	$\in$	$\{\circ_1\mathtt{t}(\alpha),\ \circ_1\mathtt{t}(\neg\alpha)\}$
$(Tr.9)$	$\mathtt{t}(\neg\alpha) = \neg_1\mathtt{t}(\alpha)$	implies	$\mathtt{t}(\circ\alpha) = \circ_1\mathtt{t}(\neg\alpha)$
$(Tr.10)$	$\mathtt{t}([a]\alpha)$	$=$	$\overline{[a]}\mathtt{t}(\alpha)$, for each $a \in \Phi$
$(Tr.11)$	$\mathtt{t}(\langle a\rangle\alpha)$	$=$	$\overline{\langle a\rangle}\mathtt{t}(\alpha)$, for each $a \in \Phi$

We will use $\mathcal{L}$ to refer to the paraconsistent systems **PI**, **mbC**, **bC** and **Ci** and, using the same notation of [BS10], the possible-translations structure associate to that systems is the pair $PT = \langle \mathscr{M}_{\mathcal{L}}, \mathtt{Tr}_{\mathcal{L}} \rangle$, where the translation in $\mathtt{Tr}_{\mathcal{L}}$ satisfies the conditions (Tr.1)–(Tr.9) and $\mathscr{M}_{\mathcal{L}}$ is formed by the following tables:

Note 3.3 (Tables of $\mathscr{M}_{\mathcal{L}}$)

$\mathcal{L}$	$\mathscr{M}_{\mathcal{L}}$ is composed by the following tables:
PI	$\sqsupset$, $\sqcap$, $\sqcup$, $\neg_1$ and $\neg_2$
mbC	$\sqsupset$, $\sqcap$, $\sqcup$, $\neg_1$, $\neg_2$, $\circ_2$ and $\circ_3$
bC	$\sqsupset$, $\sqcap$, $\sqcup$, $\neg_1$, $\neg_3$, $\circ_2$ and $\circ_3$
Ci	$\sqsupset$, $\sqcap$, $\sqcup$, $\neg_1$, $\neg_3$ and $\circ_1$

Adequate translations for each specific basilar cathodic system $\mathcal{L}^{\Phi}(a,b,c,d)$ are described in Note 3.4 below.

Note 3.4 (Restriction over translations of $\mathcal{L}^{\Phi}(a,b,c,d)$)

Logic	Restrictions over the translations
$\mathbf{PI}^{\Phi}(a,b,c,d)$	(Tr.1)–(Tr.5), (Tr.10) and (Tr.11)
$\mathbf{mbC}^{\Phi}(a,b,c,d)$	(Tr.1)–(Tr.5), (Tr.7), (Tr.10) and (Tr.11)
$\mathbf{bC}^{\Phi}(a,b,c,d)$	(Tr.1)–(Tr.4), (Tr.6), (Tr.7), (Tr.10) and (Tr.11)
$\mathbf{Ci}^{\Phi}(a,b,c,d)$	(Tr.1)–(Tr.4), (Tr.6), (Tr.8)–(Tr.11)

The three-valued multi-relational model based on the matrices given in Note 3.1 are defined as:

Definition 3.5

A *three-valued multi-relational model* $\mathfrak{M}_{\text{Thv}}$ is a pair $\langle \mathfrak{F}^{\Phi}, \overline{v} \rangle$, where $\mathfrak{F}^{\Phi}$ is a multi-relational frame and $\overline{v} : For \times W \longrightarrow \{T, t, F\}$ is a three-valued valuation determined by tables in $\mathscr{M}$, such that the following conditions are satisfied:

(i) $\overline{v}(p, w) \in \{T, t, F\}$, for $p \in Var$;

(ii) $\overline{v}(\alpha \bowtie \beta, w) = \overline{v}(\alpha, w) \bowtie \overline{v}(\beta, w)$, for $\bowtie \in \{\sqsupset, \sqcap, \sqcup\}$;

(iii) $\overline{v}(\neg_i \alpha, w) = \neg_i \overline{v}(\alpha, w)$ for $1 \leq i \leq 3$;

(iv) $\overline{v}(\circ_i \alpha, w) = \circ_i \overline{v}(\alpha, w)$ for $1 \leq i \leq 3$;

