



Concentration Inequalities and Laws of Large Numbers under Epistemic Irrelevance <sup>1</sup>

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# Concentration Inequalities and Laws of Large Numbers under Epistemic Irrelevance

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## Abstract

This paper presents concentration inequalities and laws of large numbers under weak assumptions of irrelevance, expressed through lower and upper expectations. The results are variants and extensions of De Cooman and Miranda's recent inequalities and laws of large numbers. The proofs indicate connections between concepts of irrelevance for lower/upper expectations and the standard theory of martingales.

## 1 Introduction

This paper investigates concentration inequalities and laws of large numbers under weak assumptions of “irrelevance” that are expressed using lower and upper expectations. The starting point is the assumption that, given bounded variables  $X_1, \dots, X_n$ , we have:

$$\text{for each } i \in [2, n], \text{ variables } X_1, \dots, X_{i-1} \quad (1) \\ \text{are epistemically irrelevant to } X_i.$$

Epistemic irrelevance of variables  $X_1, \dots, X_{i-1}$  to  $X_i$  obtains when [26, Def. 9.2.1]

$$\bar{E}[f(X_i)|A(X_{1:i-1})] = \bar{E}[f(X_i)] \quad (2)$$

for any bounded function  $f$  of  $X_i$  and any nonempty event  $A(X_{1:i-1})$  defined by variables  $X_{1:i-1}$ , where the functional  $\bar{E}$  is an *upper expectation* (Section 2). Here and in the remainder of the paper we simplify notation by using  $X_{1:i}$  for  $X_1, \dots, X_i$ .

A judgement of epistemic irrelevance can be interpreted as a relaxed judgement of stochastic independence, perhaps motivated by a robustness analysis or by disagreements amongst a set of decision makers. Alternatively, one might consider epistemic irrelevance as *the* appropriate concept of independence when expectations are not known precisely.

De Cooman and Miranda have recently proven a number of inequalities and laws of large numbers that also deal with judgements of irrelevance expressed through

lower/upper expectations [5]. De Cooman and Miranda's weak law of large numbers implies that, given Assumption (1), for any  $\epsilon > 0$ ,

$$P\left(\underline{\mu}_n - \epsilon \leq \frac{\sum_{i=1}^n X_i}{n} \leq \bar{\mu}_n + \epsilon\right) \geq 1 - 2e^{-\frac{n\epsilon^2/4}{(\max_i B_i)^2}},$$

where  $B_i$  is such that  $|X_i| \leq B_i$ , and

$$\underline{\mu}_n \doteq \frac{\sum_{i=1}^n \underline{E}[X_i]}{n}, \quad \bar{\mu}_n \doteq \frac{\sum_{i=1}^n \bar{E}[X_i]}{n}.$$

Moreover, De Cooman and Miranda's results and Assumption (1) imply a two-part strong law of large numbers: for any  $\epsilon > 0$ , there is  $N \in \mathbf{N}_+$  such that for any  $N' \in \mathbf{N}_+$ ,

$$\bar{P}\left(\exists n \in [N, N + N'] : \frac{\sum_{i=1}^n X_i}{n} \geq \bar{\mu} + \epsilon\right) < \epsilon, \\ \bar{P}\left(\exists n \in [N, N + N'] : \frac{\sum_{i=1}^n X_i}{n} \leq \underline{\mu} - \epsilon\right) < \epsilon.$$

This law of large numbers corresponds to a finitary version of the usual strong law of large numbers [9]; the focus on a finitary law is justified by the fact that De Cooman and Miranda do not assume countable additivity. If countable additivity holds, the finitary strong law of large numbers implies convergence of empirical means with probability one [5, Sec. 5.3].

To obtain their results, De Cooman and Miranda assume, following Walley's theory of lower previsions, that all variables are bounded, and that conglomerability (and consequently disintegrability) holds. These assumptions are discussed in more detail later.

The present paper derives laws of large numbers by exploiting concentration and martingale inequalities that are adapted to the setting of lower/upper expectations. These results use either Assumption (1) or the weaker assumption that, for each  $i \in [2, n]$  and any nonempty event  $A(X_{1:i-1})$ ,

$$\begin{aligned} E[X_i|A(X_{1:i-1})] &= E[X_i] \\ \text{and} \\ \bar{E}[X_i|A(X_{1:i-1})] &= \bar{E}[X_i]. \end{aligned} \quad (3)$$

Several results for *bounded* variables presented in this paper are basically implied by De Cooman and Miranda's work. Regarding bounded variables our contribution lies in offering tighter inequalities and alternative proof techniques that are more closely related to established methods in standard probability theory (in particular, close to Hoeffding's and Azuma's inequalities). In Section 4 we offer more significant contributions as we lift the assumption of boundedness for variables, and use martingale theory to prove laws of large numbers under elementwise disintegrability.

## 2 Expectations, disintegrability, and zero probabilities

In this section we present notation and terminology. Throughout the paper we assume that an expectation functional  $E$  maps bounded variables into real numbers, and satisfies:

(1) if  $\alpha \leq X \leq \beta$ , then  $\alpha \leq E[X] \leq \beta$ ;

(2)  $E[X + Y] = E[X] + E[Y]$ ;

where  $X, Y$  are bounded variables and  $\alpha, \beta$  are real numbers (inequalities are understood pointwise).

From such an expectation functional, a *finitely additive* probability measure  $P$  is induced by  $P(A) \doteq E[A]$  for any event  $A$ ; note that  $A$  denotes both the event and its indicator function.<sup>1</sup>

Given a set of expectation functionals, the lower and upper expectations of variable  $X$  are respectively

$$\underline{E}[X] = \inf E[X], \quad \overline{E}[X] = \sup E[X].$$

Lower and upper probabilities are defined similarly using indicator functions. Given an event  $A$ , a conditional expectation functional is constrained by  $E[X|A]P(A) = E[XA]$ . If we have a set of expectation functionals, then a set of conditional expectation functionals given an event  $A$  is produced by elementwise conditioning on event  $A$  (that is, each expectation functional is conditioned on  $A$ ).

