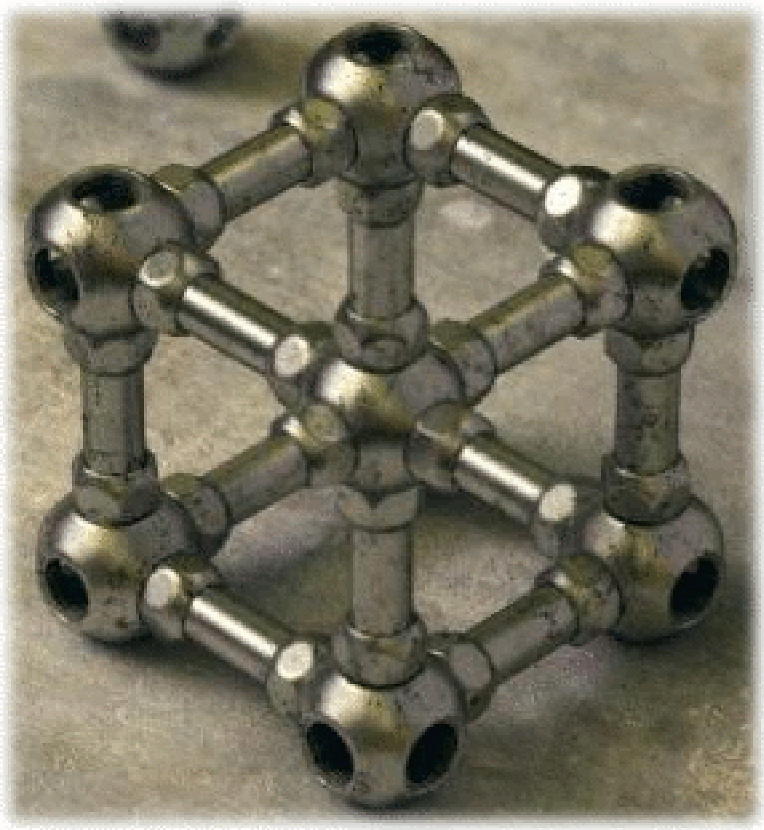

A Refuted Conjecture on Probabilistic Satisfiability [1]

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1 Introduction

The study on the thought under uncertainty is a subject from many fields, and, in computer science, it has been useful on probabilistic program and distributed system analysis. In the XIX century, Boole has[1] already studied the probability assignment to logical sentences, and we see his influence on de Finetti's theory of subject probability, in the following century. In 1965, Hailperin [2] revisited the problem, giving it a linear programming ground. In 1986, Nilsson [3] formalized the probabilistic satisfiability problem like we know it today: given logical sentences, and a probability assignment to them, we want to know if this assignment is consistent. The main analytical and numerical solution to this problem, as well its detailed history, can be seen in [4].

The probabilistic satisfiability problem (PSAT) is NP-complete. Thus the Cook-Levin's theorem [5] tells us that exists a polynomial reduction from PSAT to classical satisfiability (SAT), another NP-complete problem. Such reduction might be interesting because of the existence of good algorithms to solve SAT, these on continuous research. Another reason to study reduction from PSAT to SAT is the seek for a better understanding on the relation between logic and probability. The objective of this work is to investigate the relation between PSAT and SAT, looking for paths that enable the desired reduction. Even though such reduction it's not reached in this report, it brings significant results from the research, like a normal form for PSAT and a probabilistic entailment relation, that have shown their utility on the probabilistically satisfiability study.

The section2 formally presents the PSAT problem, in Nilsson's linear programming formulation [3], and prove its NP-completeness. The section 3 introduces the Atomic Normal Form for PSAT, which splits an instance into two partitions: a classical SAT instance and an atomic probability assignment. In section 4 a probabilistic entailment relation is suggested, and a theorem shows its utility in linking PSAT to SAT. We continue by presenting a set of formulas whose probabilistic entailment is easily verifiable, and we show some inherent properties to theses formulas. Then in section 5 we present a conjecture, under the concepts developed in the previous sections, on a possible reduction from PSAT to SAT. Such conjecture is exhaustively refuted by a counterexample.

2 The Problem

The probabilistic satisfiability (PSAT) is a decision problem, where we ask about the consistency of a probability assignment over logical formulas. Let $S = \{s_1, \ldots, s_k\}$ be a set with k logical sentences, defined on a set of n boolean variables, $X = \{x_1, \ldots, x_n\}$, with the usual operators from the classical propositional logic. Given a set a probabilities, $P = \{p_i | 0 \leq p_i \leq 1,\ 1 \leq i \leq k\}$, we say that a PSAT instance, defined by the the set S and by $p(s_i) = p_i$, with $1 \leq i \leq k$, is satisfiable if, and only if, this probability assignment is consistent.

A truth assignment (or valuation) v is initially defined as a function that associates truth values to boolean variables, formally $v : X \rightarrow \{0, 1\}$. Then we can extend its domain to the set of formulas S, as usual in classical logic [1], $v : S \rightarrow \{0, 1\}$. Let $V = \{v_1, \ldots, v_{2^n}\}$ be the set of possible truth assignments over X, and let π be a probability distribution over V. The probability of a formula s according to π is giving by $p_\pi(s) = \sum\{\pi(v_j) | v_j(s) = 1\}$. We say that the probabilities p_i, assigned to formulas from S, are consistent iff there is a distribution π over V that makes $p_\pi(s_i) = p_i, 1 \leq i \leq k$.

