$GL(m|n)$-SUPERMODULES WITH GOOD AND WEYL FILTRATIONS

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Abstract. The purpose of this paper is to prove necessary and sufficient criteria for a $GL(m|n)$-supermodule to have a good or Weyl filtration. We also introduce the notion of a Steinberg supermodule analogous to the classical notion of Steinberg module. We prove that the Steinberg supermodule inherits some properties of the Steinberg module. Some new series of finite-dimensional tilting supermodules are found.

Supergroup, supermodule, Steinberg supermodule, tilting supermodule.

Introduction

Modules with costandard (good) or standard (Weyl) filtrations and tilting modules as well, play a crucial role in the representation theory of finite-dimensional algebras and reductive algebraic groups over an algebraically closed field of positive characteristic. The concept of a tilting module can be adapted to any highest weight category $\mathcal{C}$ in the sense of [4]. It has been proven in [4] that if the partially ordered set of weights (*weight poset*) Λ of such category $\mathcal{C}$ is finite, then the full subcategory consisting of all objects of finite length is equivalent to a category of finite-dimensional modules over a quasi-hereditary algebra (a nice introduction to the theory of quasi-hereditary algebras and tilting modules over them can be found in [6]; see also [8, 20]). In many cases, like Schur algebras or rational representations of reductive algebraic groups, Λ is not finite but any finitely generated ideal Γ of the poset Λ is. In that case, the full subcategory of finite objects belonging to Γ is equivalent to a category of finite-dimensional modules over a quasi-hereditary algebra.

It was proven in [22] that the category $GL(m|n) - \mathsf{Smod}$ of left rational supermodules over the general linear supergroup $GL(m|n)$ is a highest weight category. However, the poset of weights Λ of $GL(m|n)$ does not satisfy the condition that every finitely-generated ideal Γ of Λ is finite. In fact, for every weight $\lambda \in \Lambda$, the interval $\{\mu|\mu \leq \lambda\}$ is infinite.

In [15] it was suggested how to overcome this obstacle. Since a highest weight category is an abelian category of finite type, it can be regarded as a right comodule category over a coalgebra. Equivalently, it can be viewed as a left discrete module category over a *pseudocompact* algebra. If Γ is finitely generated (or finitely cogenerated, respectively) *good* (or *cogood*, respectively) ideal (see Definitions 3.9 and 3.17 of [15]), then the corresponding highest weight category is equivalent to a category of discrete modules over an *ascending* (or *descending*, respectively) quasi-hereditary pseudocompact algebra. The theory of objects with good (*decreasing costandard*) or Weyl (*increasing standard*) filtration has been developed in §4 of [15]. The notion of a tilting object was introduced there and it was proven that the

category of discrete modules over an ascending quasi-hereditary peseudocompact algebra has enough indecomposable tilting objects. However, if Γ is infinite, it is not clear if the tilting objects are finite or not. It is natural to look for a sufficient condition on the highest weight category $\mathcal{C}$ that would guarantee that all of its indecomposable tilting objects are finite.

It makes sense to investigate first the category $GL(m|n)-\mathsf{Smod}$. The costandard and standard objects in $GL(m|n)-\mathsf{Smod}$ are (up to a parity shift) the induced supermodules $H^0(\lambda)$ and the Weyl supermodules $V(\lambda)$, respectively (cf. [22]). If $H^0(\lambda)$ is irreducible, then $H^0(\lambda)=V(\lambda)$ is tilting. Irreducible induced supermodules $H^0(\lambda)$ were characterized in [14]. Until recently, no other examples of tilting supermodules in $GL(m|n)-\mathsf{Smod}$ has been known, except in the case of the supergroup $GL(1|1)$ or in the case when $charK=0$. In both cases every indecomposable tilting supermodule is injective, projective and finite-dimensional; see [3, 15].

In this article we present a series of finite-dimensional (indecomposable) tilting supermodules, which are not irreducible, using the concept of a Steinberg supermodule. We prove that the Steinberg supermodule inherits some properties of its counterpart, a Steinberg module St_r. For example, a Steinberg supermodule remains both projective and injective when regarded as a supermodule over the corresponding Frobenius kernel in $GL(m|n)$.

