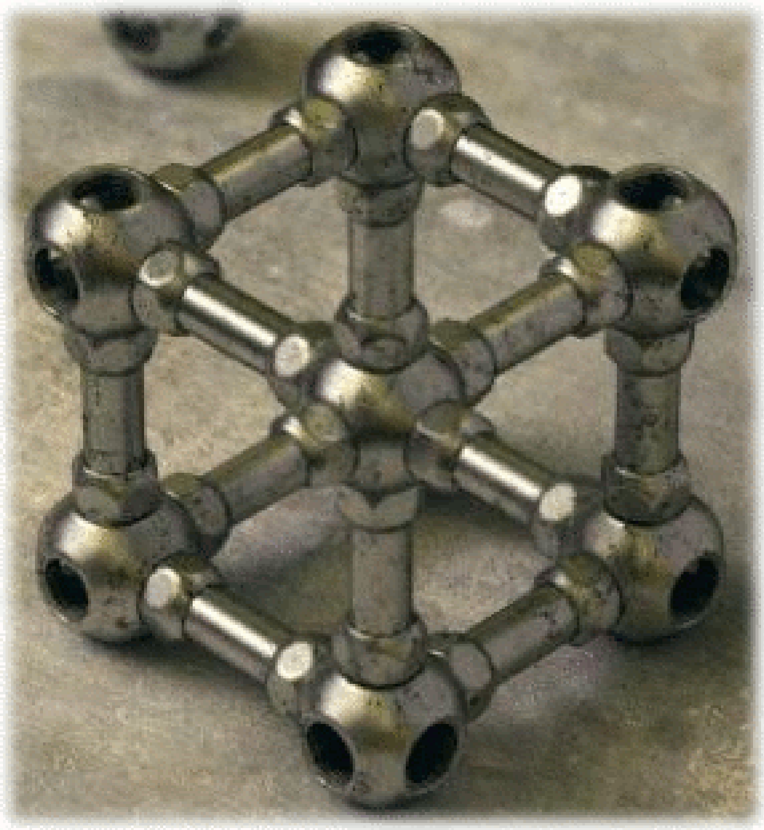

A Refuted Conjecture on Probabilistic Satisfiability [1]

Author(s):

Marcelo Finger

Glauber De Bona

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A Refuted Conjecture on Probabilistic Satisfiability*

Marcelo Finger** and Glauber De Bona***

Institute of Mathematics and Statistics, University of São Paulo,
São Paulo, Brazil
{mfinger,debona}@ime.usp.br

Abstract. In this paper, we investigate the Probabilistic Satisfiability Problem, and its relation with the classical Satisfiability Problem, looking for a possible polynomial-time reduction. For this, we present an Atomic Normal Form to the probabilistic satisfiability problem and then we define a Probabilistic Entailment relation, showing its inherent properties. At the end, we enunciate and refute a conjecture that could lead to the desired polynomial-time reduction.

Key words: probabilistic logic, probabilistic satisfiability

1 Introduction

The study of reasoning under uncertainty is a subject from many fields, and, in computer science, it has been useful on distributed system analysis and program analysis under probabilistic assumptions. In the XIX century, Boole [1] has already studied the probability assignment to logical sentences, and we see his influence on de Finetti's theory of subject probability [3]. In 1965, Hailperin [5] revisited the problem, giving it a linear programming form. In 1986, Nilsson [8] formalized the probabilistic satisfiability problem as we know it today: given logical sentences, and a probability assignment to them, we want to know if this assignment is consistent. The main analytical and numerical solutions to this problem, as well its detailed history, can be seen in [6].

The probabilistic satisfiability problem (PSAT) is NP-complete. Thus the Cook-Levin Theorem [2] tells us that there is a polynomial reduction from PSAT to classical satisfiability (SAT). Such reduction might be interesting to exploit the efficiency of SAT solvers. Another reason to study reduction from PSAT to SAT is the seek for a better understanding on the relation between logic and probability. The objective of this work is to investigate the relation between PSAT and SAT, looking for paths that enable the desired reduction.

In Section 2 we formally present the PSAT problem, in Nilsson's linear programming formulation [8]. In Section 3 we introduce the Atomic Normal Form

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for PSAT and in Section 4 a probabilistic entailment relation is suggested. Then in Section 5 we present a conjecture, under the concepts developed in the previous sections, on a possible reduction from PSAT to SAT. Such conjecture is exhaustively refuted by a counterexample.

2 The Problem

The probabilistic satisfiability (PSAT) is a decision problem, where we explore the consistency of a probability assignment over logical formulas.

Formally, let $S = \{s_1, \ldots, s_k\}$ be a set with k logical sentences, defined on a set of n Boolean variables, $X = \{x_1, \ldots, x_n\}$, with the usual operators from the classical propositional logic. A truth assignment (or valuation) v is initially defined as a function that associates truth values to Boolean variables, formally $v : X \to \{0, 1\}$. Then we can extend its domain to the set of formulas S, as usual in classical logic [1], $v : S \to \{0, 1\}$.