Now PSAT can be mathematically expressed like a linear programming problem, as introduced in [3]. Let Δ be the PSAT instance made by assigning the probabilities in P to the k formulas in S, $\Delta = \{p(s_i) = p_i | 1 \leq i \leq k\}$. We define the matrix $A_{k \times 2^n} = [a_{ij}]$, such that $a_{ij} = v_j(s_i)$, and the matrix $\overrightarrow{p}_{k \times 1} = [p_{ij}]$, such that $p_{ij} = p_i$. The instance Δ is satisfiable iff there is a vector π that hold the following restrictions:

$$A\pi = \overrightarrow{p} \quad (1)$$

$$\pi \geq 0 \quad (2)$$

$$\sum \pi = 1 \quad (3)$$

If there is a feasible solution π, then we say that π satisfies Δ, else we say that Δ is unsatisfiable. The restrictions (2) and (3) force π to be a probability distribution. The restriction (3) can be omitted if we add a entire row of $1's$ to the matrix A, $a_{k+1,j} = 1, 1 \leq j \leq 2^n$, and if we add a element $p_{k+1,1} = 1$ to the vector $\overrightarrow{p}$, what will be done from here until the end of this report.

The Carathéodory's Lemma [6] says that, if the linear programming problem (1-3) has feasible solution, then there is a solution with only $k+1$ elements of π different from zero. As pointed in [7], this lemma places PSAT in NP, because we can take a matrix $A_{k+1,k+1}$ and a vector $\pi_{k+1,1}$ as NP-certificate. Besides that, any instance from classical satisfiability (SAT), made from a set S with k sentences, can be reduced to a PSAT instance, in polynomial time, by making $p(s_i) = p_i = 1, 1 \leq i \leq k$. It follows that PSAT is NP-hard, hence NP-complete.

3 The Atomic Normal Form

Let $S = \{s_1, \ldots, s_k\}$ be a set of sentences from classical propositional logic, over the set $X = \{x_1, \ldots, x_n\}$ of boolean variables. We say that a PSAT instance,

[1] Let α and β be formulas from classical propositional logic, we have: $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$; $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$; $v(\neg\alpha) = 1$ iff $v(\alpha) = 0$; $v(\alpha \rightarrow \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$; and $v(\alpha \leftrightarrow \beta) = 1$ iff $v(\alpha) = v(\beta)$.

$\Delta = \{p(s_i) = p_i | 1 \leq i \leq l\}$, $0 \leq p_i \leq 1$, is in the *Atomic Normal Form* if it can be partitioned in two sets, (Γ, Ψ), where $\Gamma = \{p(s_i) = 1 | 1 \leq i \leq m\}$ and $\Psi = \{p(y_i) = p_i | y_i \text{ is an atom and } 1 \leq i \leq k\}$, with $0 \leq p_i \leq 1$, where $l = m+k$. The partition Γ is the SAT part of the atomic normal form, usually represented as a set of formulas, and the partition Ψ is the atomic probability assignment part. The following theorem shows how any PSAT instance can be brought to the atomic normal form, by adding a linear number of new variables.

Theorem 3.1 (Atomic Normal Form). *Let $\Delta = \{p(s_i) = p_i | 1 \leq i \leq k\}$ be a PSAT instance. Then a PSAT instance (Γ, Ψ) in the atomic normal form can be built, in polynomial time on k, such that Δ is satisfiable iff (Γ, Ψ) is satisfiable.*

Proof. To build a PSAT instance (Γ, Ψ) in the atomic normal form, from the instance $\Delta = \{p(s_i) = p_i, 1 \leq i \leq k\}$, we first add k new variables, $y_1, \dots, y_k$. Then we make $\Gamma = \{p(y_i \leftrightarrow s_i) = 1 | 1 \leq i \leq k\}$ and $\Psi = \{p(y_i) = p_i | 1 \leq i \leq k\}$. Clearly, this can be done in polynomial time on k.

Suppose there is a probability distribution π over the truth assignments $v : \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_k\} \to \{0, 1\}$ that satisfies (Γ, Ψ). Because π satisfies (Γ, Ψ), we have $p_\pi(y_i) = p_i$, $1 \leq i \leq k$. By the construction of Γ and the laws of probability, $p_\pi(y_i) = p_\pi(s_i)$, thus $p_\pi(s_i) = p_i$, $1 \leq i \leq k$. Over the truth assignments $v' : \{x_1, \dots, x_n\} \to \{0, 1\}$, we define a probability distribution π':

$$\pi'(v') = \sum \{\pi(v) | v(x_i) = v'(x_i), 1 \leq i \leq n\}$$

Hence π' is a probability distribution over $\{x_1, \dots, x_n\}$ that satisfies $p_{\pi'}(s_i) = p_i$, $1 \leq i \leq k$, and consequently π' satisfies Δ.

Now suppose there is a probability distribution π' over the truth assignments $v' : \{x_1, \dots, x_n\} \to \{0, 1\}$ that satisfies Δ. Because π' satisfies Δ, $p_{\pi'}(s_i) = p_i$, $1 \leq i \leq k$. We define a probability distribution π over the truth assignments $v : \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_k\} \to \{0, 1\}$:

$$\pi(v) = \begin{cases} \pi'(v') & \text{, if } v(x_i) = v'(x_i) \text{ and } v(y_j) = v(s_j),\ 1 \leq i \leq n \text{ and } 1 \leq j \leq k \\ 0 & \text{, other cases} \end{cases}$$

Clearly, we have $p_\pi(s_i) = p_{\pi'}(s_i)$ and $p_\pi(y_i) = p_\pi(s_i)$, $1 \leq i \leq k$. It follows that $p_\pi(y_i) = p_{\pi'}(s_i) = p_i$, $1 \leq i \leq k$, and then π satisfies Ψ. For all v, such that $\pi(v) \neq 0$, we have $v(y_i) = v(s_i)$, $1 \leq i \leq k$, thus $p_\pi(y_i \leftrightarrow s_i) = 1$, $1 \leq i \leq k$, and π satisfies Γ. Finally, π satisfies (Γ, Ψ). □

The atomic normal form allows us to see a PSAT instance (Γ, Ψ) as an interaction between a probability assignment, represented by Ψ, and a SAT instance Γ. Solutions to (Γ, Ψ) can be seen as solutions to Ψ constrained by the SAT instance Γ.