(v) $$\overline{v}(\overline{[a]}\alpha, w) = \begin{cases} t & \text{if } \overline{v}(\alpha, w') \in \{T, t\}, \text{ for all } w' \in W \text{ such that } wR_a w' \\ F & \text{if } \overline{v}(\alpha, w') = F, \text{ for some } w' \in W \text{ such that } wR_a w' \end{cases}$$

(vi) $$\overline{v}(\overline{\langle a \rangle}\alpha, w) = \begin{cases} t & \text{if } \overline{v}(\alpha, w') = \{T, t\}, \text{ for some } w' \in W \text{ such that } wR_a w'; \\ F & \text{if } \overline{v}(\alpha, w') = F, \text{ for all } w' \in W \text{ such that } wR_a w' \end{cases}$$

A sentence α is said to be *satisfied in a three-valued relational model* $\mathfrak{M}_{\text{Thv}}$, if there is a $w \in W$ such that $\overline{v}(\alpha, w) \in \{T, t\}$ (notation: $\mathfrak{M}_{\text{Thv}}, w \vDash \alpha$). A sentence α is said to be *valid in a three-valued relational model* $\mathfrak{M}_{\text{Thv}}$, if $\overline{v}(\alpha, w) \in \{T, t\}$ for all $w \in W$ (notation: $\mathfrak{M}_{\text{Thv}} \vDash \alpha$). A sentence α is said to be *valid on a frame* $\mathfrak{F}^{\Phi}$, if α is valid in all three-valued relational models based on $\mathfrak{F}^{\Phi}$ (notation: $\mathfrak{F}^{\Phi} \vDash \alpha$). $\mathfrak{F}^{\Phi}$ is said to be a *frame for* an arbitrary system **S** if every theorem of **S** is valid on $\mathfrak{F}^{\Phi}$.

In Note 3.6 the three-valued multi-relational models $\mathfrak{M}_{\text{Thv}}^{\mathcal{L}^{\Phi}(a,b,c,d)}$ adequate for mapping each basilar cathodic system $\mathcal{L}^{\Phi}(a, b, c, d)$ is specified:

Note 3.6 (The three-valued multi-relational models for $\mathcal{L}^{\Phi}(a, b, c, d)$)

$\mathcal{L}^{\Phi}(a, b, c, d)$	Three-valued multi-relational models $\mathfrak{M}_{\text{Thv}}^{\mathcal{L}^{\Phi}(a,b,c,d)}$
$\mathbf{PI}^{\Phi}(a, b, c, d)$	(i), (ii), $\overline{v}(\neg_1\alpha, w)$, $\overline{v}(\neg_2\alpha, w)$, (v) and (vi)
$\mathbf{mbC}^{\Phi}(a, b, c, d)$	(i), (ii), $\overline{v}(\neg_1\alpha, w)$, $\overline{v}(\neg_2\alpha, w)$, $\overline{v}(\circ_2\alpha, w)$, $\overline{v}(\circ_3\alpha, w)$, (v) and (vi)
$\mathbf{bC}^{\Phi}(a, b, c, d)$	(i), (ii), $\overline{v}(\neg_1\alpha, w)$, $\overline{v}(\neg_3\alpha, w)$, $\overline{v}(\circ_2\alpha, w)$, $\overline{v}(\circ_3\alpha, w)$, (v) and (vi)
$\mathbf{Ci}^{\Phi}(a, b, c, d)$	(i), (ii), $\overline{v}(\neg_1\alpha, w)$, $\overline{v}(\neg_3\alpha, w)$, $\overline{v}(\circ_1\alpha, w)$, (v) and (vi)

Now we have all the ingredients to define a modal possible-translations semantics for $\mathcal{L}^{\Phi}(a, b, c, d)$.

Definition 3.7
A *modal possible-translations structure* for a basilar cathodic system $\mathcal{L}^{\Phi}(a, b, c, d)$ is a triple $\texttt{MTP} = \langle \mathfrak{M}_{\text{Thv}}^{\mathcal{L}^{\Phi}(a,b,c,d)}, \texttt{Tr}_{\mathcal{L}^{\Phi}(a,b,c,d)}, \mathfrak{F}^{\Phi} \rangle$ such that:
(i) $\mathfrak{M}_{\text{Thv}}^{\mathcal{L}^{\Phi}(a,b,c,d)}$ is a three-valued multi-relational model in the sense of Note 3.6;
(ii) $\mathfrak{F}^{\Phi}$ is a frame for $\mathcal{L}^{\Phi}(a, b, c, d)$;
(iii) $\texttt{Tr}_{\mathcal{L}^{\Phi}(a,b,c,d)} \subseteq \texttt{Tr}_{\mathcal{L}}$ such that $\langle \mathscr{M}_{\mathcal{L}}, \texttt{Tr}_{\mathcal{L}} \rangle$ is a PTS for $\mathcal{L}$.