Let $V = \{v_1, \ldots, v_{2^n}\}$ be the set of possible truth assignments over X. A probability distribution over propositional valuations $\pi : V \to [0, 1]$ is a function that maps every valuation to a value in the real interval $[0, 1]$ such that $\sum_{i=1}^{2^n} \pi(v_i) = 1$. The probability of a formula s according to π is giving by $p_\pi(s) = \sum\{\pi(v_j) | v_j(s) = 1\}$.

Let $P = \{p_i | 0 \leq p_i \leq 1,\ 1 \leq i \leq k\}$ be a set of probabilities. We say that the probability assignment $p(s_i) = p_i$ is consistent if, and only if, there is a distribution π over V that makes $p_\pi(s_i) = p_i, 1 \leq i \leq k$. Finally a PSAT instance, defined by the set S and by $p(s_i) = p_i$, with $1 \leq i \leq k$, is satisfiable iff this probability assignment is consistent.

Now PSAT can be expressed as a linear programming problem, as introduced in [8]. Let Δ be the PSAT instance made by assigning the probabilities in P to the k formulas in S, $\Delta = \{p(s_i) = p_i | 1 \leq i \leq k\}$. We define the matrix $A_{k \times 2^n} = [a_{ij}]$, such that $a_{ij} = v_j(s_i)$, and the matrix $p_{k \times 1} = [p_i]$. So Δ is satisfiable iff there is a vector π that hold the restrictions:

$$A\pi = p \ ; \tag{1}$$

$$\pi \geq 0 \ ; \tag{2}$$

$$\sum \pi = 1 \ . \tag{3}$$

If there is a feasible solution π, then we say that π satisfies Δ, else we say that Δ is unsatisfiable. The restrictions (2) and (3) force π to be a probability distribution. The restriction (3) can be omitted if we add an entire row of $1's$ to the matrix A, $a_{k+1,j} = 1, 1 \leq j \leq 2^n$, and if we add an element $p_{k+1,1} = 1$ to the vector p, what will be done from here until the end of this paper.

According to Carathéodory's Lemma [9], if the linear programming problem (1-3) has feasible solution, then there is a solution with only $k + 1$ elements of

[1] Let α and β be formulas from classical propositional logic, we have: $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$; $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$; $v(\neg\alpha) = 1$ iff $v(\alpha) = 0$; $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$; and $v(\alpha \leftrightarrow \beta) = 1$ iff $v(\alpha) = v(\beta)$.

π different from zero. As pointed in [4], this result places PSAT in NP, because we can take a matrix $A_{k+1,k+1}$ and a vector $\pi_{k+1,1}$ as NP-certificate. Besides that, any instance from classical satisfiability (SAT), made from a set S with k sentences, can be reduced to a PSAT instance, in polynomial time, by making $p(s_i) = p_i = 1, 1 \le i \le k$. It follows that PSAT is NP-hard, hence NP-complete.

3 The Atomic Normal Form

Let $S = \{s_1, \ldots, s_k\}$ be a set of sentences from classical propositional logic, over the set $X = \{x_1, \ldots, x_n\}$ of Boolean variables. We say that a PSAT instance, $\Delta = \{p(s_i) = p_i | 1 \le i \le k\}$, $0 \le p_i \le 1$, is in the *Atomic Normal Form* if it can be partitioned in two sets, (Γ, Ψ), where $\Gamma = \{p(s_i) = 1 | 1 \le i \le m\}$ and $\Psi = \{p(y_i) = p_i | y_i$ is an atom and $1 \le i \le l\}$, with $0 \le p_i \le 1$, where $k = m + l$. The partition Γ is the SAT part of the atomic normal form, usually represented as a set of formulas, and the partition Ψ is the atomic probability assignment part. The following Theorem shows how any PSAT instance can be brought to the atomic normal form, by adding a linear number of new variables.

Theorem 1 (Atomic Normal Form). *Let $\Delta = \{p(s_i) = p_i | 1 \le i \le k\}$ be a PSAT instance. Then a PSAT instance (Γ, Ψ) in the atomic normal form can be built, in polynomial time on k, such that Δ is satisfiable iff (Γ, Ψ) is satisfiable.*

Proof. To build a PSAT instance (Γ, Ψ) in the atomic normal form, from the instance $\Delta = \{p(s_i) = p_i, 1 \le i \le k\}$, we first add k new variables, $y_1, \ldots, y_k$. Then we make $\Gamma = \{p(y_i \leftrightarrow s_i) = 1 | 1 \le i \le k\}$ and $\Psi = \{p(y_i) = p_i | 1 \le i \le k\}$. Clearly, this can be done in polynomial time on k.

Suppose there is a probability distribution π over the truth assignments $v : \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_k\} \to \{0, 1\}$ that satisfies (Γ, Ψ). Because π satisfies (Γ, Ψ), we have $p_\pi(y_i) = p_i$, $1 \le i \le k$. By the construction of Γ and the laws of probability, $p_\pi(y_i) = p_\pi(s_i)$, thus $p_\pi(s_i) = p_i$, $1 \le i \le k$. Over the truth assignments $v' : \{x_1, \ldots, x_n\} \to \{0, 1\}$, we define a probability distribution π':

$$\pi'(v') = \sum \{\pi(v) | v(x_i) = v'(x_i), 1 \le i \le n\} \ .$$

Hence π' is a probability distribution over $\{x_1, \ldots, x_n\}$ that satisfies $p_{\pi'}(s_i) = p_i$, $1 \le i \le k$, and consequently π' satisfies Δ.