Let v' be a truth assignment over $\{y_1, \dots, y_k\}$. We say that v' is consistent with Γ, a SAT instance over the variables $\{y_1, \dots, y_k\} \cup \{x_1, \dots, x_n\}$, if there is a truth assignment $v : \{y_1, \dots, y_k\} \cup \{x_1, \dots, x_n\} \to \{0, 1\}$ that satisfies Γ, such that $v(y_i) = v'(y_i)$, $1 \leq i \leq k$.

Lemma 3.2. *Let (Γ, Ψ) be a PSAT instance in the atomic normal form, where $\Psi = \{p(y_i) = p_i, 1 \leq i \leq k\}$, and let Γ be a SAT instance over the variables $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_k\}$. If (Γ, Ψ) is satisfied by the probability distribution π, then every truth assignment $v : \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_k\} \to \{0, 1\}$,* *such that $\pi(v) > 0$, extend a truth assignment $v' : \{y_1, \dots, y_k\} \to \{0, 1\}$, with $v(y_i) = v'(y_i)$, $1 \le i \le k$, such that v' is consistent with Γ.*

Proof. For all formula $s_i \in \Gamma$, we have $p_\pi(s_i) = 1$. Hence, if a truth assignment $v : \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_k\} \to \{0, 1\}$ has $\pi(v) > 0$, v must satisfy all formulas $s_i \in \Gamma$, because if v didn't satisfy any $s_i \in \Gamma$, we would have $p_\pi(s_i) < 1$, which is a contradiction. Building a truth assignment $v' : \{y_1, \dots, y_k\} \to \{0, 1\}$, with $v'(y_i) = v(y_i)$, $1 \le i \le k$, it must be consistent with Γ by definition, because v satisfies Γ. □

Theorem 3.3. *Let $\Psi = \{p(y_i) = p_i | 1 \le i \le k\}$ be a probability assignment. A instance PSAT (Γ, Ψ) in the atomic normal form is satisfiable iff there is a matrix A_Ψ, with $k+1$ rows and up to $k+1$ columns, that, with a vector π, obeys to the restrictions (1) and (2), such that each column of A_Ψ (unless the bottom row) corresponds to a truth assignment over $\{y_1, \dots, y_k\}$ consistent with Γ.*

Proof. Suppose Γ has m formulas over the variables $x_1, \dots, x_n$. Because (Γ, Ψ) is satisfiable, there is a matrix A, $(m+k+1) \times (2^{n+k})$ and a probability distribution π, over the 2^{n+k} truth assignment, that satisfy the restrictions (1) and (2). We build A' deleting all columns corresponding to truth assignments v_j, such that $\pi(v_j) = 0$, and we build π' deleting each π_j equal to zero, corresponding to $\pi(v_j) = 0$. Clearly, A' and π' satisfy (1) and (2). For all formula $s_i \in \Gamma$, we have $p_{\pi'}(s_i) = 1 = p_i$, but we added $p_{m+k+1} = 1$ to represent the restriction (3). Because the columns of A' represent truth assignments v_j, such that $\pi(v_j) > 0$, these truth assignments must satisfy Γ. Now the rows corresponding to formulas in Γ must be equal to the row (m+k+1), with only 1's, and can be extracted from A', with the corresponding p_i's. We have then the matrix A'', with $(k+1)$ rows that, with π', satisfies (1) and (2). Thus, by Carathéodory's lemma [6], there is a matrix A_Ψ with $k+1$ rows and up to $k+1$ columns that satisfies (1) and (2). And, by the lemma (3.2), the truth assignments corresponding to the columns of A_Ψ must be consistent with Γ.

Now suppose there is a square matrix A_Ψ, with dimension $k+1$, that obeys the constrains (1) and (2), with solution π, and with columns representing truth assignments over $y_1, \dots, y_k$ consistent with Γ. We build A' adding m rows with $1's$, corresponding to formulas in Γ, and we add m 1's, corresponding to theses formulas, to $\vec{p}$. Now each column of A' represents a truth assignment that satisfies Γ. Then we build A'', from A', adding m columns with $1's$, and we insert m $0's$ in π, corresponding to theses columns, obtaining π'. Clearly, A'' and π' satisfy the restrictions (1) and (2), hence (Γ, Ψ) is satisfiable. □

4 A Probabilistic Entailment Relation

A set of formulas Γ entails (or logically implies) a formula α, $\Gamma \models \alpha$, if every truth assignment that satisfies each formula in Γ also satisfies α or, equivalently, if $\Gamma \cup \{\neg\alpha\}$ is unsatisfiable. In this section, we look for a analogous entailment for the probabilistic logic, exploring PSAT instances in the atomic normal form.

Let Ψ be a atomic probability assignment and let α be a formula. We say that Ψ probabilistic entails α, denoted by $\Psi \models\!\!\approx \alpha$, iff the PSAT instance in the atomic normal form $(\{\neg\alpha\}, \Psi)$ is (probabilistically) unsatisfiable. In other words, if $(\{\neg\alpha\}, \Psi)$ is unsatisfiable, then $p(\neg\alpha) = 1$ and $p(\alpha) = 0$ are not consistent with

Ψ; and because the probabilities are non-negative, $\Psi \models\!\approx \alpha$ implies $p(\alpha) > 0$, for any probability distribution that satisfies Ψ.