It was proven in [9] (see also II.10.5.2 (1) of [11]) that, for every (dominant) weight λ, the tensor product $St_r\otimes I(\lambda)^{[r]}$ is isomorphic to $I((p^r-1)\rho+p^r\lambda)$, where $I(\mu)$ is the injective envelope of an irreducible module $L(\mu)$ of the highest weight μ. We prove an analogous statement for Steinberg supermodules. More specifically, the tensor product of the r-th Steinberg supermodule with the r-th *even* Frobenius twist of an indecomposable injective module over $GL(m|n)_{res}$ is both injective and indecomposable. The most difficult part of the proof is to show that this tensor product is injective. To prove it, we use the aforementioned fact that the Steinberg supermodule is injective over the corresponding Frobenius kernel and some spectral sequence arguments (see Lemma 10.9 and Theorem 10.10). Our series of tilting supermodules is obtained in a similar way; they are tensor products of Steinberg supermodules with even Frobenius twists of tilting modules over $GL(m|n)_{res}$. This generalizes Proposition (2.1) from [8] for general linear supergroups.

The paper is organized as follows. In the first seven sections and in the ninth section we give all necessary definitions, notations and derive auxiliary results. In the eighth section we prove two criteria for a $GL(m|n)$-supermodule to have a decreasing good or increasing Weyl filtration (both possibly infinite). The results of this section are interesting on their own and they are used in the last section to describe completely all cases when a symmetric or an exterior (super)power of the standard $GL(m|n)$-supermodule W have a good or Weyl filtration. In the tenth section we introduce the notion of a Steinberg supermodule and prove some of its important properties mentioned earlier.

1. Hopf superalgebras

We follow definitions and notations from [17]. For the convenience of the reader we will recall some of them here. Let A be an (associative) superalgebra. The category of left (or right, respectively) A-supermodules (with ungraded morphisms) is denoted by ${}_A\mathsf{SMod}$ (or by SMod_A, respectively). If A is a supercoalgebra, then

the category of left (or right, respectively) A-supercomodules (with ungraded morphisms) is denoted by ${}^A\mathsf{SMod}$ (or by SMod^A, respectively). These categories are not abelian, but their *underlying even categories* ${}_A\mathsf{Smod}, \mathsf{Smod}_A, {}^A\mathsf{Smod}$, Smod^A, consisting of the same objects but considered only with even morphisms, are.

Let H be a Hopf superalgebra. A Hopf subsuperalgebra R of H is called *right normal* if $\sum(-1)^{|r||h_1|}s_H(h_1)rh_2 \in R$ for all $r \in R$ and $h \in H$ (cf. [16]). Symmetrically, R is called *left normal* if $\sum(-1)^{|r||h_2|}h_1rs_H(h_2) \in R$ for all $r \in R$ and $h \in H$. If R is both left and right normal, then R is called just *normal*. If H is supercocommutative and the antipode s_H is bijective, then the right normality is equivalent to the left normality.

Let M be a left H-supermodule. Denote by M_R a subsuperspace of M that is generated by the elements rm for $m \in M$ and $r \in R^+ = \ker \epsilon_R$.

Lemma 1.1. *If R is left normal in H, then M_R is a H-subsupermodule.*

Proof. For every $r \in R^+$ we have

$$(hr)m = \sum(\sum(-1)^{|r||h_2|}h_1rs_H(h_2))h_3m \in M_R.$$

□

Let R be a finite-dimensional Hopf superalgebra. The dual superspace R^* has a natural structure of Hopf superalgebra given by

$$(\phi\psi)(r) = \sum(-1)^{|\psi||r_1|}\phi(r_1)\psi(r_2), \text{ where } \phi, \psi \in R^*, r \in R \text{ and } \Delta_R(r) = \sum r_1 \otimes r_2;$$

$\Delta_{R^*}(\phi) = \sum \phi_1 \otimes \phi_2$ whenever $\phi(rs) = \sum(-1)^{|\phi_2||r|}\phi_1(r)\phi_2(s)$ for every $r, s \in R$; $s_{R^*}(\phi)(r) = \phi(s_R(r))$, and $\epsilon_{R^*}(\phi) = \phi(1_R)$.