Now suppose there is a probability distribution π' over the truth assignments $v' : \{x_1, \ldots, x_n\} \to \{0, 1\}$ that satisfies Δ. Because π' satisfies Δ, $p_{\pi'}(s_i) = p_i$, $1 \le i \le k$. We define a probability distribution π over the truth assignments $v : \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_k\} \to \{0, 1\}$:

$$\pi(v) = \begin{cases} \pi'(v') \text{ , if } v(x_i) = v'(x_i) \text{ and } v(y_j) = v(s_j),\ 1 \le i \le n \text{ and } 1 \le j \le k \\ \quad 0 \text{ , other cases} \end{cases} .$$

Clearly, we have $p_\pi(s_i) = p_{\pi'}(s_i)$ and $p_\pi(y_i) = p_\pi(s_i)$, $1 \le i \le k$. It follows that $p_\pi(y_i) = p_{\pi'}(s_i) = p_i$, $1 \le i \le k$, and then π satisfies Ψ. For all v, such that $\pi(v) \ne 0$, we have $v(y_i) = v(s_i)$, $1 \le i \le k$, thus $p_\pi(y_i \leftrightarrow s_i) = 1$, $1 \le i \le k$, and π satisfies Γ. Finally, π satisfies (Γ, Ψ). □

The atomic normal form allows us to see a PSAT instance (Γ, Ψ) as an interaction between a probability assignment (Ψ) and a SAT instance (Γ). Solutions to (Γ, Ψ) can be seen as solutions to Ψ constrained by the SAT instance Γ.

4 A Probabilistic Entailment Relation

Let Ψ be an atomic probability assignment and let α be a formula. We say that Ψ probabilistically entails α, denoted by $\Psi \mathrel{|}\!\approx \alpha$, iff the PSAT instance in the atomic normal form $(\{\neg\alpha\}, \Psi)$ is (probabilistically) unsatisfiable. In other words, if $(\{\neg\alpha\}, \Psi)$ is unsatisfiable, then $p(\neg\alpha) = 1$ and $p(\alpha) = 0$ are not consistent with Ψ; and because the probabilities are non-negative, $\Psi \mathrel{|}\!\approx \alpha$ implies $p(\alpha) > 0$, for any probability distribution that satisfies Ψ.

We denote by $\Psi^{\mathrel{|}\!\approx}$ the set of all formulas α such that $\Psi \mathrel{|}\!\approx \alpha$. The following Theorem shows, by this set, the role probabilistic entailment plays in PSAT study.

Theorem 2. *Let $\Sigma = (\Gamma, \Psi)$ be a PSAT instance in the atomic normal form. Σ is satisfiable iff for each $\alpha \in \Psi^{\mathrel{|}\!\approx}$, $\Gamma \cup \{\alpha\}$ is classically satisfiable.*

Proof. Suppose there is an $\alpha \in \Psi^{\mathrel{|}\!\approx}$, but $\Gamma \cup \{\alpha\}$ is classically unsatisfiable, then $\Gamma \vDash \neg\alpha$. From $\alpha \in \Psi^{\mathrel{|}\!\approx}$, we obtain that $(\{\neg\alpha\}, \Psi)$ is probabilistically unsatisfiable, and thus (Γ, Ψ) is also probabilistically unsatisfiable.

Conversely, suppose (Γ, Ψ) is probabilistically unsatisfiable. Let γ be the conjunction of all formulas in Γ. Obviously $(\{\gamma\}, \Psi)$ is also unsatisfiable. It follows that $\neg\gamma \in \Psi^{\mathrel{|}\!\approx}$ and, clearly, $\Gamma \cup \{\neg\gamma\}$ is classically unsatisfiable. □

This motivates the study of the probabilistic entailment properties.

4.1 Probabilistic Entailment Properties

We first note an initial relation between $\mathrel{|}\!\approx$ and $\vDash$:

Lemma 1. *If $\Psi \mathrel{|}\!\approx \alpha$ and $\alpha \vDash \beta$, then $\Psi \mathrel{|}\!\approx \beta$.*

Proof. From $\Psi \mathrel{|}\!\approx \alpha$, we know that $p(\alpha) > 0$, and $\alpha \vDash \beta$ yields $p(\alpha) \leq p(\beta)$. So $p(\beta) > 0$, and therefore $\Psi \mathrel{|}\!\approx \beta$. □

However we note that $\Psi \mathrel{|}\!\approx \alpha$ and $\Psi \mathrel{|}\!\approx \beta$ don't imply $\Psi \mathrel{|}\!\approx \alpha \wedge \beta$. As counterexample, we take $p(\alpha) = p(\beta) = 0.4$ and $p(\alpha \vee \beta) = 0.8$, from where we obtain $p(\alpha \wedge \beta) = 0$ and thus $\Psi \mathrel{|}\!\not\approx \alpha \wedge \beta$. In this counterexample, we made $p(\alpha \vee \beta) = p(\alpha) + p(\beta)$, but it's only possible when $p(\alpha) + p(\beta) \leq 1$. This leads us to the next Lemma.