We denote by $\Psi^{\models\!\approx}$ the set of all formulas α such that $\Psi \models\!\approx \alpha$. The following theorem shows the special role this set and the probabilistic entailment play in PSAT study.

Theorem 4.1. *Let $\Sigma = (\Gamma, \Psi)$ be a PSAT instance in the atomic normal form. Σ is satisfiable iff for each $\alpha \in \Psi^{\models\!\approx}$, $\Gamma \cup \{\alpha\}$ is classically satisfiable.*

Proof. Suppose there is an $\alpha \in \Psi^{\models\!\approx}$, but $\Gamma \cup \{\alpha\}$ is classically unsatisfiable, then $\Gamma \models \neg\alpha$. From $\alpha \in \Psi^{\models\!\approx}$, we obtain that $(\{\neg\alpha\}, \Psi)$ is probabilistically unsatisfiable, and thus (Γ, Ψ) is also probabilistically unsatisfiable.

Conversely, suppose (Γ, Ψ) is probabilistically unsatisfiable. Let γ be the conjunction of all formulas in Γ. Obviously $(\{\gamma\}, \Psi)$ also is unsatisfiable. It follows that $\neg\gamma \in \Psi^{\models\!\approx}$, and $\Gamma \cup \{\neg\gamma\}$ clearly is classically unsatisfiable. □

This motivates the study of the probabilistic entailment properties.

4.1 Probabilistic Entailment Properties

We first note an initial relation between $\models\!\approx$ and $\models$:

Lemma 4.2. *If $\Psi \models\!\approx \alpha$ and $\alpha \models \beta$, then $\Psi \models\!\approx \beta$.*

Proof. From $\Psi \models\!\approx \alpha$, we know that $p(\alpha) > 0$, and $\alpha \models \beta$ yields $p(\alpha) \leq p(\beta)$. So $p(\beta) > 0$, and therefore $\Psi \models\!\approx \beta$. □

However we note that $\Psi \models\!\approx \alpha$ and $\Psi \models\!\approx \beta$ don't imply $\Psi \models\!\approx \alpha \wedge \beta$. As counterexample, we take $p(\alpha) = p(\beta) = 0.4$ and $p(\alpha \vee \beta) = 0.8$, from where we obtain $p(\alpha \wedge \beta) = 0$ and thus $\Psi \not\models\!\approx \alpha \wedge \beta$. In this counterexample, we made $p(\alpha \vee \beta) = p(\alpha) + p(\beta)$, but it's only possible when $p(\alpha) + p(\beta) \leq 1$. This leads us to the next lemma.

Because in the atomic normal form the probabilities are assigned to atoms, and consequently to their negations, it's useful to define literal, so we can talk about probabilities overs literals. A literal x is an atom or its negation, and $\bar{x}$ denotes the negation of x.

Lemma 4.3. *Let Ψ be an atomic probability assignment such that, for literals y and z, $p(y) + p(z) > 1$. Then $\Psi \models\!\approx y \wedge z$.*

Proof. As direct consequence of Kolmogorov's probability axioms, we know that

$$p(y) + p(z) = p(y \vee z) + p(y \wedge z)$$

As $p(y) + p(z) > 1$, and always $p(y \vee z) \leq 1$, we obtain that $p(y \wedge z) > 0$, and thus $\Psi \models\!\approx y \wedge z$. □

However, as $p(y_1) + \ldots + p(y_k) > 1$ doesn't imply $\Psi \models\!\approx y_1 \wedge \ldots \wedge y_k$, we look for a suitable generalization for the lemma (4.3).

Let $y_1, \ldots, y_k$ be literals and let j be an integer, with $1 \leq j \leq k$. We define:

$$C^j(y_1, \ldots, y_k) = \bigvee \{y_{i_1} \wedge \ldots \wedge y_{i_j} | 1 \leq i_1 < \ldots < i_j \leq k\} \tag{4}$$

For example, $C^1(y,z,w) = y \vee z \vee w$, $C^2(y,z,w) = (y \wedge z) \vee (y \wedge w) \vee (z \vee w)$ and $C^3(y,z,w) = y \wedge z \wedge w$. It is useful to define $C^0(y_1,\ldots,y_k) = 1$, as the conjunction neutral element. We call formulas in the format (4) a *C-formula.*

From the commutativity of the logical operators $\wedge$ and $\vee$, we obtain that the literals order in (4) is irrelevant. Besides that, let's look at the following C-formulas properties, related to the entailment:

Lemma 4.4. *Let $y_1,\ldots,y_k$ be a set of literals and let j, j', k and k' be non-negative integers:*

(a) if $0 \leq j' < j$ then $C^j(y_1,\ldots,y_k) \vDash C^{j'}(y_1,\ldots,y_k)$.

(b) if $k' > k$ then $C^j(y_1,\ldots,y_k) \vDash C^j(y_1,\ldots,y_{k'})$.

Proof.

(a) Let $Y = \{y_1,\ldots,y_k\}$ be a set of literals. Note that the truth assignment v satisfies $C^j(y_1,\ldots,y_k)$ iff there is a set $Y' \subseteq Y$, such that $|Y'| = j$ and $v(y) = 1$ for all $y \in Y'$. Obviously, if v satisfies $C^j(y_1,\ldots,y_k)$, then, for each $1 \leq j' < j$, there is a set $Y'' \subseteq Y' \subseteq Y$ such that $|Y''| = j'$ and $v(y) = 1$ for all $y \in Y''$. So such truth assignment v must also satisfy $C^{j'}(y_1,\ldots,y_k)$.