The consequence relation in a MPTS is based in the combination of consequence relations of logics with lower semantical complexity, as defined below.

Definition 3.8
Let $\Gamma \cup \{\alpha\}$ be a set of $\mathcal{L}^{\Phi}(a, b, c, d)$-formulas and $\vDash_{\mathfrak{F}^{\Phi}}$ a consequence relation in $\mathfrak{F}^{\Phi}$. A consequence relation in MPTS, denoted by $\vDash_{\text{MPTS}}$, is defined as:

$$\Gamma \vDash_{\text{MPTS}} \alpha \quad \text{iff} \quad \texttt{t}(\Gamma) \vDash_{\mathfrak{F}^{\Phi}} \texttt{t}(\alpha)$$

for all translations $\texttt{t} \in \texttt{Tr}_{\mathcal{L}^{\Phi}(a,b,c,d)}$.

From this we can prove the following results:

Theorem 3.9
Let $\Gamma \cup \{\alpha\}$ be a set of $\mathcal{L}^{\Phi}(a, b, c, d)$-formulas and $\mathcal{F}^{\Phi}$ a class of frames for $\mathcal{L}^{\Phi}(a, b, c, d)$, then:

$$\Gamma \vdash_{\mathcal{L}^{\Phi}(a,b,c,d)} \alpha \text{ implies } \Gamma \vDash_{\text{MPTS}} \alpha$$

Proof. It is easy to see that for each $\texttt{t} \in \texttt{Tr}_{\mathcal{L}^{\Phi}(a,b,c,d)}$ applied to the axioms of $\mathcal{L}^{\Phi}(a, b, c, d)$ outputs valid sentences in $\mathfrak{M}_{\text{Thv}}^{\mathcal{L}^{\Phi}(a,b,c,d)}$ and that rules preserve validity. ∎

Since the valuations of the consistency connective $\circ$ depends of the valuation of negation $\neg$, it is convenient to define a non-canonical measure of complexity ℓ of formulas including also the multimodal cases.

Definition 3.10
Let $\mathbf{S}$ be a system, and For be a set of sentences of $\mathbf{S}$. The function $\ell : For \longrightarrow \mathbb{N}$ denote the *complexity length* of sentences, and is defined as:

(i) $\ell(p) = 0$, for $p \in Var$;
(ii) $\ell(\neg\alpha) = \ell(\alpha) + 1$;
(iii) $\ell(\alpha \bowtie \beta) = \mathtt{max}\{\ell(\alpha), \ell(\beta)\} + 1$, for $\bowtie \in \{\supset, \wedge, \vee\}$;
(iv) $\ell(\circ\alpha) = \ell(\alpha) + 2$;

$$(v) \quad \ell([a]\alpha) = \begin{cases} 0 & \text{if } \ a = \mathit{0} \\ \ell(\alpha) & \text{if } \ a = \mathit{1} \\ \ell(\alpha) + 1 & \text{if } \ a \in \Phi_0, \ a \neq \mathit{0} \ \text{ and } \ a \neq \mathit{1} \\ \ell([b][c]\alpha) & \text{if } \ a = b \odot c \\ \ell([b]\alpha \wedge [c]\alpha) & \text{if } \ a = b \cup c \end{cases}$$

$$(vi) \quad \ell(\langle a \rangle\alpha) = \begin{cases} 0 & \text{if } \ a = \mathit{0} \\ \ell(\alpha) & \text{if } \ a = \mathit{1} \\ \ell(\alpha) + 1 & \text{if } \ a \in \Phi_0, \ a \neq \mathit{0} \ \text{ and } \ a \neq \mathit{1} \\ \ell(\langle b \rangle\langle c \rangle\alpha) & \text{if } \ a = b \odot c \\ \ell(\langle b \rangle\alpha \vee \langle c \rangle\alpha) & \text{if } \ a = b \cup c \end{cases}$$

The next four lemmas prove the representability of the bi-valued relational models by means of appropriate translations and three-valued relational models for the systems $\mathbf{PI}^\Phi(a,b,c,d)$, $\mathbf{mbC}^\Phi(a,b,c,d)$, $\mathbf{bC}^\Phi(a,b,c,d)$ and $\mathbf{Ci}^\Phi(a,b,c,d)$.