Because in the atomic normal form the probabilities are assigned to atoms, and consequently to their negations, it's useful to define literal, so we can talk about probabilities overs literals. A literal x is an atom or its negation, and $\bar{x}$ denotes the negation of x.

Lemma 2. *Let Ψ be an atomic probability assignment such that, for literals y and z, $p(y) + p(z) > 1$. Then $\Psi \approx\!\!\!\!\mid\; y \wedge z$.*

Proof. As direct consequence of Kolmogorov's probability axioms, we know that

$$p(y) + p(z) = p(y \vee z) + p(y \wedge z) \ .$$

As $p(y) + p(z) > 1$, and always $p(y \vee z) \leq 1$, we obtain that $p(y \wedge z) > 0$, and thus $\Psi \approx\!\!\!\!\mid\; y \wedge z$. □

However, as $p(y_1) + \ldots + p(y_k) > 1$ doesn't imply $\Psi \approx\!\!\!\!\mid\; y_1 \wedge \ldots \wedge y_k$, we look for a suitable generalization for the Lemma 2.

Let $y_1, \ldots, y_k$ be literals and let j be an integer, with $1 \leq j \leq k$. We define:

$$C^j(y_1, \ldots, y_k) = \bigvee\{y_{i_1} \wedge \ldots \wedge y_{i_j} | 1 \leq i_1 < \ldots < i_j \leq k\} \ . \tag{4}$$

For example, $C^1(y, z, w) = y \vee z \vee w$, $C^2(y, z, w) = (y \wedge z) \vee (y \wedge w) \vee (z \vee w)$ and $C^3(y, z, w) = y \wedge z \wedge w$. It is useful to define $C^0(y_1, \ldots, y_k) = 1$, as the conjunction neutral element. We call formulas in the format (4) a *C-formula.*

From the commutativity of the logical operators $\wedge$ and $\vee$, we obtain that the literals order in (4) is irrelevant. Besides that, let's look at the following C-formulas properties, related to the entailment:

Lemma 3. *Let $y_1, \ldots, y_k$ be literals and let j, j', k and k' be non-negative integers:*

(a) if $0 \leq j' < j$ then $C^j(y_1, \ldots, y_k) \vDash C^{j'}(y_1, \ldots, y_k)$;
(b) if $k' > k$ then $C^j(y_1, \ldots, y_k) \vDash C^j(y_1, \ldots, y_{k'})$.

Proof.

(a) Let $Y = \{y_1, \ldots, y_k\}$ be a set of literals. Note that the truth assignment v satisfies $C^j(y_1, \ldots, y_k)$ iff there is a set $Y' \subseteq Y$ such that $|Y'| = j$ and $v(y) = 1$, for all $y \in Y'$. Obviously, if v satisfies $C^j(y_1, \ldots, y_k)$, then, for each $1 \leq j' < j$, there is a set $Y'' \subseteq Y' \subseteq Y$ such that $|Y''| = j'$ and $v(y) = 1$ for all $y \in Y''$. So such truth assignment v must also satisfy $C^{j'}(y_1, \ldots, y_k)$.
(b) When $k' > k$, $C^j(y_1, \ldots, y_{k'})$ can be written in the format $\alpha \vee C^j(y_1, \ldots, y_k)$. So each truth assignment that satisfies $C^j(y_1, \ldots, y_k)$ also satisfies $C^j(y_1, \ldots, y_{k'})$. □

Let $Y = \{y_1, \ldots, y_k\}$ be a set of literals. Another important C-formulas property, to be used in section 5, is related to adding opposite literals, $z, \bar{z} \notin Y$, in $C^j(Y)$, where $C^j(Y)$ denotes $C^j(y_1, \ldots, y_k)$:

Lemma 4. *Let Y be a set of literals, and let z be a literal, such that $z, \bar{z} \notin Y$. Then $C^j(Y) \equiv C^{j+1}(Y \cup \{z, \bar{z}\})$.*

Proof. Expanding the formula $C^{j+1}(Y \cup \{z, \bar{z}\})$ and ruling out the conjunctions that imply $z \wedge \bar{z}$, we have 2 kinds of conjunctions: the ones where j literals are in Y and the remaining literal is in $\{z, \bar{z}\}$, and those with all literals in Y. Or: $C^{j+1}(Y \cup \{z, \bar{z}\}) \equiv z \wedge C^j(Y) \vee \bar{z} \wedge C^j(Y) \vee C^{j+1}(Y) \equiv C^j(Y)$, by Lemma 3a. □

Before we enunciate the next Theorem, we need the following Lemma, where C_k^j denotes $C^j(y_1,\ldots,y_k)$:

Lemma 5. *Let i and k be integers, with $0 \le i < k$. Then*

$$p(y_{k+1} \wedge C_k^i) + p(C_k^{i+1}) = p(C_{k+1}^{i+1}) + p(y_{k+1} \wedge C_k^{i+1}) \ .$$