(b) When $k' > k$, $C^j(y_1,\ldots,y_{k'})$ can be written in the format $\alpha \vee C^j(y_1,\ldots,y_k)$. So each truth assignment that satisfies $C^j(y_1,\ldots,y_k)$ also satisfies $C^j(y_1,\ldots,y_{k'})$.

□

Let $Y = \{y_1,\ldots,y_n\}$ be a set of literals. Another important C-formulas property, to be used in section 5, is related to adding opposite literals, $z, \bar{z} \notin Y$, in $C^j(Y)$, where $C^j(Y)$ denotes $C^j(y_1,\ldots,y_k)$:

Lemma 4.5. *Let Y be a set of literals, and let z be a literal, such that $z, \bar{z} \notin Y$. Then $C^j(Y) \equiv C^{j+1}(Y \cup \{z, \bar{z}\})$.*

Proof. Expanding the formula $C^{j+1}(Y \cup \{z,\bar{z}\})$ and ruling out the conjunctions that imply $z \wedge \bar{z}$, we have conjunctions with $j+1$ literals, where j literals are in Y and the remaining literal is in $\{z, \bar{z}\}$. In other words, $C^{j+1}(Y \cup \{z,\bar{z}\}) \equiv z \wedge C^j(Y) \vee \bar{z} \wedge C^j(Y) \equiv C^j(Y)$. □

Before we enunciate the next theorem, we need the following lemma, where C^j_k denotes $C^j(y_1,\ldots,y_k)$:

Lemma 4.6. *Let i and k be integers, with $0 \leq i < k$, $p(y_{k+1} \wedge C^i_k) + p(C^{i+1}_k) = p(C^{i+1}_{k+1}) + p(y_{k+1} \wedge C^{i+1}_k)$*

Proof. Directly from Kolmogorov's axioms, we have:

$$p(y_{k+1} \wedge C^i_k) + p(C^{i+1}_k) = p(y_{k+1} \wedge C^i_k \vee C^{i+1}_k) + p(y_{k+1} \wedge C^i_k \wedge C^{i+1}_k)$$

From lemma (4.4), we know that $C^{i+1}_k \vDash C^i_k$. Then:

$$p(y_{k+1} \wedge C^i_k \wedge C^{i+1}_k) = p(y_{k+1} \wedge C^{i+1}_k)$$

From the C-formulas definition, we note that:

$$y_{k+1} \wedge C^i_k \vee C^{i+1}_k \equiv C^{i+1}_{k+1}$$

Finally, we obtain:

$$p(y_{k+1} \wedge C_k^i) + p(C_k^{i+1}) = p(C_{k+1}^{i+1}) + p(y_{k+1} \wedge C_k^{i+1})$$

$\square$

Theorem 4.7. *Let $\{y_1, \ldots, y_k\}$ be a set of literals. Then:*

$$p(y_1) + \ldots + p(y_k) = p(C^1(y_1, \ldots, y_k)) + \ldots + p(C^k(y_1, \ldots, y_k))$$

Proof. The proof proceeds by induction in k, with C_k^j denoting $C^j(y_1, \ldots, y_k)$:
Induction basis: $k = 1$, $p(y_1) = C^1(y_1)$ trivially.
Induction hypothesis: $k = j \geq 1$, $p(y_1) + \ldots + p(y_j) = p(C_j^1) + \ldots + p(C_j^j)$.
Induction step: $k = j + 1$; starting from the induction hypothesis, we sum $p(y_{j+1})$ to both sides of equality:

$$p(y_1) + \ldots + p(y_j) + p(y_{j+1}) = p(C_j^1) + \ldots + p(C_j^j) + p(y_{j+1})$$

As y_{j+1} is equivalent to $y_{j+1} \wedge C_j^0$, we apply the lemma (4.6):

$$p(C_j^1) + p(y_{j+1}) = p(C_{j+1}^1) + p(y_{j+1} \wedge C_j^1)$$

So we obtain:

$$p(y_1) + \ldots + p(y_{j+1}) = p(C_{j+1}^1) + p(y_{j+1} \wedge C_j^1 t) + p(C_j^2) + \ldots + p(C_j^j)$$

In an analogous way, we repeat the lemma (4.6) application $j - 1$ times, obtaining:

$$p(y_1) + \ldots + p(y_{j+1}) = p(C_{j+1}^1) + \ldots + p(C_{j+1}^j) + p(y_{j+1} \wedge C_j^j)$$

Noting that $y_{j+1} \wedge C_j^j = C_{j+1}^{j+1}$, we finally have:

$$p(y_1) + \ldots + p(y_{j+1}) = p(C^1(y_1, \ldots, y_{j+1})) + \ldots + p(C^{j+1}(y_1, \ldots, y_{j+1}))$$

$\square$

Having presented the C-formulas, and with theorem (4.7) in hand, we can enunciate the theorem that finally generalizes the lemma (4.3):

Theorem 4.8. *Let $\{y_1, \ldots, y_k\}$ be a set of literals, and let Ψ be a probability assignment to these literals. If $\sum_{i=1}^k p(y_i) > j - 1$, then $C^j(y_1, \ldots, y_k) \in \Psi^{\approx}$.*

Proof. In one hand, from theorem (4.7), we have:

$$\sum_{i=1}^{k} p(y_i) = \sum_{i=1}^{k} p(C^i(y_1, \ldots, y_k)) > j - 1$$

As $p(\alpha) \leq 1$ for all formula α, $\sum_{i=1}^{j-1} p(C^i(y_1, \ldots, y_k)) \leq j-1$, thus $\sum_{i=j}^{k} p(C^i(y_1, \ldots, y_k)) > 0$.