Lemma 3.11
Given a $\mathbf{PI}^\Phi(a,b,c,d)$-valuation v in $\mathfrak{M}_{\text{Biv}}^{\text{PI}^\Phi\text{(a,b,c,d)}}$ and a frame $\mathfrak{F}^\Phi$ for $\mathbf{PI}^\Phi(a,b,c,d)$ it is possible to define a translation $\mathtt{t}$ in $\mathtt{Tr}_{\text{PI}^\Phi\text{(a,b,c,d)}}$, a valuation $\overline{v}$ and a three-valued relational model $\mathfrak{M}_{\text{Thv}}^{\text{PI}^\Phi\text{(a,b,c,d)}}$ such that, for every formula α in $\mathbf{PI}^\Phi(a,b,c,d)$ and all $w \in W$:

(i) $\overline{v}(\mathtt{t}(\alpha), w) = t$ iff $v(\alpha, w) = 1$
(ii) $\overline{v}(\mathtt{t}(\alpha), w) = F$ iff $v(\alpha, w) = 0$

Proof. Consider $\overline{v} : For \times W \longrightarrow \{T, t, F\}$ defined as:

$$(Val) \quad \overline{v}(p, w) = \begin{cases} F & \text{if } v(p, w) = 0 \\ t & \text{if } v(p, w) = 1 \end{cases}$$

Clearly $\overline{v}$ can be extended homomorphically to all formulas in the matrix $\mathscr{M}_{\text{PI}^\Phi\text{(a,b,c,d)}}$. Now define the intended translation in the following way:

($T1$) $\mathtt{t}(p) = p$
($T2$) $\mathtt{t}(\alpha \supset \beta) = \mathtt{t}(\alpha) \sqsupset \mathtt{t}(\beta)$
($T3$) $\mathtt{t}(\alpha \wedge \beta) = \mathtt{t}(\alpha) \sqcap \mathtt{t}(\beta)$
($T4$) $\mathtt{t}(\alpha \vee \beta) = \mathtt{t}(\alpha) \sqcup \mathtt{t}(\beta)$

$$(T5) \quad \mathtt{t}(\neg\alpha) = \begin{cases} \neg_1 \mathtt{t}(\alpha) & \text{if } v(\neg\alpha, w) = 0 \\ \neg_2 \mathtt{t}(\alpha) & \text{if } v(\neg\alpha, w) = 1 \end{cases}$$

($T6$) $\mathtt{t}([a]\alpha) = \overline{[a]}\mathtt{t}(\alpha)$, for $a \in \Phi$
($T7$) $\mathtt{t}(\langle a \rangle\alpha) = \overline{\langle a \rangle}\mathtt{t}(\alpha)$, for $a \in \Phi$

Note that the collection of translations is determined by restrictions (Tr.1)–(Tr.5), (Tr.10) and (Tr.11) which characterize translations of $\mathbf{PI}^{\Phi}(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d})$. The three-valued model $\mathfrak{M}^{\mathrm{PI}^{\Phi}(\mathrm{a,b,c,d})}_{\mathrm{Thv}}$ is obtained by extending $\mathfrak{F}^{\Phi}$ with the valuation $\bar{v}$ defined above. The result is proven by induction on ℓ.