Proof. Directly from Kolmogorov's axioms, we have:

$$p(y_{k+1} \wedge C_k^i) + p(C_k^{i+1}) = p(y_{k+1} \wedge C_k^i \vee C_k^{i+1}) + p(y_{k+1} \wedge C_k^i \wedge C_k^{i+1}) \ .$$

From Lemma 3, we know that $C_k^{i+1} \models C_k^i$. Then:

$$p(y_{k+1} \wedge C_k^i \wedge C_k^{i+1}) = p(y_{k+1} \wedge C_k^{i+1}) \ .$$

From the C-formulas definition, we note that:

$$y_{k+1} \wedge C_k^i \vee C_k^{i+1} \equiv C_{k+1}^{i+1} \ .$$

Finally, we obtain:

$$p(y_{k+1} \wedge C_k^i) + p(C_k^{i+1}) = p(C_{k+1}^{i+1}) + p(y_{k+1} \wedge C_k^{i+1}) \ .$$

□

Theorem 3. *Let $\{y_1,\ldots,y_k\}$ be a set of literals. Then:*

$$p(y_1) + \ldots + p(y_k) = p(C^1(y_1,\ldots,y_k)) + \ldots + p(C^k(y_1,\ldots,y_k)) \ .$$

Proof. The proof proceeds by induction in k, with C_k^j denoting $C^j(y_1,\ldots,y_k)$:
Induction basis: $k = 1$, $p(y_1) = C^1(y_1)$ trivially.
Induction hypothesis: $k = j \ge 1$, $p(y_1) + \ldots + p(y_j) = p(C_j^1) + \ldots + p(C_j^j)$.
Induction step: $k = j+1$; starting from the induction hypothesis, we sum $p(y_{j+1})$ to both sides of equality:

$$p(y_1) + \ldots + p(y_j) + p(y_{j+1}) = p(C_j^1) + \ldots + p(C_j^j) + p(y_{j+1}) \ .$$

As y_{j+1} is equivalent to $y_{j+1} \wedge C_j^0$, we apply Lemma 5:

$$p(C_j^1) + p(y_{j+1}) = p(C_{j+1}^1) + p(y_{j+1} \wedge C_j^1) \ .$$

So we obtain:

$$p(y_1) + \ldots + p(y_{j+1}) = p(C_{j+1}^1) + p(y_{j+1} \wedge C_j^1) + p(C_j^2) + \ldots + p(C_j^j) \ .$$

In an analogous way, we apply Lemma 5 $j-1$ times, obtaining:

$$p(y_1) + \ldots + p(y_{j+1}) = p(C_{j+1}^1) + \ldots + p(C_{j+1}^j) + p(y_{j+1} \wedge C_j^j) \ .$$

Noting that $y_{j+1} \wedge C_j^j \equiv C_{j+1}^{j+1}$, we finally have:

$$p(y_1) + \ldots + p(y_{j+1}) = p(C^1(y_1,\ldots,y_{j+1})) + \ldots + p(C^{j+1}(y_1,\ldots,y_{j+1})) \ ,$$

as desired. □

Having presented the C-formulas, and with Theorem 3 in hand, we can enunciate the Theorem that finally generalizes the Lemma 2:

Theorem 4. *Let* $\{y_1, \ldots, y_k\}$ *be a set of literals, and let* Ψ *be a probability assignment to these literals. If* $\sum_{i=1}^{k} p(y_i) > j - 1$*, then* $C^j(y_1, \ldots, y_k) \in \Psi^{\approx}$*.*

Proof. In one hand, from Theorem 3, we have:

$$\sum_{i=1}^{k} p(y_i) = \sum_{i=1}^{k} p(C^i(y_1, \ldots, y_k)) > j - 1 \ .$$

As $p(\alpha) \leq 1$ for all formula α, $\sum_{i=1}^{j-1} p(C^i(y_1, \ldots, y_k)) \leq j - 1$, for any $j \leq k$, it follows that $\sum_{i=j}^{k} p(C^i(y_1, \ldots, y_k)) > 0$.

In other hand, from Lemma 3, it follows that $C^k(y_1, \ldots, y_k) \vDash \ldots \vDash C^j(y_1, \ldots, y_k)$, because $j \leq k$, and hence:

$$p(C^j(y_1, \ldots, y_k)) \geq \ldots \geq p(C^k(y_1, \ldots, y_k)) \ ;$$
$$(k - j + 1)p(C^j(y_1, \ldots, y_k)) \geq \textstyle\sum_{i=j}^{k} p(C^i(y_1, \ldots, y_k)) > 0 \ ;$$
$$p(C^j(y_1, \ldots, y_k)) > 0 \ .$$

We conclude that $\Psi \mathrel{|}\!\approx C^j(y_1, \ldots, y_k)$, and therefore $C^j(y_1, \ldots, y_k) \in \Psi^{\approx}$. □