In other hand, from lemma (4.4), it follows that $C^k(y_1, \ldots, y_k) \vDash \ldots \vDash C^j(y_1, \ldots, y_k)$, because $j \leq k$, and hence:

$$p(C^j(y_1, \ldots, y_k)) \geq \ldots \geq p(C^k(y_1, \ldots, y_k))$$
$$(k - j + 1)p(C^j(y_1, \ldots, y_k)) \geq \textstyle\sum_{i=j}^{k} p(C^i(y_1, \ldots, y_k)) > 0$$
$$p(C^j(y_1, \ldots, y_k)) > 0$$

We conclude that $\Psi \approx C^j(y_1, \ldots, y_k)$, and therefore $C^j(y_1, \ldots, y_k) \in \Psi^{\approx}$. $\square$

5 A Conjecture Refutation on Probabilistic Satisfiability

Let (Γ, Ψ) be a PSAT instance. In one hand, from theorem (4.1), if $\{\alpha\} \cup \Gamma$ is classically unsatisfiable, for a C-formula $\alpha \in \Psi^{\approx}$, then (Γ, Ψ) is probabilistically unsatisfiable. In other hand, if each formula $\alpha \in \Psi^{\approx}$ were implied by a formula $C^j(y_1, \ldots, y_k)$, such that $\sum_{i=1}^{k} p(y_i) > j - 1$, the probabilistic unsatisfiability of (Γ, Ψ) would yield the classical unsatisfiability of $\{C^j(y_1, \ldots, y_k)\} \cup \Gamma$, for one C-formula in that condition. Having this in mind, we conjecture that, for all unsatisfiable PSAT instance (Γ, Ψ), there is a formula $C^j(y_1, \ldots, y_k)$, with $\sum_{i=1}^{k} p(y_i) > j-1$, such that $\{C^j(y_1, \ldots, y_k)\} \cup \Gamma$ is unsatisfiable. The following lemma refutes such conjecture.

Lemma 5.1. *There is an unsatisfiable PSAT instance* (Γ, Ψ) *such that, for each formula* $C^j(y_1, \ldots, y_k)$*, with* $\sum_{i=1}^{k} p(y_i) > j - 1$*,* $\{C^j(y_1, \ldots, y_k)\} \cup \Gamma$ *is classically satisfiable.*

Proof. Our proof will be built by presenting an example: an unsatisfiable PSAT instance $\Delta = (\Gamma, \Psi)$ where, for each formula $C^j(y_1, \ldots, y_k)$, with $\sum_{i=1}^{k} p(y_i) > j - 1$, $\{C^j(y_1, \ldots, y_k)\} \cup \Gamma$ is classically satisfiable.

Let's consider the PSAT instance $\Delta = (\Gamma, \Psi)$, where Γ is a set with 1 formula, from classical propositional logic, over 4 boolean variables $\{x_1, \ldots, x_4\}$. To simplify the writting, if α and β are formulas, then $\alpha\beta$ denotes $\alpha \wedge \beta$, and $\bar{\alpha}$ denotes $\neg\alpha$:

$$\Gamma = \{x_1x_2x_3x_4 \vee x_1\bar{x}_2\bar{x}_3\bar{x}_4 \vee \bar{x}_1x_2\bar{x}_3\bar{x}_4 \vee \bar{x}_1\bar{x}_2x_3\bar{x}_4 \vee x_1\bar{x}_2x_3x_4\}$$

And Ψ is the following probability assignment to the boolean variables:

$$\Psi = \{p(x_1) = 0,47, \quad p(x_2) = 0,40, \quad p(x_3) = 0,46, \quad p(x_4) = 0,05\}$$

It follows that, from Kolmogorov's axioms:

$$p(\bar{x}_1) = 0,53, \quad p(\bar{x}_2) = 0,60, \quad p(\bar{x}_3) = 0,54 \quad \text{e} \quad p(\bar{x}_4) = 0,95$$

Let $v_k : \{x_1, \ldots, x_4\} \to \{0,1\}$, $1 \leq k \leq 5$, be the only truth assignments to satisfy Γ, such that v_1 satisfies $x_1x_2x_3x_4$, v_2 satisfies $x_1\bar{x}_2\bar{x}_3\bar{x}_4$, v_3 satisfies $\bar{x}_1x_2\bar{x}_3\bar{x}_4$, v_4 satisfies $\bar{x}_1\bar{x}_2x_3\bar{x}_4$, and v_5 satisfies $x_1\bar{x}_2x_3x_4$.

To note the unsatisfiability of $\Delta = (\Gamma, \Psi)$, we present it in its linear programming format, where the matrix $A_{5\times5}$ has as columns the truth assignments that satisfy Γ, $a_{ij} = v_j(x_i)$, with $1 \leq i \leq 4$ and $1 \leq j \leq 5$, adding a bottom row of 1's corresponding to the restriction (3):

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The vector $\overrightarrow{p}_{5\times1} = [p_{ij}]$ is such that $p_{i,1} = p(x_i)$, $1 \leq i \leq 4$, and $p_{5,1} = 1$ is corresponding to the restriction (3). If there is a probability distribution π over the truth assignments $v_1, \dots, v_5$ that satisfies the PSAT instance $\Delta = (\Gamma, \Psi)$, then the following linear programming problem must have a feasible solution:

$$A\pi = \vec{p} \quad (5)$$

$$\pi \geq 0 \quad (6)$$

Solving the possible and determined linear system (5), we obtain $\pi(v_5) = -0,23$, which contradicts the restriction (6). Thus $\Delta = (\Gamma, \Psi)$ is unsatisfiable.