### 2.1 Disintegrability and factorization

We will employ an assumption of *disintegrability* in our proofs; namely,

$$\overline{E}[W] \leq \overline{E}[\overline{E}[W|Z]] \quad (4)$$

for any  $W \geq 0, Z \geq 0$  of interest, where  $W$  and  $Z$  may stand for sets of (non-negative) variables. Note that disintegrability can fail for a single finitely additive

<sup>1</sup>A probability measure defined on a field completely characterizes an expectation functional on bounded functions that are measurable with respect to the field and vice-versa [26, Theorem 3.2.2].

probability measure over an infinite space [6, 10]; that is, there is a finitely additive probability measure  $P$  such that

$$E_P[W] > E_P[E_P[W|Z]].$$

One way to obtain disintegrability is to restrict attention to simple variables; that is, variables that take on finitely many distinct values. In particular, indicator functions are simple variables; hence simple variables suffice to express convergence of relative frequencies, and our results apply then.

Another way to obtain disintegrability for every probability measure  $P$  is to adopt countable additivity [1]. That is, assume that if

$$A_1 \supset A_2 \supset \dots$$

is a countable sequence of events, then

$$\bigcap_i A_i = \emptyset \quad \text{implies} \quad \lim_{n \rightarrow \infty} \overline{P}(A_n) = 0. \quad (5)$$

This assumption says that if  $\bigcap_i A_i = \emptyset$ , then  $\lim_{n \rightarrow \infty} P(A_n) = 0$  for every possible probability measure.

A third way to obtain disintegrability is simply to impose it. One may consider disintegrability a ‘‘rationality’’ requirement.

- The theories of coherent behavior by Heath and Sudderth [14] and by Lane and Sudderth [19] follow this path by axiomatizing the *strategic* measures of Dubins and Savage [11], and thus prescribing probability measures that disintegrate appropriately along some predefined partitions. This would be sufficient for our purposes, but there are limitations in the approach as summarized by Kadane et al [16]. The disintegrability of strategic measures has actually been used to prove various laws of large numbers in a finitely additive setting [17].
- Another scheme that imposes disintegrability is Walley's theory of lower previsions; in that theory, Expression (4) is a consequence of axioms for ‘‘coherent’’ behavior. This is the path adopted by De Cooman and Miranda, who consequently have Expression (4) at their disposal.

When disintegrability holds, recursive application of Expression (4) yields: if  $f_i(X_i) \geq 0$  for  $i \in \{1, \dots, n\}$ , then

$$\begin{aligned} & \overline{E} \left[ \prod_{i=1}^n f_i(X_i) \right] \\ & \leq \overline{E} \left[ \dots \overline{E} \left[ \overline{E} \left[ \prod_{i=1}^n f_i(X_i) | X_{1:n-1} \right] | X_{1:n-2} \right] \dots \right]; \end{aligned}$$

Assumption (1) then implies an inequality we use later: for bounded and nonnegative functions,

$$\overline{E} \left[ \prod_{i=1}^n f_i(X_i) \right] \leq \prod_{i=1}^n \overline{E}[f_i(X_i)]. \quad (6)$$

## 2.2 Zero probabilities, full conditional measures and weak irrelevance

It should be noted that the definition of epistemic irrelevance (Expression (2)) does not contain any clause concerning zero probabilities. Indeed, Walley's theory of lower previsions follows de Finetti in adopting *full conditional measures*, and in this setting Expression (2) can be imposed without concerns about zero probabilities. Recall that a full conditional measure  $P : \mathcal{B} \times (\mathcal{B} \setminus \emptyset) \rightarrow \mathfrak{R}$ , where  $\mathcal{B}$  is a Boolean algebra, is a set-function that for every nonempty event  $C$  satisfies [10, 18]:

- (1)  $P(C|C) = 1$ ;
- (2)  $P(A|C) \geq 0$  for all  $A$ ;
- (3)  $P(A \cup B|C) = P(A|C) + P(B|C)$  for all disjoint  $A$  and  $B$ ;
- (4)  $P(A \cap B|C) = P(A|B \cap C)P(B|C)$  for all  $A$  and  $B$  such that  $B \cap C \neq \emptyset$ .

Full conditional measures are not adopted in the usual Kolmogorovian theory, and if countable additivity is adopted and conditioning is defined through Radon-Nykodym derivatives, it may be impossible to satisfy the axioms for full conditional measures [23, 24]. Thus there are some differences between epistemic irrelevance (at least as defined by Walley) and the usual Kolmogorovian set-up, besides the obvious set-valued/point-valued distinction.

Suppose that one wishes to deal with sets of probability measures and associated lower/upper expectations, but chooses to adopt the Kolmogorovian set-up for each measure. That is, each measure satisfies countable additivity and thus disintegrability, and conditioning is left undefined when the conditioning event has probability zero. It might seem reasonable to amend Expression (2) as follows:

$$\overline{E}[f(X_i)|A(X_{1:i-1})] = \overline{E}[f(X_i)] \quad (7) \\ \text{if } \underline{P}(A(X_{1:i-1})) > 0.$$

This condition is a natural for theories that do not define conditioning on events of lower probability zero, such as Giron and Rios' theory [13]. Alas, this weaker condition is really too weak to produce laws of large numbers, as the following example shows.

**Example 1** Suppose  $X_1, X_2, \dots$  assume values in  $\{0, 1, 2\}$ , and

$$\underline{P}(X_i = x) = 0,$$

$$\overline{P}(X_i = x) = 1/2$$

for  $x \in \{0, 1, 2\}$ . Consequently,  $E[X_i] \in [1/2, 3/2]$ . Suppose additionally that

$$P(X_i = x|X_{i-1} = x, X_{1:i-2}) = 1$$

for  $x \in \{0, 1, 2\}$ ; that is, the  $i$ th variable reproduces the value of the  $(i-1)$ th variable. They are obviously dependent variables. However, all events have lower probability zero, so variables  $X_{1:i-1}$  would be irrelevant to  $X_i$  by Expression (7).