5 A Conjecture and its Refutation

Let (Γ, Ψ) be a PSAT instance. In one hand, from Theorem 2, if $\{\alpha\} \cup \Gamma$ is classically unsatisfiable, for a C-formula $\alpha \in \Psi^{\approx}$, then (Γ, Ψ) is probabilistically unsatisfiable. In other hand, if each formula $\alpha \in \Psi^{\approx}$ were implied by a formula $C^j(y_1, \ldots, y_k)$, such that $\sum_{i=1}^{k} p(y_i) > j - 1$, the probabilistic unsatisfiability of (Γ, Ψ) would yield the classical unsatisfiability of $\{C^j(y_1, \ldots, y_k)\} \cup \Gamma$, for one C-formula in that condition. Having this in mind, we conjecture the following:

Conjecture 1 *If* (Γ, Ψ) *is an unsatisfiable PSAT instance, then there is a C-formula* $C^j(y_1, \ldots, y_k)$*, with* $\sum_{i=1}^{k} p(y_i) > j - 1$*, such that* $\{C^j(y_1, \ldots, y_k)\} \cup \Gamma$ *is classically unsatisfiable.*

Refutation. Our refutation will be built by presenting a counterexample PSAT instance. Let's consider the PSAT instance $\Delta = (\Gamma, \Psi)$, where Γ is a set with 1 formula, from classical propositional logic, over 4 boolean variables $x_1, \ldots, x_4$. To simplify the writing, if α and β are formulas, then $\alpha\beta$ denotes $\alpha \wedge \beta$, and $\bar{\alpha}$ denotes $\neg\alpha$:

$$\Gamma = \{x_1 x_2 x_3 x_4 \vee x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \vee \bar{x}_1 x_2 \bar{x}_3 \bar{x}_4 \vee \bar{x}_1 \bar{x}_2 x_3 \bar{x}_4 \vee x_1 \bar{x}_2 x_3 x_4\} \ .$$

And Ψ is the following probability assignment to the boolean variables:

$$\Psi = \{p(x_1) = 0.47, \quad p(x_2) = 0.40, \quad p(x_3) = 0.46 \quad \text{and} \quad p(x_4) = 0.05\} \ .$$

It follows that, from Kolmogorov's axioms:

$$p(\bar{x}_1) = 0.53, \quad p(\bar{x}_2) = 0.60, \quad p(\bar{x}_3) = 0.54 \quad \text{and} \quad p(\bar{x}_4) = 0.95 \ .$$

Let $v_k : \{x_1, \ldots, x_4\} \rightarrow \{0, 1\}$, $1 \leq k \leq 5$, be the only truth assignments to satisfy Γ, such that v_1 satisfies $x_1x_2x_3x_4$, v_2 satisfies $x_1\bar{x}_2\bar{x}_3\bar{x}_4$, v_3 satisfies $\bar{x}_1x_2\bar{x}_3\bar{x}_4$, v_4 satisfies $\bar{x}_1\bar{x}_2x_3\bar{x}_4$, and v_5 satisfies $x_1\bar{x}_2x_3x_4$.

To note the unsatisfiability of $\Delta = (\Gamma, \Psi)$, we will show a formula $\alpha \in \Psi^{\approx}$, such that $\{\alpha\} \cup \Gamma$ is unsatisfiable. As $p(x_1) + p(x_2) + p(x_3) = 1.33$, by Theorem 3 we have $p(C^1(x_1, x_2, x_3)) + p(C^2(x_1, x_2, x_3)) + p(C^3(x_1, x_2, x_3)) = 1.33$. With $p(C^1(x_1, x_2, x_3)) \leq 1$ and, by Lemma 3a, $p(C^2(x_1, x_2, x_3)) \geq p(C^3(x_1, x_2, x_3))$, it follows $p(C^2(x_1, x_2, x_3)) \geq 0.165$. Thus $p(\bar{x}_4) + p(C^2(x_1, x_2, x_3)) > 1$ and $p(\bar{x}_4 \wedge C^2(x_1, x_2, x_3)) > 0$. Let α be $\bar{x}_4 \wedge C^2(x_1, x_2, x_3)$. We note that $\alpha \in \Psi^{\approx}$ and $\{\alpha\} \cup \Gamma$ is unsatisfiable. Then, by Theorem 2, Δ is probabilistically unsatisfiable.

Now we have to exhaustively show that, for each formula $C^j(y_1, \ldots, y_k)$ with $\sum_{i=1}^{k} p(y_i) > j - 1$, $\{C^j(y_1, \ldots, y_k)\} \cup \Gamma$ is satisfiable. Remembering Lemma 3a, being Y a set of literals, if $0 \leq j' < j$, then $C^j(Y) \vDash C^{j'}(Y)$. For each set of literals $Y = \{y_1, \ldots, y_k\}$, we define $j_{max}(Y) = \lceil \sum_{i=1}^{k} p(y_i) \rceil$ and denote $C^{j_{max}(Y)}(Y)$ by $C^{j_{max}}(Y)$. So if $0 \leq j < j_{max}(Y)$, then $C^{j_{max}}(Y) \vDash C^j(Y)$, and if $j_{max}(Y) < j$, then $\sum_{i=1}^{k} p(y_i) \leq j - 1$. Thus for each set of literals Y, it's enough to verify the satisfiability of $\{C^{j_{max}}(Y)\} \cup \Gamma$.