Now we have to show exhaustively that, for each formula $C^j(y_1, \dots, y_k)$ with $\sum_{i=1}^{k} p(y_i) > j-1$, $\{C^j(y_1, \dots, y_k)\} \cup \Gamma$ is satisfiable. Remembering the lemma (4.4.a), being Y a set of literals, if $0 \leq j' < j$, then $C^j(Y) \vDash C^{j'}(Y)$. For each set of literals $Y = \{y_1, \dots, y_k\}$, we define $j_{max}(Y) = \lceil \sum_{i=1}^{k} p(y_i) \rceil$ and denote $C^{j_{max}(Y)}(Y)$ by $C^{j_{max}}(Y)$. So if $0 \leq j < j_{max}(Y)$, then $C^{j_{max}}(Y) \vDash C^j(Y)$, and if $j_{max}(Y) < j$, then $\sum_{i=1}^{k} p(y_i) \leq j-1$. Thus for each set of literals Y, it's enough to verify the satisfiability of $\{C^{j_{max}}(Y)\} \cup \Gamma$.

With 4 variables, we have 8 different literals, that yield $2^8 = 256$ possible sets of literals to be checked. We easily note that the empty set doesn't need to be verified, because $\sum_{y \in \emptyset} p(y) = 0 = j_{max}(\emptyset)$, and $C^0(Y) = TRUE$ is satisfied by any truth assignment. Furthermore, if $z, \bar{z} \notin Y$, then $j_{max}(Y \cup \{z, \bar{z}\}) = \lceil p(z) + p(\bar{z}) + \sum_{y \in Y} p(y) \rceil = j_{max}(Y) + 1$. So, by lemma (4.5), $C^{j_{max}}(Y \cup \{z, \bar{z}\}) \equiv C^{j_{max}}(Y)$, thus we don't need to check sets with 2 opposite literals. The 4 tables below show the 80 remaining possible sets of literals over $\{x_1, \dots, x_4\}$, organized by the set length. Each row presents a set Y of literals, the sum of these literals probabilities, $\sum_{y \in Y} p(y)$, the j_{max} defined by this set and the truth assignment that satisfies $\{C^{j_{max}}(Y)\} \cup \Gamma$. Each truth assignment is represented by the conjunction it is the only one to satisfy.

Table 1: Sets with 4 literals

set of literals Y	$\sum_{y \in Y} p(y)$	j_{max}	$v_i \mid v_i \vDash \{C^{j_{max}}(Y)\} \cup \Gamma$
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4\}$	2.62	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4\}$	1.72	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, \bar{x}_2, x_3, \bar{x}_4\}$	2.54	3	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_1, \bar{x}_2, x_3, x_4\}$	1.64	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_1, x_2, \bar{x}_3, \bar{x}_4\}$	2.42	3	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, x_2, \bar{x}_3, x_4\}$	1.52	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, x_2, x_3, \bar{x}_4\}$	2.34	3	$\bar{x}_1x_2\bar{x}_3\bar{x}4$
$\{\bar{x}_1, x_2, x_3, x_4\}$	1.44	2	$x_1x_2x_3x_4$
$\{x_1, \bar{x}_2, \bar{x}_3, \bar{x}_4\}$	2.56	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_1, \bar{x}_2, \bar{x}_3, x_4\}$	1.66	2	$x_1x_2x_3x_4$
$\{x_1, \bar{x}_2, x_3, \bar{x}_4\}$	2.48	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_1, \bar{x}_2, x_3, x_4\}$	1.58	2	$x_1x_2x_3x_4$
$\{x_1, x_2, \bar{x}_3, \bar{x}_4\}$	2.36	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_1, x_2, \bar{x}_3, x_4\}$	1.46	2	$x_1x_2x_3x_4$
$\{x_1, x_2, x_3, \bar{x}_4\}$	2.28	3	$x_1x_2x_3x_4$
$\{x_1, x_2, x_3, x_4\}$	1.38	2	$x_1x_2x_3x_4$

Table 2: Sets with 3 literals

set of literals Y	$\sum_{y \in Y} p(y)$	j_{max}	$v_i \vert v_i \models \{C^{j_{max}}(Y)\} \cup \Gamma$
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$	1.67	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, \bar{x}_2, x_3\}$	1.59	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_1, x_2, \bar{x}_3\}$	1.47	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, x_2, x_3\}$	1.39	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{x_1, \bar{x}_2, \bar{x}_3\}$	1.61	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_1, \bar{x}_2, x_3\}$	1.53	2	$x_1\bar{x}_2x_3x_4$
$\{x_1, x_2, \bar{x}_3\}$	1.41	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_1, x_2, x_3\}$	1.33	2	$x_1x_2x_3x_4$
$\{\bar{x}_1, \bar{x}_2, \bar{x}_4\}$	2.08	3	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_1, \bar{x}_2, x_4\}$	1.18	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_1, x_2, \bar{x}_4\}$	1.88	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, x_2, x_4\}$	0.98	1	$x_1x_2x_3x_4$
$\{x_1, \bar{x}_2, \bar{x}_4\}$	2.02	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_1, \bar{x}_2, x_4\}$	1.12	2	$x_1x_2x_3x_4$
$\{x_1, x_2, \bar{x}_4\}$	1.82	2	$x_1x_2x_3x_4$
$\{x_1, x_2, x_4\}$	0.92	1	$x_1x_2x_3x_4$
$\{\bar{x}_1, \bar{x}_3, \bar{x}_4\}$	2.02	3	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, \bar{x}_3, x_4\}$	1.12	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, x_3, \bar{x}_4\}$	1.94	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_1, x_3, x_4\}$	1.04	2	$x_1x_2x_3x_4$
$\{x_1, \bar{x}_3, \bar{x}_4\}$	1.96	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_1, \bar{x}_3, x_4\}$	1.06	2	$x_1x_2x_3x_4$
$\{x_1, x_3, \bar{x}_4\}$	1.88	2	$x_1x_2x_3x_4$
$\{x_1, x_3, x_4\}$	0.98	1	$x_1x_2x_3x_4$
$\{\bar{x}_2, \bar{x}_3, \bar{x}_4\}$	2.09	3	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_2, \bar{x}_3, x_4\}$	1.19	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_2, x_3, \bar{x}_4\}$	2.01	3	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_2, x_3, x_4\}$	1.11	2	$x_1x_2x_3x_4$
$\{x_2, \bar{x}_3, \bar{x}_4\}$	1.89	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_2, \bar{x}_3, x_4\}$	0.99	1	$x_1x_2x_3x_4$
$\{x_2, x_3, \bar{x}_4\}$	1.81	2	$x_1x_2x_3x_4$
$\{x_2, x_3, x_4\}$	0.91	1	$x_1x_2x_3x_4$