Lemma 1.2. *The functor $\Phi : R \to R^*$ is a self-duality on the category of finite-dimensional Hopf superalgebras.*

Proof. Follow the arguments in I.8(1) of [11]. □

Additionally, if M is a left R-supermodule, then M is a right R^*-supercomodule via the linear map $M \to Hom_K(R, M) \simeq M \otimes R^*$ which maps $m \in M$ to $l_m : R \to M$ given by $l_m(r) = (-1)^{|m||r|}rm$ for $r \in R$. Symmetrically, if M is a right R-supercomodule, then M is a left R^*-supermodule by $\phi m = \sum(-1)^{|\phi||m_1|}\phi(r_2)m_1$, where $\tau_M(m) = \sum m_1 \otimes r_2$, $\phi \in R^*$ and $m \in M$. Furthemore, all of the above statements remain valid after replacing the right coaction by the left action and vice-versa.

The proof of the following lemma is easy and is left for the reader.

Lemma 1.3. *There is an equivalence of categories $\mathsf{SMod}^R \simeq {}_{R^*}\mathsf{SMod}$ that preserves parities of morphisms. Symmetrically, ${}^R\mathsf{SMod} \simeq \mathsf{SMod}_{R^*}$.*

In particular, the category SMod^R has enough projective objects. Thus for any $M, N \in \mathsf{SMod}^R$ we have $Ext^n_{\mathsf{SMod}^R}(M, N) \simeq H^n(Hom_{\mathsf{SMod}^R}(P^\bullet_M, N))$, where $P^\bullet_M \to M \to 0$ is a projective resolution of M (cf. Proposition 8.2 in XX of [12]). Moreover, the resolution $P^\bullet_M$ can be chosen in Smod^R.

2. Algebraic supergroups and their linear representations

Let SAlg_K denote the category of supercommutative K-superalgebras with even morphisms. Let B be a supercommutative Hopf superalgebra. Then the representable functor $A \to Hom_{\mathsf{SAlg}_K}(B, A)$ from SAlg_K to the category of sets is a natural group functor. It is denoted by $G = SSp\ B$ and called an *affine supergroup*. If B is finitely generated, then G is called an *algebraic supergroup*. Any (closed) subsupergroup H of G is uniquely defined by a Hopf superideal I_H of B such that an element $g \in G(A)$ belongs to $H(A)$ if and only if $g(I_H) = 0$. For example, the *largest even subsupergroup* G_{ev} of G corresponds to the ideal BB_1. The restriction of G_{ev} to the full subcategory of commutative K-algebras is denoted by G_{res}.

Let W be a finite-dimensional superspace. The group functor $A \to \mathsf{End}_A(W \otimes A)_0^*$ is an algebraic supergroup. It is called a *general linear supergroup* and denoted by $GL(W)$. If $\dim W_0 = m, \dim W_1 = n$, then $GL(W)$ is also denoted by $GL(m|n)$.

Fix a homogeneous basis of W, say consisting of w_i for $1 \leq i \leq m+n$, where $|w_i| = 0$ provided $1 \leq i \leq m$, and $|w_i| = 1$ otherwise. It is easy to see that $K[GL(m|n)]$ is a localization of $K[c_{ij}|1 \leq i, j \leq m+n]$, where $|c_{ij}| = |w_i| + |w_j|$, by some element d. More precisely, the generic matrix $C = (c_{ij})_{1 \leq i,j \leq m+n}$ has a block form

$$\begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix},$$

where the $m \times m$ and $n \times n$ blocks C_{00} and C_{11} are even, and the $m \times n$ and $n \times m$ blocks C_{01} and C_{10} are odd. Then

$$\Delta_{GL(m|n)}(c_{ij}) = \sum_{1 \leq k \leq m+n} c_{ik} \otimes c_{kj},\ \epsilon_{GL(m|n)}(c_{ij}) = \delta_{ij},$$

and $d = \det(C_{00}) \det(C_{11})$.