In this example, Expression (6) fails. For instance,

$$\overline{E}[X_1 X_2] = 5/2 > 9/4 = (3/2)(3/2) = \overline{E}[X_1] \overline{E}[X_2].$$

Moreover the example illustrates a failure of any sensible law of large numbers, as for any  $\epsilon > 0$ ,

$$P\left(1/2 - \epsilon \leq \frac{\sum_{i=1}^n X_i}{n} \leq 3/2 + \epsilon\right) \in [0, 1/2],$$

because the inequality inside the probability is only satisfied when  $\{X_1 = 1\}$  obtains.

We might thus consider an alternative to Expression (7):

$$\overline{E}[f(X_i)|A(X_{1:i-1})] = \overline{E}[f(X_i)] \quad (8) \\ \text{if } \overline{P}(A(X_{1:i-1})) > 0.$$

The concept of irrelevance conveyed by Expression (8) does lead to Expression (6). To see this, note that for nonnegative  $X$  and  $Y$ , we have

$$\overline{E}[XY] \leq \sup_P E_P[\overline{E}[XY|Y]] \\ = \sup_P E_P[A \overline{E}[XY|Y] + A^c \overline{E}[XY|Y]],$$

using disintegrability and defining  $A$  as the set of all values of  $Y$  such that  $\overline{P}(A^c) = 0$ . Hence  $P(A^c) = 0$  for every  $P$  and using Expression (8):

$$\overline{E}[XY] \leq \sup_P E_P[AY \overline{E}[X|Y]] \\ = \sup_P E_P[AY \overline{E}[X]] \\ = \sup_P E_P[AY] \overline{E}[X] \\ = \overline{E}[X] \sup_P E_P[Y] \\ = \overline{E}[X] \overline{E}[Y].$$

[As a digression, note that one might *define* conditional expectations as  $\underline{E}[X|A] = \inf_{P:P(A)>0} E_P[X|A]$  and  $\overline{E}[X|A] = \sup_{P:P(A)>0} E_P[X|A]$ . This form of conditioning has been advocated by several authors [27, 28], and it is quite similar to Walley's concept of regular

extension [26, Ap. J]. For such a form of conditioning, Expression (8) seems to be the natural definition of irrelevance.]

In short, more than one combination of definitions and assumptions lead to the results presented in the remainder of this paper. For instance, Expression (6) obtains when Assumption (1) holds *and* disintegrability holds (because all variables are simple, *or* because countable additivity is assumed, *or* because disintegrability is imposed). Alternatively, Expression (6) obtains when Expression (8) holds for any  $i \in [2, n]$ , any bounded function  $f$  of  $X_i$ , and any event  $A(X_{1:i-1})$ , and additionally disintegrability holds.

Similar remarks concerning zero probabilities can be directed at Assumption (3). We say that *weak irrelevance* obtains when either:

- For any  $i \in [2, n]$  and any nonempty event  $A(X_{1:i-1})$ ,

$$\begin{aligned} \underline{E}[X_i | A(X_{1:i-1})] &= \underline{E}[X_i] \\ &\text{and} \\ \overline{E}[X_i | A(X_{1:i-1})] &= \overline{E}[X_i] \end{aligned}$$

[this is Assumption (3), and it requires full conditional measures].

or:

- For any  $i \in [2, n]$  and any event  $A(X_{1:i-1})$ ,

$$\begin{aligned} \underline{E}[X_i | A(X_{1:i-1})] &= \underline{E}[X_i] \text{ if } \overline{P}(A(X_{1:i-1})) > 0 \\ &\text{and} \\ \overline{E}[X_i | A(X_{1:i-1})] &= \overline{E}[X_i] \text{ if } \overline{P}(A(X_{1:i-1})) > 0. \end{aligned}$$

### 3 Bounded variables

Take variables  $X_1, \dots, X_n$  such that  $|X_i| \leq B_i$  and define

$$\gamma_n \doteq \sum_{i=1}^n B_i^2 > 0.$$

We start by deriving two concentration inequalities.

#### 3.1 Concentration inequalities

The following inequality is a counterpart of Hoeffding inequality [8, 15] in the context of lower/upper expectations; it is slightly tighter than similar inequalities by De Cooman and Miranda [5]. It is interesting to note that the proof is remarkably similar to the proof of the original Hoeffding inequality.

**Theorem 1** *If bounded variables  $X_1, \dots, X_n$  satisfy Expression (6), then if  $\gamma_n > 0$ ,*

$$\overline{P}\left(\sum_{i=1}^n (X_i - \overline{E}[X_i]) \geq \epsilon\right) \leq e^{-2\epsilon^2/\gamma_n},$$

$$\overline{P}\left(\sum_{i=1}^n (X_i - \overline{E}[X_i]) \leq -\epsilon\right) \leq e^{-2\epsilon^2/\gamma_n}.$$

*Proof.* By Markov inequality, if  $X \geq 0$ , then for any  $\epsilon > 0$  we have  $P(X \geq \epsilon) \leq E[X]/\epsilon$ . Consequently, for  $s > 0$ , any variable  $X$  satisfies

$$\overline{P}(X \geq \epsilon) = \overline{P}(e^{sX} \geq e^{s\epsilon}) \leq e^{-s\epsilon} \overline{E}[\exp(sX)].$$

Using this inequality and Expression (6):

$$\begin{aligned} \overline{P}\left(\sum_{i=1}^n (X_i - \overline{E}[X_i]) \geq \epsilon\right) &\leq e^{-s\epsilon} \overline{E}\left[\exp\left(\sum_{i=1}^n s(X_i - \overline{E}[X_i])\right)\right] \\ &\leq e^{-s\epsilon} \prod_{i=1}^n \overline{E}[\exp(s(X_i - \overline{E}[X_i]))]. \end{aligned}$$