With 4 variables, we have 8 different literals, that yield $2^8 = 256$ possible sets of literals to be checked. We easily note that the empty set doesn't need to be verified, because $\sum_{y \in \emptyset} p(y) = 0 = j_{max}(\emptyset)$, and $C^0(Y) = TRUE$ is satisfied by any truth assignment. Considering now sets with one literal, whose $j_{max} = 1$, note that each of the 8 literals is true either in v_3, or in v_5, so we don't need to check theses sets also. Furthermore, if $z, \bar{z} \notin Y$, then $j_{max}(Y \cup \{z, \bar{z}\}) = \lceil p(z) + p(\bar{z}) + \sum_{y \in Y} p(y) \rceil = j_{max}(Y) + 1$. So, by Lemma 4, $C^{j_{max}}(Y \cup \{z, \bar{z}\}) \equiv C^{j_{max}}(Y)$, thus we don't need to check sets with 2 opposite literals. The 3 tables below show the 72 remaining possible sets of literals over $\{x_1, \ldots, x_4\}$, organized by the set length. Each row presents a set Y of literals, the sum of these literals probabilities, $\sum_{y \in Y} p(y)$, the j_{max} defined by this set and the truth assignment that satisfies $\{C^{j_{max}}(Y)\} \cup \Gamma$. Each truth assignment is represented by the conjunction it is the only one to satisfy.

And so we finish the refutation, with an unsatisfiable PSAT instance, (Γ, Ψ), where for each formula $C^j(Y)$, such that $\sum_{y \in Y} p(y) > j - 1$, we have shown a truth assignment that satisfies $\{C^j(Y)\} \cup \Gamma$. □

Note that Theorems 3 and 4 also hold for formulas, not only for atoms. The formula α presented as "witness" for the unsatisfiability of PSAT instance (Γ, Ψ), from Conjecture 1 refutation, could be written as a C-formula of formulas: $\alpha \equiv \neg x_4 \wedge C^2(x_1, x_2, x_3) \equiv C^2(\neg x_4, C^2(x_1, x_2, x_3))$. As $p(\neg x_4) + (C^2(x_1, x_2, x_3)) > 1$, by Theorem 4, $\Psi \approx C^2(\neg x_4, C^2(x_1, x_2, x_3))$, but this C-formula is inconsistent with Γ, as we have shown.

Table 1. Sets with 4 literals

set of literals Y	$\sum_{y\in Y} p(y)$	j_{max}	v_i	set of literals Y	$\sum_{y\in Y} p(y)$	j_{max}	v_i
$\{\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4\}$	2.62	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_1,\bar{x}_2,\bar{x}_3,x_4\}$	1.72	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1,\bar{x}_2,x_3,\bar{x}_4\}$	2.54	3	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$	$\{\bar{x}_1,\bar{x}_2,x_3,x_4\}$	1.64	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_1,x_2,\bar{x}_3,\bar{x}_4\}$	2.42	3	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_1,x_2,\bar{x}_3,x_4\}$	1.52	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1,x_2,x_3,\bar{x}_4\}$	2.34	3	$\bar{x}_1x_2\bar{x}_3\bar{x}4$	$\{\bar{x}_1,x_2,x_3,x_4\}$	1.44	2	$x_1x_2x_3x_4$
$\{x_1,\bar{x}_2,\bar{x}_3,\bar{x}_4\}$	2.56	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_1,\bar{x}_2,\bar{x}_3,x_4\}$	1.66	2	$x_1x_2x_3x_4$
$\{x_1,\bar{x}_2,x_3,\bar{x}_4\}$	2.48	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_1,\bar{x}_2,x_3,x_4\}$	1.58	2	$x_1x_2x_3x_4$
$\{x_1,x_2,\bar{x}_3,\bar{x}_4\}$	2.36	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_1,x_2,\bar{x}_3,x_4\}$	1.46	2	$x_1x_2x_3x_4$
$\{x_1,x_2,x_3,\bar{x}_4\}$	2.28	3	$x_1x_2x_3x_4$	$\{x_1,x_2,x_3,x_4\}$	1.38	2	$x_1x_2x_3x_4$