1. The atomic case follows from (Val) and (T1).
2. Consider that the induction hypothesis is valid for all formula α with $\ell(\alpha) \leq k$, for some k:

   (IHa) $\bar{v}(\mathtt{t}(\alpha), w) = t$ iff $v(\alpha, w) = 1$
   (IHb) $\bar{v}(\mathtt{t}(\alpha), w) = F$ iff $v(\alpha, w) = 0$
3. The cases where $\alpha = \beta \supset \gamma$, $\alpha = \beta \wedge \gamma$ and $\alpha = \beta \vee \gamma$ the result follows easily by induction hypothesis. The more complicated case $\alpha = \neg\beta$ can be found in lemma 5.13. of [BS10].
4. Consider $\alpha = [a]\beta$:
   - Part A: $\bar{v}(\mathtt{t}([a]\beta), w) = t$ iff $v([a]\beta, w) = 1$.
     $\bar{v}(\mathtt{t}([a]\beta), w) = t$ iff$_{(T6)}$ $\bar{v}(\overline{[a]}\mathtt{t}(\beta), w) = t$ iff$_{\text{Def. 3.5(v)}}$ $\bar{v}(\mathtt{t}(\beta), w') = t$ for each $w' \in W$ such that $wR_a w'$ iff$_{(IHb)}$ $v(\beta, w) = 1$ for each $w' \in W$ such that $wR_a w'$ iff $v([a]\beta, w) = 1$.
   - Part B: $\bar{v}(\mathtt{t}([a]\beta), w) = F$ iff $v([a]\beta, w) = 0$.
     Analogous to Part A.
5. The case where $\alpha = \langle a \rangle\beta$ is analogous to the previous case.

■

Lemma 3.12
Given a $\mathbf{mbC}^{\Phi}(a, b, c, d)$-valuation v in $\mathfrak{M}^{\mathrm{mbC}^{\Phi}(\mathrm{a,b,c,d})}_{\mathrm{Biv}}$ and a frame $\mathfrak{F}^{\Phi}$ for $\mathbf{mbC}^{\Phi}(a, b, c, d)$ it is possible to find a translation $\mathtt{t}$ in $\mathtt{Tr}_{\mathrm{mbC}^{\Phi}(\mathrm{a,b,c,d})}$, a valuation $\bar{v}$ in a three-valued relational model $\mathfrak{M}^{\mathrm{mbC}^{\Phi}(\mathrm{a,b,c,d})}_{\mathrm{Thv}}$ such that every formula α in $\mathbf{mbC}^{\Phi}(a, b, c, d)$ and for all $w \in W$:
(*i*) $\bar{v}(\mathtt{t}(\alpha), w) = T$ implies $v(\neg\alpha, w) = 0$
(*ii*) $\bar{v}(\mathtt{t}(\alpha), w) = F$ iff $v(\alpha, w) = 0$

Proof. The argument is essentially similar to the previous Lemma, but here the language includes the connective $\circ$. We just emphasize the subtleties concerning $\circ$.

Consider $\bar{v} : For \times W \longrightarrow \{T, t, F\}$ defined as:

$$(Val) \quad \bar{v}(p, w) = \begin{cases} F & \text{if } v(p, w) = 0 \\ T & \text{if } v(\neg p, w) = 0 \\ t & \text{if } v(p, w) = 1 \end{cases}$$

Clearly $\bar{v}$ can be homomorphically extended to all formulas in the matrix $\mathscr{M}_{\mathrm{mbC}^{\Phi}(\mathrm{a,b,c,d})}$. The translations differ from the previous lemma in the following details:

$$(T5) \quad \mathtt{t}(\neg\alpha) = \begin{cases} \neg_1\mathtt{t}(\alpha) & \text{if } v(\neg\alpha) = 0 \text{ or } v(\alpha) = 0 = v(\neg\neg\alpha) \\ \neg_2\mathtt{t}(\alpha) & \text{if } v(\neg\alpha) = 1 \end{cases}$$

$$(T8) \quad \mathtt{t}(\circ\alpha) = \begin{cases} \circ_3\mathtt{t}(\alpha) & \text{if } v(\circ\alpha) = 0 \\ \circ_2\mathtt{t}(\neg\alpha) & \text{if } v(\circ\alpha) = 1 \text{ and } v(\neg\alpha) = 0 \\ \circ_2\mathtt{t}(\alpha) & \text{if } v(\circ\alpha) = 1 \end{cases}$$

The collection of translations allowed is now determined by restrictions (Tr.1), (Tr.2), (Tr.3), (Tr.4), (Tr.5), (Tr.7), (Tr.10) and (Tr.11) that characterize the translations of $\mathbf{mbC}^{\Phi}(a,b,c,d)$. The model $\mathfrak{M}^{\mathrm{mbC}^{\Phi}(\mathrm{a,b,c,d})}_{\mathrm{Thv}}$ is obtained by extending $\mathfrak{F}^{\Phi}$ with the valuation $\overline{v}$ defined above. The statement is proven by induction on the length of complexity of ℓ.