We now use Hoeffding's result (Expression (11)) that if variable  $X$  satisfies  $a \leq X \leq b$  and  $E[X] \leq 0$ , then  $E[\exp(sX)] \leq \exp(s^2(b-a)^2/8)$  for any  $s > 0$ . Thus for any  $P$ ,  $E_P[\exp(s(X_i - \overline{E}[X_i]))] \leq \exp(s^2 B_i^2/8)$ , and then  $\overline{E}[\exp(s(X_i - \overline{E}[X_i]))] \leq \exp(s^2 B_i^2/8)$ . Consequently,

$$\overline{P}\left(\sum_{i=1}^n (X_i - \overline{E}[X_i]) \geq \epsilon\right) \leq e^{-s\epsilon} e^{s^2 \gamma_n/8} \leq e^{-2\epsilon^2/\gamma_n},$$

where the last inequality is obtained by taking  $s = 4\epsilon/\gamma_n$ . This proves the first inequality in the theorem; the second inequality is proved by taking  $\overline{P}(\sum_{i=1}^n ((-X_i) - \overline{E}[-X_i]) \geq \epsilon)$  and noting that  $\underline{E}[X_i] = -\overline{E}[-X_i]$ .  $\square$

We now move to weak irrelevance and obtain an analogue of Azuma's inequality [2, 7]. It is again interesting to note that the proof is remarkably similar to the proof of the original Azuma inequality. De Cooman and Miranda [5, Sec. 4.1] show that their inequalities are valid under weak irrelevance; the next inequality is slightly tighter than theirs.

**Theorem 2** *If bounded variables  $X_1, \dots, X_n$  satisfy weak irrelevance and disintegrability (Expression (4)) holds, then if  $\gamma_n > 0$ ,*

$$\overline{P}\left(\sum_{i=1}^n (X_i - \overline{E}[X_i]) \geq \epsilon\right) \leq e^{-2\epsilon^2/\gamma_n},$$

$$\bar{P}\left(\sum_{i=1}^n (X_i - \underline{E}[X_i]) \leq -\epsilon\right) \leq e^{-2\epsilon^2/\gamma_n}.$$

*Proof.* Using both Markov's inequality (as in the proof of Theorem 1) and disintegrability, for any  $s > 0$  we get

$$\begin{aligned} \bar{P}\left(\sum_{i=1}^n (X_i - \bar{E}[X_i]) \geq \epsilon\right) &\leq e^{-s\epsilon} \bar{E}\left[\exp\left(\sum_{i=1}^n s(X_i - \bar{E}[X_i])\right)\right] \\ &\leq e^{-s\epsilon} \bar{E}\left[\bar{E}\left[\exp\left(\sum_{i=1}^n s(X_i - \bar{E}[X_i])\right) \mid X_{1:n-1}\right]\right] \\ &\leq e^{-s\epsilon} \bar{E}\left[\exp\left(\sum_{i=1}^{n-1} s(X_i - \bar{E}[X_i])\right) h(X_{1:n-1})\right], \end{aligned}$$

where

$$h(X_{1:n-1}) = \bar{E}[\exp(s(X_n - \bar{E}[X_n])) \mid X_{1:n-1}].$$

Due to weak irrelevance,

$$E_P[X_n \mid X_{1:n-1}] \leq \bar{E}[X_n \mid X_{1:n-1}] = \bar{E}[X_n];$$

consequently, for any  $P$ ,

$$E_P[X_n - \bar{E}[X_n] \mid X_{1:n-1}] \leq 0.$$

We now use Hoeffding's result (Expression (11)) that if variable  $X$  satisfies  $a \leq X \leq b$  and  $E[X] \leq 0$ , then  $E[\exp(sX)] \leq \exp(s^2(b-a)^2/8)$  for any  $s > 0$ . Thus for any  $P$  we have

$$E_P[\exp(s(X_n - \bar{E}[X_n])) \mid X_{1:n-1}] \leq \exp(s^2 B_n^2/8)$$

and then  $h(X_{1:n-1}) \leq \exp(s^2 B_n^2/8)$ . Thus

$$\begin{aligned} \bar{P}\left(\sum_{i=1}^n (X_i - \bar{E}[X_i]) \geq \epsilon\right) &\leq e^{-s\epsilon} \bar{E}\left[\exp\left(\sum_{i=1}^n s(X_i - \bar{E}[X_i])\right)\right] \\ &\leq e^{-s\epsilon} \bar{E}\left[\exp\left(\sum_{i=1}^{n-1} s(X_i - \bar{E}[X_i])\right) \exp(s^2 B_n^2/8)\right] \\ &\leq e^{-s\epsilon} \exp(s^2 B_n^2/8) \bar{E}\left[\exp\left(\sum_{i=1}^{n-1} s(X_i - \bar{E}[X_i])\right)\right]. \end{aligned}$$

These inequalities can be iterated to produce:

$$\bar{P}\left(\sum_{i=1}^n (X_i - \bar{E}[X_i]) \geq \epsilon\right) \leq e^{-s\epsilon} \exp\left(s^2 \sum_{i=1}^n B_i^2/8\right).$$

Finally, by taking  $s = 4\epsilon/\gamma_n$ ,

$$\bar{P}\left(\sum_{i=1}^n (X_i - \bar{E}[X_i]) \geq \epsilon\right) \leq e^{-2\epsilon^2/\gamma_n}.$$

The second inequality in the theorem is proved by noting that weak irrelevance of  $X_1, \dots, X_n$  implies weak irrelevance of  $-X_1, \dots, -X_n$  (as  $\underline{E}[X_i] = -\bar{E}[-X_i]$ ), and then by taking  $\bar{P}(\sum_{i=1}^n ((-X_i) - \bar{E}[-X_i]) \geq \epsilon)$ .  $\square$