The element $Ber(C) = \det(C_{00} - C_{01}C_{11}^{-1}C_{10}) \det(C_{11})^{-1}$ is called the *Berezinian*. It is a group-like element of the Hopf superalgebra $K[GL(m|n)]$ (cf. [1]). It is easy to show that $K[GL(m|n)] = K[c_{ij}|1 \leq i, j \leq m+n]_{Ber(C)}$.

By definition, the category of left (or right, respectively) G-supermodules coincides with the category of right (or left, respectively) $K[G]$-supercomodules. Denote them by $G - \mathsf{SMod}$ and $\mathsf{SMod} - G$ respectively. The corresponding even underlying categories are denoted by $G - \mathsf{Smod}$ and $\mathsf{Smod} - G$, respectively. For example, the right supercomodule structure of $GL(W)$-supermodule W is defined by

$$\tau_W(w_i) = \sum_{1 \leq j \leq m+n} w_j \otimes c_{ji}.$$

There is an endofunctor $M \to \Pi M$, called the *parity shift*, in all above categories such that ΠM coincides with M as a $K[G]$-comodule and $(\Pi M)_i = M_{i+1}$ for $i \in \mathbb{Z}_2$.

Every G_{res}-module can be regarded as a *purely even* G_{ev}-supermodule. Furthemore, $G_{res} - \mathsf{mod}$ is a full subcategory of both $G_{ev} - \mathsf{Smod}$ and $G_{ev} - \mathsf{SMod}$ and $Ext^n_{G_{res}}(M, N) = Ext^n_{G_{ev}}(M, N)$ for every $M, N \in G_{res} - \mathsf{mod}$ and $n \geq 0$.

There is a one-to-one correspondence between G-supermodule structures on a finite-dimensional superspace M and *linear representations* $G \to GL(M)$ (cf. [11, 22]). If $\tau_M(m) = \sum m_1 \otimes f_2 \in M \otimes K[G]$, then $g \in G(A)$ acts on $M \otimes A$ by the even A-linear automorphism $\tau(g)(m \otimes 1) = \sum m_1 \otimes g(f_2)$. In other words, M is a G-supermodule if and only if the group functor G acts on the functor $M_a = M \otimes ?$

in such a way that for every $A \in \mathsf{SAlg}_K$ the group $G(A)$ acts on $M_a(A) = M \otimes A$ by even A-linear automorphisms.

A subsupergroup H is called *(faithfully) exact* in G if the induction functor ind_H^G is (faithfully) exact. The exactness of H is equivalent to the condition that the restriction functor $M \to M|_H$ takes injectives to injectives (cf. [11, 22]). Besides, H is faithfully exact if and only if G/H is an affine superscheme (see Theorem 5.2 of [23]).

In what follows, we use the following criterion for G/H to be an affine superscheme (cf. Corollary 8.15 of [17]). A superscheme G/H is affine if and only if G_{ev}/H_{ev} is affine if and only if G_{res}/H_{res} is affine. For example, if H is a normal or finite subsupergroup of G, then G/H is affine, hence H is faithfully exact in G.

Let B denote a *standard Borel subsupergroup* of $G = GL(W)$ that consists of all lower triangular matrices. Let V be its largest unipotent subsupergroup that consists of all lower triangular matrices with units on their diagonals.

Let T be a maximal torus of B consisting of all diagonal matrices and let $X(T)$ be its *character group*. The group $X(T)$ can be naturally identified with $\mathbb{Z}^{m+n}$ so that an element $\lambda \in X(T)$ has a form $(\lambda_1, \ldots, \lambda_{m+n})$, where $\lambda_i \in \mathbb{Z}$ for $1 \leq i \leq m+n$. Let ϵ_i denote a character

$$(0, \ldots, \underbrace{1}_{i-\text{th place}}, \ldots, 0) \text{ for } 1 \leq i \leq m+n.$$

Let B^{opp} denote the transpose of B, that is B^{opp} is a Borel subsupergroup consisting of all upper triangulat matrices. Then the transpose V^{opp} of V is the largest unipotent subsupergroup of B^{opp}.