1. The atomic case follows from (Val) and (T1).
2. Assume the induction hypothesis for all formulas α with $\ell(\alpha) \leq k$, for some k:

   (IHa) $\overline{v}(\mathtt{t}(\alpha), w) = T$ implies $v(\neg\alpha, w) = 0$
   (IHb) $\overline{v}(\mathtt{t}(\alpha), w) = F$ iff $v(\alpha, w) = 0$

3. For the cases $\alpha = \beta \supset \gamma$, $\alpha = \beta \wedge \gamma$, $\alpha = \beta \vee \gamma$, $\alpha = \neg\beta$ and $\alpha = \circ\beta$, see lemma 5.14. in [BS10].
4. Consider $\alpha = [a]\beta$
   - Part A: $\overline{v}(\mathtt{t}([a]\beta), w) = T$ implies $v(\neg([a]\beta), w) = 0$
     If $\overline{v}(\mathtt{t}([a]\beta), w) = T$ then, by Definition 3.5 (v), $\overline{v}(\overline{[a]}\mathtt{t}(\beta), w) = T$. Given that such valuation is impossible, so the result is valid by vacuity.
   - Part B: $\overline{v}(\mathtt{t}([a]\beta), w) = F$ iff $v([a]\beta, w) = 0$
     Immediate from Definition 3.5 (v) and (IHb).
5. The case $\alpha = \langle a\rangle\beta$ is analogous to the previous.

■

Lemma 3.13
Given a $\mathbf{bC}^{\Phi}(a,b,c,d)$-valuation v in $\mathfrak{M}^{\mathrm{bC}^{\Phi}(\mathrm{a,b,c,d})}_{\mathrm{Biv}}$ and a frame $\mathfrak{F}^{\Phi}$ for $\mathbf{bC}^{\Phi}(a,b,c,d)$ it is possible to find a translation $\mathtt{t}$ in $\mathtt{Tr}_{\mathrm{bC}^{\Phi}(\mathrm{a,b,c,d})}$, a valuation $\overline{v}$ and a model $\mathfrak{M}^{\mathrm{bC}^{\Phi}(\mathrm{a,b,c,d})}_{\mathrm{Thv}}$ such that, for each formula α in $\mathbf{bC}^{\Phi}(a,b,c,d)$ and for all $w \in W$:
(*i*) $\overline{v}(\mathtt{t}(\alpha), w) = T$ implies $v(\neg\alpha, w) = 0$
(*ii*) $\overline{v}(\mathtt{t}(\alpha), w) = F$ iff $v(\alpha, w) = 0$

Proof. Consider $\overline{v} : For \times W \longrightarrow \{T, t, F\}$ defined as:

$$(Val) \quad \overline{v}(p, w) = \begin{cases} F & \text{if } v(p, w) = 0 \\ T & \text{if } v(\neg p, w) = 0 \\ t & \text{if } v(p, w) = 1 \end{cases}$$

Again, $\overline{v}$ can be homomorphically extended to all formulas in the matrix $\mathscr{M}_{bC^{\Phi}(a,b,c,d)}$. The translations differ from the previous lemma in some details, as:

$$(T5) \quad \mathtt{t}(\neg\alpha) = \begin{cases} \neg_3\mathtt{t}(\alpha) & \text{if } v(\alpha) = 1 = v(\neg\alpha) \\ \neg_1\mathtt{t}(\alpha) & \text{if } v(\alpha) = 0 \end{cases}$$

$$(T8) \quad \mathtt{t}(\circ\alpha) = \begin{cases} \circ_3\mathtt{t}(\alpha) & \text{if } v(\circ\alpha) = 0 \\ \circ_2\mathtt{t}(\neg\alpha) & \text{if } v(\circ\alpha) = 1 \text{ and } v(\neg\alpha) = 0 \\ \circ_2\mathtt{t}(\alpha) & \text{if } v(\circ\alpha) = 1 \end{cases}$$