### 3.2 Laws of large numbers

Theorem 1 leads to simple proofs of laws of large numbers already stated by De Cooman and Miranda [5]. To start, take Assumption (1). Using subadditivity of upper probability and Theorem 1,

$$\bar{P}\left(\left(\sum_{i=1}^n X_i \geq n\bar{\mu}_n + \epsilon\right) \cup \left(\sum_{i=1}^n X_i \leq n\bar{\mu}_n - \epsilon\right)\right) \leq 2e^{-\frac{2\epsilon^2}{\gamma_n}},$$

where as before,  $\bar{\mu}_n \doteq (1/n) \sum_{i=1}^n \underline{E}[X_i]$  and  $\bar{\mu}_n \doteq (1/n) \sum_{i=1}^n \bar{E}[X_i]$ . By noting that  $\underline{P}(A) = 1 - \bar{P}(A^c)$  for any event  $A$ , by including the endpoints of relevant inequalities, and by using  $n\epsilon$  instead of  $\epsilon$ :

$$\begin{aligned} \underline{P}\left(\bar{\mu}_n - \epsilon \leq \frac{\sum_{i=1}^n X_i}{n} \leq \bar{\mu}_n + \epsilon\right) &\geq \\ \underline{P}\left(\bar{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \bar{\mu}_n + \epsilon\right) &\geq 1 - 2e^{-\frac{2n\epsilon^2}{B^2}}, \end{aligned}$$

where we define  $B \doteq \max_i B_i$ . By taking limits, we obtain a weak law of large numbers:

$$\lim_{n \rightarrow \infty} \underline{P}\left(\bar{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \bar{\mu}_n + \epsilon\right) = 1.$$

An analogue of De Cooman and Miranda's finitary strong law of large numbers can be deduced as well from the previous inequalities, as follows. Here and in the remainder of the paper,  $n$ ,  $N$  and  $N'$  denote positive integers. For all  $\epsilon > 0$ ,  $N > 0$  and  $N' > 0$ ,

$$\begin{aligned} \bar{P}\left(\exists n \in [N, N + N'] : \frac{\sum_{i=1}^n X_i}{n} \geq \bar{\mu}_n + \epsilon\right) &\leq \sum_{n=N}^{N+N'} \bar{P}\left(\frac{\sum_{i=1}^n X_i}{n} \geq \bar{\mu}_n + \epsilon\right) \\ &\leq \sum_{n=N}^{N+N'} e^{-2n\epsilon^2/B^2} \\ &= \left(e^{-2N\epsilon^2/B^2}\right) \sum_{n=0}^{N'} e^{-2n\epsilon^2/B^2} \\ &= \left(e^{-2N\epsilon^2/B^2}\right) \frac{1 - e^{2(N'+1)\epsilon^2/B^2}}{1 - e^{-2\epsilon^2/B^2}} \\ &< \frac{e^{-2N\epsilon^2/B^2}}{1 - e^{-2\epsilon^2/B^2}}. \end{aligned}$$

Consequently,

$$\bar{P}\left(\exists n \in [N, N + N'] : \frac{\sum_{i=1}^n X_i}{n} \geq \bar{\mu} + \epsilon\right) < \epsilon,$$

provided that  $N$  is a positive integer such that

$$N > -(B^2/(2\epsilon^2)) \ln \epsilon(1 - e^{-2\epsilon^2/B^2}).$$

An analogous argument leads to

$$\bar{P}\left(\exists n \in [N, N + N'] : \frac{\sum_{i=1}^n X_i}{n} \leq \bar{\mu} - \epsilon\right) < \epsilon.$$

By superadditivity of upper probability, we obtain a perhaps more intuitive statement of the strong law of large numbers: for all  $\epsilon > 0$ , there is  $N$  such that for any  $N'$ ,

$$\underline{P}\left(\forall n \in [N, N + N'] : \underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \bar{\mu}_n + \epsilon\right) > 1 - 2\epsilon,$$

thus reproducing De Cooman and Miranda's strong laws.

We now present a pair of weak/strong laws of large numbers under weak irrelevance. De Cooman and Miranda prove a similar pair of laws by resorting to their previous results on *forward irrelevant natural extensions* [5, Sec. 4.1]. The proof offered now is perhaps more direct, using our analogue of Azuma's inequality.

**Theorem 3** *If bounded variables  $X_1, \dots, X_n$  satisfy weak irrelevance and Expression (4) holds, then for any  $\epsilon > 0$ ,*

$$\underline{P}\left(\underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \bar{\mu}_n + \epsilon\right) \geq 1 - 2e^{-2n\epsilon^2/B^2},$$

and there is  $N$  such that for any  $N'$ ,

$$\underline{P}\left(\forall n \in [N, N + N'] : \underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \bar{\mu}_n + \epsilon\right) > 1 - 2\epsilon.$$

*Proof.* Using subadditivity of upper probability and Theorem 2, and defining again  $B \doteq \max_i B_i$ ,

$$\bar{P}\left(\left(\sum_{i=1}^n X_i \geq n\bar{\mu}_n + \epsilon\right) \cup \left(\sum_{i=1}^n X_i \leq n\underline{\mu}_n - \epsilon\right)\right) \leq 2e^{-\frac{2n\epsilon^2}{B^2}},$$

and we obtain the first expression in the theorem. To produce the second inequality (strong law), note:

$$\begin{aligned} & \bar{P}\left(\exists n \in [N, N + N'] : \frac{\sum_{i=1}^n X_i}{n} \geq \bar{\mu} + \epsilon\right) \\ & \leq \sum_{n=N}^{N+N'} \bar{P}\left(\frac{\sum_{i=1}^n X_i}{n} \geq \bar{\mu} + \epsilon\right) \\ & \leq \sum_{n=N}^{N+N'} e^{-2n\epsilon^2/B^2} \\ & < \frac{e^{-2N\epsilon^2/B^2}}{1 - e^{-2\epsilon^2/B^2}}. \end{aligned}$$

Again,

$$\bar{P}\left(\exists n \in [N, N + N'] : \frac{\sum_{i=1}^n X_i}{n} \geq \bar{\mu} + \epsilon\right) < \epsilon$$

provided that  $N$  is a positive integer such that

$$N > -(B^2/(2\epsilon^2)) \ln \epsilon(1 - e^{-2\epsilon^2/B^2}).$$

This is ‘‘half’’ of the second expression in the theorem; the other ‘‘half’’ is proved analogously.  $\square$

The theorem easily implies the following concise weak law of large numbers, by taking limits:

$$\lim_{n \rightarrow \infty} \underline{P}\left(\underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \bar{\mu}_n + \epsilon\right) = 1.$$

## 4 Laws of large numbers without boundedness

We now consider variables without bounds in their ranges under the assumption of weak irrelevance; the resulting laws of large numbers are the main contribution of the paper. We will assume in this section that countable additivity holds (Expression (5)). This assumption of countable additivity implies disintegrability; that is,  $E_P[W] = E_P[E_P[W|Z]]$  for any  $P$ ,  $W$  and  $Z$ . Thus our setup is close to the standard (Kolmogorovian) one, where any expectation functional is a linear monotone and monotonically convergent functional that can be expressed through Lebesgue integration. We only depart from the Kolmogorovian tradition in explicitly letting a *set* of such functionals to be permissible given a set of assessments.