Table 3: Sets with 2 literals

set of literals Y	$\sum_{y\in Y} p(y)$	j_{max}	$v_i \vert v_i \models \{C^{j_{max}}(Y)\} \cup \Gamma$
$\{\bar{x}_1, \bar{x}_2\}$	1.13	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{\bar{x}_1, x_2\}$	0.93	1	$x_1x_2x_3x_4$
$\{x_1, \bar{x}_2\}$	1.07	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_1, x_2\}$	0.87	1	$x_1x_2x_3x_4$
$\{\bar{x}_1, \bar{x}_3\}$	1.07	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, x_3\}$	0.99	1	$x_1x_2x_3x_4$
$\{x_1, \bar{x}_3\}$	1.01	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_1, x_3\}$	0.93	1	$x_1x_2x_3x_4$
$\{\bar{x}_1, \bar{x}_4\}$	1.48	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_1, x_4\}$	0.58	1	$x_1x_2x_3x_4$
$\{x_1, \bar{x}_4\}$	1.42	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_1, x_4\}$	0.52	1	$x_1x_2x_3x_4$
$\{\bar{x}_2, \bar{x}_4\}$	1.55	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_2, x_4\}$	0.65	1	$x_1x_2x_3x_4$
$\{x_2, \bar{x}_4\}$	1.35	2	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{x_2, x_4\}$	0.45	1	$x_1x_2x_3x_4$
$\{\bar{x}_2, \bar{x}_3\}$	1.14	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_2, x_3\}$	1.06	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{x_2, \bar{x}_3\}$	0.94	1	$x_1x_2x_3x_4$
$\{x_2, x_3\}$	0.86	1	$x_1x_2x_3x_4$
$\{\bar{x}_3, \bar{x}_4\}$	1.49	2	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{\bar{x}_3, x_4\}$	0.59	1	$x_1x_2x_3x_4$
$\{x_3, \bar{x}_4\}$	1.41	2	$\bar{x}_1\bar{x}_2x_3\bar{x}_4$
$\{x_3, x_4\}$	0.51	1	$x_1x_2x_3x_4$

Table 4: Sets with 1 literal

set of literals Y	$\sum_{y\in Y} p(y)$	j_{max}	$v_i \vert v_i \models \{C^{j_{max}}(Y)\} \cup \Gamma$
$\{x_1\}$	0.47	1	$x_1x_2x_3x_4$
$\{\bar{x}_1\}$	0.53	1	$\bar{x}_1x_2\bar{x}_3\bar{x}_4$
$\{x_2\}$	0.40	1	$x_1x_2x_3x_4$
$\{\bar{x}_2\}$	0.60	1	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_3\}$	0.46	1	$x_1x_2x_3x_4$
$\{\bar{x}_3\}$	0.54	1	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$
$\{x_4\}$	0.05	1	$x_1x_2x_3x_4$
$\{\bar{x}_4\}$	0.95	1	$x_1\bar{x}_2\bar{x}_3\bar{x}_4$

And so we finish the proof, with an unsatisfiable PSAT instance, (Γ, Ψ), where for each formula $C^j(Y)$, such that $\sum_{y\in Y} p(y) > j-1$, we have shown a truth assignment that satisfies $\{C^j(Y)\} \cup \Gamma$. $\square$

6 Conclusion

In this report, we investigated the relation between probabilistic satisfiability and classical satisfiability. For this, we presented the Atomic Normal Form, which splits PSAT into an atomic probability assignment and a SAT instance. Then we could define a Probabilistic Entailment relation ($\models_{\approx}$) and study its properties. With the theorem (4.1), we remarked the role of probabilistic entailment in linking PSAT to SAT. We defined C-formulas, which with theorem (4.8) have shown ubiquity in the probabilistic satisfiability study. Finally, we conjectured the completeness of only looking at C-formulas probabilistically entailed to decide the satisfiability of a PSAT instance, and we refuted it with a counterexample.

The exponential number of C-formulas to investigate doesn't allow its exhaustive use in a possible polynomial reduction from PSAT to SAT, which were our initial objective. However, the founded results seem to be useful on PSAT study, bringing it back to logic. The Atomic Normal Form makes an important step in separating the problem logical part (SAT) from the probability assignment. Such normal form might be useful to standardize PSAT instances, in order to compare numeric outputs from algorithms that solve the problem. The introduced probabilistic entailment relation enables the presentation of a "witness" formula for the unsatisfiability of a PSAT instance. Such formula can be used either in proving probabilistic unsatisfiability, using classical unsatisfiability, or in algorithms that solve PSAT.

As the efficiency of algorithms that solve PSAT is considerably lower than those from algorithms for another NP-complete problems (like SAT), we believe there is a lot of work to be done. A possible approach would be the polynomial reduction to SAT, which seems to be closer with the atomic normal form and the probabilistic entailment relation. It would also be interesting to use the concepts we presented here together with linear algebra techniques to explore PSAT, as it can be seen as a linear programming problem.

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