The same procedure used in Lemma 3.11 and Lemma 3.12. ■

Lemma 3.14
Given a $\mathbf{Ci}^{\Phi}(a,b,c,d)$-valuation v in $\mathfrak{M}_{\mathrm{Biv}}^{\mathrm{Ci}^{\Phi}(\mathrm{a,b,c,d})}$ and a frame $\mathfrak{F}^{\Phi}$ for $\mathbf{Ci}^{\Phi}(a,b,c,d)$ it is possible to define a translation $\mathtt{t}$ in $\mathtt{Tr}_{\mathrm{Ci}^{\Phi}(\mathrm{a,b,c,d})}$, a valuation $\overline{v}$ and a Kripke model $\mathfrak{M}_{\mathrm{Thv}}^{\mathrm{Ci}^{\Phi}(\mathrm{a,b,c,d})}$ such that, for every α in $\mathbf{Ci}^{\Phi}(a,b,c,d)$ and for all $w \in W$:

(i) $\overline{v}(\mathtt{t}(\alpha), w) = T$ implies $v(\neg\alpha, w) = 0$
(ii) $\overline{v}(\mathtt{t}(\alpha), w) = F$ iff $v(\alpha, w) = 0$

Proof. Consider $\overline{v} : For \times W \longrightarrow \{T, t, F\}$ defined as:

$$(Val) \quad \overline{v}(p,w) = \begin{cases} F & \text{if } v(p,w) = 0 \\ T & \text{if } v(\neg p, w) = 0 \\ t & \text{if } v(p,w) = 1 \end{cases}$$

Once more, $\overline{v}$ can be extended to all formulas in the matrix $\mathscr{M}_{\mathrm{Ci}^{\Phi}(\mathrm{a,b,c,d})}$ in the usual way. The translations differ from the previous lemma in the following points:

$$(T5) \quad \mathtt{t}(\neg\alpha) = \begin{cases} \neg_3 \mathtt{t}(\alpha) & \text{if } v(\alpha) = 1 = v(\neg\alpha) \\ \neg_1 \mathtt{t}(\alpha) & \text{if } v(\alpha) = 0 \end{cases}$$

$$(T8) \quad \mathtt{t}(\circ\alpha) = \begin{cases} \circ_1 \mathtt{t}(\neg\alpha) & \text{if } v(\circ\alpha) = 1 \\ \circ_1 \mathtt{t}(\alpha) & \text{if } v(\circ\alpha) = 0 \end{cases}$$

To conclude, use the same procedure used in Lemma 3.11 and Lemma 3.12. ∎

From the Representability Lemma we can now prove completeness by means of modal possible-translations semantics.

Corollary 3.15
Let $\Gamma \cup \{\alpha\}$ a set of formulas of $\mathcal{L}^{\Phi}(a,b,c,d)$, then:

$$\Gamma \vDash_{\mathrm{MTP}} \alpha \ \text{ implies } \ \Gamma \vdash_{\mathcal{L}^{\Phi}(a,b,c,d)} \alpha$$

This result has several important consequences. The most conspicuous one is that cathodic (multi)modal logics, the class of paraconsistent (multi)modal logics we have defined, can be seen as non-trivial combinations of three-valued (multi)modal logics (considered by several authors, see e.g. [Fit91]), despite being also characterized by multi-relational frames generalizing the familiar Kripke frames. As mentioned before, in order to obtain such combinations we cannot, in principle, rely in the methods of fibring by functions, algebraic fibring and modulated fibring as in [CCG+07]. This topic will be investigated in a future paper.

Motivations and applications for cathodic modal logics are discussed in [BS09b], but it would seem evident that multimodal logics with restricted negations, as our cathodic systems, have considerable potential applications in expressing degrees of belief, knowledge, norms, conflicting obligations and other situations where classical negations can be excessive. Our modal possible-translations semantics open a whole area of research, not only due to the expressive power of this semantic interpretation but also due to the purely logical tools (and problems therein) that such models represent. Description Logics are used in computer science as an important knowledge representation formalism, generalizing the notion of semantic networks. It is interesting to notice that Description Logics are notational variants of propositional multimodal logics $\mathbf{K}_m$ as shown by K. Schild in [Sch91]. From this perspective cathodic systems can be a very suitable generalization for Description Logics, adding the necessary expressivity to help solving problem of system designers in handling error conditions. This is also a topic of (hopefully promising) further research.

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