We will use a sequence of variables  $\{Y_n\}$  defined as follows:

$$Y_n \doteq \sum_{i=1}^n X_i - E_P[X_i|X_{1:i-1}].$$

The key observation is that  $Y_n$  is a function of all variables  $X_{1:n}$  such that

$$\begin{aligned} E_P[Y_n|X_{1:n-1}] &= \left(\sum_{i=1}^{n-1} X_i - E_P[X_i|X_{1:i-1}]\right) + \\ & \quad E_P[X_n - E_P[X_n|X_{1:n-1}]|X_{1:n-1}] \\ &= Y_{n-1} + \\ & \quad E_P[X_n|X_{1:n-1}] - E_P[X_n|X_{1:n-1}] \\ &= Y_{n-1}; \end{aligned}$$

so,  $\{Y_n\}$  is a *martingale* with respect to  $P$ . Thus,

$$\begin{aligned} & E_P[(Y_n - Y_{n-1})^2|X_{1:n-1}] \\ &= E_P[Y_n^2|X_{1:n-1}] - 2E_P[Y_{n-1}Y_n|X_{1:n-1}] + Y_{n-1}^2 \\ &= E_P[Y_n^2|X_{1:n-1}] - 2Y_{n-1}E_P[Y_n|X_{1:n-1}] + Y_{n-1}^2 \\ &= E_P[Y_n^2|X_{1:n-1}] - 2Y_{n-1}Y_{n-1} + Y_{n-1}^2 \\ &= E_P[Y_n^2|X_{1:n-1}] - Y_{n-1}^2. \end{aligned}$$

And by taking expectations on both sides and noting that  $Y_i - Y_{i-1} = X_i - E_P[X_i|X_{1:i-1}]$ , we get

$$E_P[Y_n^2] = E_P[(X_n - E_P[X_n|X_{1:n-1}])^2] + E_P[Y_{n-1}^2].$$

Iterating this expression, we obtain:

$$E_P[Y_n^2] = \sum_{i=1}^n E_P[(X_i - E_P[X_i|X_{1:i-1}])^2]. \quad (9)$$

With these preliminaries, we have:

**Theorem 4** *Assume countable additivity. If variables  $X_1, \dots, X_n$  satisfy weak irrelevance, and  $\underline{E}[X_i]$  and  $\overline{E}[X_i]$  are finite quantities such that  $\overline{E}[X_i] - \underline{E}[X_i] \leq \delta$ , and the variance of any  $X_i$  is no larger than a finite quantity  $\sigma^2$ , then for any  $\epsilon > 0$ ,*

$$\underline{P}\left(\underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \overline{\mu}_n + \epsilon\right) \geq 1 - \frac{\sigma^2 + \delta^2}{\epsilon^2 n},$$

and there is  $N > 0$  such that for any  $N' > 0$ ,

$$\underline{P}\left(\forall n \in [N, N+N'] : \underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \overline{\mu}_n + \epsilon\right) > 1 - 2\epsilon.$$

Consequently,

$$\forall \epsilon > 0 : \lim_{n \rightarrow \infty} \underline{P}\left(\underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \overline{\mu}_n + \epsilon\right) = 1,$$

$$\underline{P}\left(\limsup_{n \rightarrow \infty} \left(\frac{\sum_{i=1}^n X_i}{n} - \overline{\mu}_n\right) \leq 0\right) = 1,$$

$$\underline{P}\left(\liminf_{n \rightarrow \infty} \left(\frac{\sum_{i=1}^n X_i}{n} - \underline{\mu}_n\right) \geq 0\right) = 1.$$

*Proof.* For a fixed  $P$  and for all  $\epsilon > 0$ ,

$$\begin{aligned} & P\left(\underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \overline{\mu}_n + \epsilon\right) \\ &= P\left(\sum_{i=1}^n \underline{E}[X_i] - \epsilon n < \sum_{i=1}^n X_i < \sum_{i=1}^n \overline{E}[X_i] + \epsilon n\right) \\ &\geq P\left(\sum_{i=1}^n E_P[X_i|X_{1:i-1}] - \epsilon n < \sum_{i=1}^n X_i \right. \\ &\quad \left. < \sum_{i=1}^n E_P[X_i|X_{1:i-1}] + \epsilon n\right) \\ &\quad \text{(using weak irrelevance)} \\ &= P\left(-\epsilon < \frac{\sum_{i=1}^n X_i - E_P[X_i|X_{1:i-1}]}{n} < \epsilon\right) \\ &= P(-\epsilon < Y_n/n < \epsilon) \\ &= P(|Y_n/n| < \epsilon). \end{aligned}$$

Applying Chebyshev's inequality and Expression (9),

$$\begin{aligned} P(|Y_n/n| \geq \epsilon) &\leq \frac{E_P[Y_n^2]}{\epsilon^2 n^2} \\ &= \frac{\sum_{i=1}^n E_P[(X_i - E_P[X_i|X_{1:i-1}])^2]}{\epsilon^2 n^2}. \end{aligned}$$