Any simple B-supermodule has the form K_λ or ΠK_λ for some $\lambda \in X(T)$, where the $K[B]$-comodule structure of K_λ is given by $1 \mapsto 1 \otimes \prod_{1 \leq i \leq m+n} c_{ii}^{\lambda_i}$. The *induced supermodule* $ind_B^G \Pi^a K_\lambda$ is denoted by $H^0(\lambda^a)$ for $a = 0, 1$. Observe that $H^0(\lambda^a) = \Pi^a H^0(\lambda) \neq 0$ if and only if λ is a *dominant* weight, that is $\lambda_1 \geq \ldots \geq \lambda_m, \lambda_{m+1} \geq \ldots \geq \lambda_{m+n}$. If it is the case, then $H^0(\lambda^a)$ has a simple socle $L(\lambda^a) = \Pi^a L(\lambda)$ and any irreducible $GL(W)$-supermodule is isomorphic to the socle of some $H^0(\lambda^a)$. The set of dominant weights is denoted by $X(T)^+$.

As it has been shown in [22], the even category of $GL(W)$-supermodules is a highest weight category with $H^0(\lambda^a)$ as costandard objects, subject to a *Bruhat-Tits order* such that $\mu \leq \lambda$ if and only if

$$\sum_{1 \leq i \leq k} \mu_i \leq \sum_{1 \leq i \leq k} \lambda_i \text{ for } 1 \leq i \leq m+n-1, \text{ and } \sum_{1 \leq i \leq m+n} \mu_i = \sum_{1 \leq i \leq m+n} \lambda_i$$

if and only if

$$\lambda - \mu \in \sum_{1 \leq i < j \leq m+n} \mathbb{Z}_{\geq 0}(\epsilon_i - \epsilon_j).$$

The standard object of the highest weight λ in $GL(W) - \mathsf{Smod}$ is denoted by $V(\lambda)$.

Let Γ be a finitely generated ideal in $X(T)^+$. We say that a $GL(W)$-supermodule M belongs to Γ whenever any composition factor of M is isomorphic to some $L(\lambda^a)$ with $\lambda \in \Gamma$. For any $GL(W)$-supermodule N there exists a unique maximal subsupermodule of N which belongs to Γ. It is denoted by $O_\Gamma(N)$. Also, there exists a unique minimal subsupermodule N' of N such that N/N' belongs to Γ. The subsupermodule N' is denoted by $O^\Gamma(N)$. For notions of (decreasing) good and (increasing) Weyl filtrations we refer the reader to §4 of [15].

Let $H_{ev}(\lambda)$ and $V_{ev}(\lambda)$ denote costandard and standard objects in the highest weight category $GL(W)_{res} - \mathsf{mod}$, respectively (subject to the same partial order). Then the category $GL(W)_{ev} - \mathsf{Smod}$ is also a highest weight category with costandard objects $\Pi^a H_{ev}(\lambda)$ and standard objects $\Pi^a V_{ev}(\lambda)$ for $a = 0, 1$.

3. Superalgebras of distributions

Let G be an algebraic supergroup and $Dist(G)$ be the *superalgebra of distributions* of G. As a superspace $Dist(G)$ coincides with $\bigcup_{k\geq 0} Dist_k(G) \subseteq K[G]^*$, where $Dist_k(G) = (K[G]/\mathfrak{m}^{k+1})^*$ and $\mathfrak{m} = \ker \epsilon_{K[G]} = K[G]^+$. $Dist(G)$ is a supercocommutative Hopf superalgebra with a bijective antipode (see [11, 23] for more details). For example, the comultiplication $\Delta_{Dist(G)}$ maps an element $\phi \in Dist_k(G)$ to $\sum \phi_1 \otimes \phi_2 \in Dist_k(G)^{\otimes 2}$ if and only if

$$\phi(fh) = \sum (-1)^{|\phi_2||f|} \phi_1(f)\phi_2(h)$$

for every $f, h \in K[G]$.