Table 2. Sets with 3 literals

set of literals Y	$\sum_{y\in Y} p(y)$	j_{max}	v_i	set of literals Y	$\sum_{y\in Y} p(y)$	j_{max}	v_i
$\{\bar{x}_1,\bar{x}_2,\bar{x}_3\}$	1.67	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_1,\bar{x}_2,x_3\}$	1.59	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_1,x_2,\bar{x}_3\}$	1.47	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_1,x_2,x_3\}$	1.39	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{x_1,\bar{x}_2,\bar{x}_3\}$	1.61	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_1,\bar{x}_2,x_3\}$	1.53	2	$x_1\bar{x}_2x_3x_4$
$\{x_1,x_2,\bar{x}_3\}$	1.41	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_1,x_2,x_3\}$	1.33	2	$x_1x_2x_3x_4$
$\{\bar{x}_1,\bar{x}_2,\bar{x}_4\}$	2.08	3	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$	$\{\bar{x}_1,\bar{x}_2,x_4\}$	1.18	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_1,x_2,\bar{x}_4\}$	1.88	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_1,x_2,x_4\}$	0.98	1	$x_1x_2x_3x_4$
$\{x_1,\bar{x}_2,\bar{x}_4\}$	2.02	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_1,\bar{x}_2,x_4\}$	1.12	2	$x_1x_2x_3x_4$
$\{x_1,x_2,\bar{x}_4\}$	1.82	2	$x_1x_2x_3x_4$	$\{x_1,x_2,x_4\}$	0.92	1	$x_1x_2x_3x_4$
$\{\bar{x}_1,\bar{x}_3,\bar{x}_4\}$	2.02	3	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_1,\bar{x}_3,x_4\}$	1.12	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1,x_3,\bar{x}_4\}$	1.94	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$	$\{\bar{x}_1,x_3,x_4\}$	1.04	2	$x_1x_2x_3x_4$
$\{x_1,\bar{x}_3,\bar{x}_4\}$	1.96	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_1,\bar{x}_3,x_4\}$	1.06	2	$x_1x_2x_3x_4$
$\{x_1,x_3,\bar{x}_4\}$	1.88	2	$x_1x_2x_3x_4$	$\{x_1,x_3,x_4\}$	0.98	1	$x_1x_2x_3x_4$
$\{\bar{x}_2,\bar{x}_3,\bar{x}_4\}$	2.09	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_2,\bar{x}_3,x_4\}$	1.19	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_2,x_3,\bar{x}_4\}$	2.01	3	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$	$\{\bar{x}_2,x_3,x_4\}$	1.11	2	$x_1x_2x_3x_4$
$\{x_2,\bar{x}_3,\bar{x}_4\}$	1.89	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_2,\bar{x}_3,x_4\}$	0.99	1	$x_1x_2x_3x_4$
$\{x_2,x_3,\bar{x}_4\}$	1.81	2	$x_1x_2x_3x_4$	$\{x_2,x_3,x_4\}$	0.91	1	$x_1x_2x_3x_4$

Table 3. Sets with 2 literals

set of literals Y	$\sum_{y\in Y} p(y)$	j_{max}	v_i	set of literals Y	$\sum_{y\in Y} p(y)$	j_{max}	v_i
$\{\bar{x}_1,\bar{x}_2\}$	1.13	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$	$\{\bar{x}_1,x_2\}$	0.93	1	$x_1x_2x_3x_4$
$\{x_1,\bar{x}_2\}$	1.07	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_1,x_2\}$	0.87	1	$x_1x_2x_3x_4$
$\{\bar{x}_1,\bar{x}_3\}$	1.07	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_1,x_3\}$	0.99	1	$x_1x_2x_3x_4$
$\{x_1,\bar{x}_3\}$	1.01	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_1,x_3\}$	0.93	1	$x_1x_2x_3x_4$
$\{\bar{x}_1,\bar{x}_4\}$	1.48	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_1,x_4\}$	0.58	1	$x_1x_2x_3x_4$
$\{x_1,\bar{x}_4\}$	1.42	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{x_1,x_4\}$	0.52	1	$x_1x_2x_3x_4$
$\{\bar{x}_2,\bar{x}_4\}$	1.55	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_2,x_4\}$	0.65	1	$x_1x_2x_3x_4$
$\{x_2,\bar{x}_4\}$	1.35	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$	$\{x_2,x_4\}$	0.45	1	$x_1x_2x_3x_4$
$\{\bar{x}_2,\bar{x}_3\}$	1.14	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_2,x_3\}$	1.06	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{x_2,\bar{x}_3\}$	0.94	1	$x_1x_2x_3x_4$	$\{x_2,x_3\}$	0.86	1	$x_1x_2x_3x_4$
$\{\bar{x}_3,\bar{x}_4\}$	1.49	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$	$\{\bar{x}_3,x_4\}$	0.59	1	$x_1x_2x_3x_4$
$\{x_3,\bar{x}_4\}$	1.41	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$	$\{x_3,x_4\}$	0.51	1	$x_1x_2x_3x_4$

6 Conclusion

In this paper, we investigated the relation between probabilistic satisfiability and classical satisfiability. For this, we presented the Atomic Normal Form and defined a Probabilistic Entailment relation ($\approx$). We defined C-formulas, which have shown ubiquity in the probabilistic satisfiability study. Finally, we conjectured the completeness of only looking at C-formulas probabilistically entailed to decide the satisfiability of a PSAT instance, and we refuted it with a counterexample.

The exponential number of C-formulas to investigate doesn't allow its exhaustive use in a possible polynomial reduction from PSAT to SAT. However, the founded results seem to be useful on PSAT study, bringing it back to logic. The Atomic Normal Form might be useful to standardize PSAT instances, in order to compare numeric outputs from algorithms that solve the problem. The introduced probabilistic entailment relation enables the presentation of a "witness" formula for the unsatisfiability of a PSAT instance, which can be used in proving probabilistic unsatisfiability, using classical unsatisfiability.

As the efficiency of algorithms that solve PSAT is considerably lower than those from algorithms for another NP-complete problems (like SAT), we believe there is a lot of work to be done. A possible approach would be the polynomial reduction to SAT, using the concepts we presented here together with linear algebra techniques to explore PSAT, as it can be seen as a linear programming problem.

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