Now write  $(X_i - E_P[X_i|X_{1:i-1}])^2$  as

$$((X_i - E_P[X_i]) + (E_P[X_i] - E_P[X_i|X_{1:i-1}]))^2,$$

and then:

$$\begin{aligned} & \sum_{i=1}^n E_P[(X_i - E_P[X_i|X_{1:i-1}])^2] \\ &= \sum_{i=1}^n E_P[(X_i - E_P[X_i])^2] \\ &\quad + 2E_P[(X_i - E_P[X_i])(E_P[X_i] - E_P[X_i|X_{1:i-1}])] \\ &\quad + E_P[(E_P[X_i] - E_P[X_i|X_{1:i-1}])^2] \\ &\leq \sum_{i=1}^n \sigma^2 + \delta^2 \\ &\quad + 2(E_P[X_i] - E_P[X_i|X_{1:i-1}])E_P[X_i - E_P[X_i]] \\ &= \sum_{i=1}^n \sigma^2 + \delta^2. \end{aligned}$$

Hence

$$\sum_{i=1}^n E_P[(X_i - E_P[X_i|X_{1:i-1}])^2] \leq n(\sigma^2 + \delta^2), \quad (10)$$

and combining these inequalities, we obtain:

$$P(|Y_n/n| \geq \epsilon) \leq \frac{\sigma^2 + \delta^2}{\epsilon^2 n},$$

and then

$$P\left(\underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \overline{\mu}_n + \epsilon\right) \geq 1 - \frac{\sigma^2 + \delta^2}{\epsilon^2 n}$$

for any  $P$ , as desired. By taking the limit as  $n$  grows without bound, we obtain

$$\lim_{n \rightarrow \infty} \underline{P}\left(\underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \overline{\mu}_n + \epsilon\right) = 1.$$

The proof of the strong law of large numbers uses the same strategy, but replaces the appeal to Chebyshev's inequality by an appeal to the Kolmogorov-Hajek-Renyi inequality (described in the Appendix), following the proof of the strong law of large numbers by Whittle [29, Thm. 14.2.3]. So, for a fixed  $P$  and for all  $\epsilon > 0$ , we proceed as previously to obtain:

$$\begin{aligned} & P\left(\forall n \in [N, N+N'] : \underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \overline{\mu}_n + \epsilon\right) \\ &\geq P\left(\forall n \in [N, N+N'] : -\epsilon < \frac{Y_n}{n} < \epsilon\right) \\ &= P(\forall n \in [N, N+N'] : |Y_n/n| < \epsilon). \end{aligned}$$



As  $\{Y_N, Y_{N+1}, \dots, Y_{N+N'}\}$  forms a martingale, we use the Kolmogorov-Hajek-Renyi inequality to produce:

$$\begin{aligned}
& P(\forall n \in [N, N+N'] : |Y_n/n| < \epsilon) \\
& \geq 1 - \frac{\sum_{i=1}^N E_P[(X_i - E_P[X_i|X_{1:i-1}])^2]}{\epsilon^2 N^2} \\
& \quad - \frac{\sum_{i=N+1}^{N+N'} E_P[(X_i - E_P[X_i|X_{1:i-1}])^2]}{\epsilon^2 i^2} \\
& \geq 1 - \frac{\sigma^2 + \delta^2}{\epsilon^2 N} - \sum_{i=N+1}^{N+N'} \frac{\sigma^2 + \delta^2}{\epsilon^2 i^2} \\
& \quad \text{(using Expression (10))} \\
& \geq 1 - \frac{\sigma^2 + \delta^2}{\epsilon^2 N} - \sum_{i=N+1}^{\infty} \frac{\sigma^2 + \delta^2}{\epsilon^2 i^2} \\
& \geq 1 - \frac{\sigma^2 + \delta^2}{\epsilon^2} \left( \frac{1}{N} + \int_N^{\infty} 1/i^2 di \right) \\
& = 1 - \frac{\sigma^2 + \delta^2}{\epsilon^2} \left( \frac{1}{N} + \frac{1}{N} \right) \\
& = 1 - 2 \frac{\sigma^2 + \delta^2}{\epsilon^2 N}.
\end{aligned}$$

Consequently, for integer  $N > (\sigma^2 + \delta^2)/\epsilon^3$ , we obtain the desired inequality

$$P\left(\forall n \in [N, N+N'] : \underline{\mu}_n - \epsilon < \frac{\sum_{i=1}^n X_i}{n} < \bar{\mu}_n + \epsilon\right) > 1 - 2\epsilon.$$

As we assume countable additivity for every  $P$ , the proof of the Kolmogorov-Hajek-Renyi can be extended to an infinite intersection of (decreasing) events expressed as  $\{\forall j \geq 1 : |X_j| < \epsilon_j\}$ ; thus

$$\begin{aligned}
& \forall \epsilon > 0 : \forall \delta > 0 : \exists N > 0 : \\
& P\left(\forall m \geq N : \frac{\sum_{i=1}^m X_i - \bar{E}[X_i]}{m} > \epsilon\right) \geq 1 - \delta,
\end{aligned}$$

and this is equivalent to:

$$\forall \epsilon > 0 : \lim_{N \rightarrow \infty} P\left(\forall m \geq N : \frac{\sum_{i=1}^m X_i - \bar{E}[X_i]}{m} > \epsilon\right) = 1.$$

As the events in these probability values form an increasing sequence, we have, for all  $\epsilon > 0$ ,

$$P\left(\exists N > 0 : \forall m \geq N : \frac{\sum_{i=1}^m X_i - \bar{E}[X_i]}{m} > \epsilon\right) = 1.$$

Now this is equivalent to  $\forall k > 0 : P(A_k) = 1$ , where  $A_k = \{\exists N > 0 : \forall m \geq N : (1/m) \sum_{i=1}^m X_i - \bar{E}[X_i] > 1/k\}$ , and because  $P(\cup_{k>0} \neg A_k) \leq \sum_{k>0} P(\neg A_k) = 0$ , we have  $P(\forall k > 0 : A_k) = 1$ , so

$$P\left(\forall k > 0 : \exists N > 0 : \forall m \geq N : \frac{\sum_{i=1}^m X_i - \bar{E}[X_i]}{m} > \epsilon\right) = 1.$$

This is exactly the desired expression

$$P\left(\limsup_{n \rightarrow \infty} \left(\frac{\sum_{i=1}^n X_i}{n} - \bar{\mu}_n\right) \leq 0\right) = 1.$$

A similar argument proves the last inequality in the theorem, starting from:

$$\begin{aligned}
& \forall \epsilon > 0 : \forall \delta > 0 : \exists N > 0 : \\
& P\left(\forall m \geq N : \frac{\sum_{i=1}^m X_i - \underline{E}[X_i]}{m} < -\epsilon\right) \geq 1 - \delta.
\end{aligned}$$

□

## 5 Discussion

The concentration inequalities and laws of large numbers proved in this paper assume rather weak conditions of epistemic irrelevance. When compared to usual laws of large numbers, both premises and consequences are weaker: expectations are not assumed precisely known, and convergence is interval-valued.

Theorems 1 and 2 and their ensuing laws of large numbers are implied by De Cooman and Miranda's seminal work [5] (and their results generalize several previous efforts [12]). Actually, De Cooman and Miranda start from a weaker condition of *forward factorization* that implies both Assumption (1) and weak irrelevance. The possible advantage of our proof techniques for these two theorems is that they are rather close to well-known methods in standard probability theory, such as Hoeffding's inequality (it should be noted that De Cooman and Miranda already indicate the similarity between their inequalities and Hoeffding's).

The most significant results of the paper employ weak irrelevance to produce concentration inequalities (Theorem 2) and laws of large numbers (Theorems 3 and 4). The latter theorem is possibly the most valuable contribution. The strategy for most proofs is to translate assumptions of weak irrelevance into facts regarding martingales, and to adapt results for martingales to this setting. This strategy keeps the proof relatively short and close to well-known results in probability theory. The connection between lower/upper expectations and the theory of martingales seems rather natural [4, 25], but the relationship between epistemic irrelevance and martingales does not appear to have been explored in depth so far. We note that the basic constraint defining martingales (that is,  $E[Y_n|X_{1:n-1}] = Y_{n-1}$ ) is preserved by convex combination of mixtures; therefore, the study of martingales seems appropriate when one deals with convex sets of probability measures — certainly it seems less contorted than the analysis through stochastic independence, as stochastic independence is *not* preserved by convex combination.

The proofs presented in this paper need assumptions of disintegrability that can be easily satisfied if countable additivity is adopted. It is an open question whether similar results can be proven without disintegrability, particularly when one deals with unbounded variables.

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## A Two auxiliary inequalities

The following inequality is a simple extension of a basic result by Hoeffding [8, 15]: If variable  $X$  satisfies  $a \leq X \leq b$  and  $E[X] \leq 0$ , then for any  $s > 0$ ,

$$E[\exp(sX)] \leq \exp(s^2(b-a)^2/8). \quad (11)$$

First, the inequality is clearly valid if  $a = b$ , or if  $a = 0$ , or if  $b < 0$ . From now on, suppose  $b \geq 0 > a$ . By convexity of the exponential function,

$$\exp(sx) \leq \frac{x-a}{b-a}e^{sb} + \frac{b-x}{b-a}e^{sa} \quad \text{for } x \in [a, b].$$

Given monotonicity of expectations and  $E[X] \leq 0$ ,

$$E[\exp(sX)] \leq \frac{b}{b-a}e^{sa} - \frac{a}{b-a}e^{sb} \doteq \exp(\phi(s(b-a)))$$

for  $\phi(u) = -pu + \log(1-p+pe^u)$  with  $p = -a/(b-a)$  (and note that  $p \in (0, 1]$  in the situation under consideration). Given that  $\phi(0) = \phi'(0) = 0$  and  $\phi''(u) \leq 1/4$  for  $u > 0$  (as the maximum of  $\phi''(u)$  is  $1/4$ , attained at  $e^u = (1-p)/p$ ), we can use Taylor's theorem as follows. For some  $v \in (0, u)$ ,  $\phi(u) = \phi(0) + u\phi'(0) + (u^2/2)\phi''(v) \leq (u^2/8)$  and consequently  $\phi(s(b-a)) \leq s^2(b-a)^2/8$ . By putting together these inequalities, we obtain Expression (11).

We now review the Kolmogorov-Hajek-Renyi inequality, almost exactly as proved by Whittle [29]; this is presented just to indicate the role of (elementwise) disintegrability in the derivation. Let  $\{X_i\}$  be a martingale with  $X_0 = 0$ , and let  $\{\epsilon_i\}$  be a sequence  $0 = \epsilon_0 \leq \epsilon_1 \leq \dots$ ; the inequality is

$$P(\forall j \in [1, n] : |X_j| < \epsilon_j) \geq 1 - \sum_{i=1}^n \frac{E[(X_i - X_{i-1})^2]}{\epsilon_i^2}.$$

To prove this inequality, define

$$A_n \doteq \{\forall j \in [1, n] : |X_j| < \epsilon_j\}.$$

Using  $\xi_i = X_i - X_{i-1}$ , and again denoting an event and its indicator function by the same symbol, we have

$$\begin{aligned} P(A_n) &= E_P[A_n] \\ &= E_P[A_{n-1}\{|X_n| < \epsilon_n\}] \\ &\geq E_P[A_{n-1}(1 - X_n^2/\epsilon_n^2)] \\ &\quad (\text{as } \{|X| < \epsilon\} \geq 1 - X^2/\epsilon^2) \\ &= E_P[A_{n-1}(1 - (X_{n-1}^2 + \xi_n^2)/\epsilon_n^2)] \\ &\quad (\text{by the martingale property}) \\ &\geq E_P[A_{n-2}(1 - X_{n-1}^2/\epsilon_{n-1}^2)] - E_P[\xi_n^2/\epsilon_n^2] \\ &\quad (\text{as } \epsilon_{n-1} \leq \epsilon_n \text{ and } \\ &\quad \{|X| < \epsilon\}(1 - X^2/\epsilon^2) \geq (1 - X^2/\epsilon^2)). \end{aligned}$$

Iteration of the last inequality yields the result. Note that it was necessary to apply disintegrability of  $P$  when applying the martingale property (that is, elementwise disintegrability is used).

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