If H is a subsupergroup of G, then $Dist(H)$ is a Hopf subsuperalgebra of $Dist(G)$. More precisely, $\phi \in Dist(G)$ belongs to $Dist(H)$ if and only if $\phi(I_H) = 0$. Furthemore, if H is a normal subsupergroup of G, then for every $f \in I_N$ we have $\sum (-1)^{|f_1||f_2|} f_2 \otimes s_G(f_1) f_3 \in I_N \otimes K[G]$ (cf. p. 731 of [23]). Therefore $Dist(H)$ is normal in $Dist(G)$. If G is *connected* (*pseudoconnected* in the terminology of [23]; see also §3 of [10]), then the converse statement is also valid (follow arguments in the proof of Lemma 5.1 of [10]).

4. Unipotent and finite supergroups

An algebraic supergroup G is called *unipotent* if every simple G-supermodule M is trivial, that is $M \simeq \Pi^a K$ for $a = 0, 1$ (cf. [24, 16]).

We refer the reader to [5] for more detailed introduction to the theory of finite or infinitesimal algebraic supergroups. A finite supergroup G is called *infinitesimal* if $K[G]^+$ is nilpotent. Therefore $Dist(G) = K[G]^*$ and, by Lemma 1.3, the category $G - \mathsf{SMod}$ is equivalent to the category ${}_{Dist(G)}\mathsf{SMod}$. The functor $F : G - \mathsf{SMod} \to \mathsf{SMod}_K$ such that $F(M) = M/M_{Dist(G)}$ is right exact. Denote its n-th left derived functor $L_nF(M)$ by $H_n(G, M)$. In other words, $H_n(G, M) = H_n(F(P^\bullet_M))$ for any projective resolution $P^\bullet_M$ of M. Theorem 7.1' from chapter XX of [12] implies that $Ext^n_G(M, K)^* \simeq H_n(G, M)$ for every $n \geq 0$.

Lemma 4.1. *If G is unipotent, then every injective G-supermodule is isomorphic to $K[G]^{\oplus I} \bigoplus \Pi K[G]^{\oplus J}$ for some (possibly infinite) index sets I and J.*

If G is also infinitesimal, then every projective G-supermodule is isomorphic to $Dist(G)^{\oplus I} \bigoplus \Pi Dist(G)^{\oplus J}$.

Proof. Every injective G-supermodule is a direct sum of indecomposable injective hulls of simple supermodules. On the other hand, the socle of a G-supermodule M is equal to M^G. Since $K[G]^G = K$, up to an isomorphism, $K[G]$ and $\Pi K[G]$ are unique indecomposable injectives. If G is infinitesimal and P is a projective G-supermodule, then $radP = Dist(G)^+ P$. Since $Dist(G)/Dist(G)^+ = K$, the claim follows. □

Lemma 4.2. *If G is unipotent, then a G-supermodule M is injective if and only if $H^1(G, M) = 0$. If G is also infinitesimal, then a G-supermodule M is projective if*

and only if $H_1(G,M)=0$. Moreover, if M is injective (or projective, respectively), then $H^n(G,M)=0$ (or $H_n(G,M)=0$, respectively) for every $n\geq 1$.

Proof. $Ext^1_G(K,M)=0$ implies $Ext^1_G(N,M)=0$ for every finite-dimensional G-supermodule N. Following arguments in the proof of Proposition 3.4 in [22], we obtain that $Ext^1_G(N,M)=0$ for every N. The proof of the second statement is similar. □

Lemma 4.3. *Let $N\trianglelefteq H$ and H be unipotent. Then an H-supermodule M is injective if and only if both the H/N-supermodule M^N and the N-supermodule $M|_N$ are injective.*

Proof. If $M|_N$ is an injective N-supermodule, then the spectral sequence (cf. Proposition 3.1(3) of [18])

$$E_2^{n,m}=H^n(H/N,H^m(N,M))\Rightarrow H^{n+m}(H,M)$$

degenerates, which yields $H^n(H/N,M^N)\simeq H^n(H,M)$ for every $n\geq 1$. Lemma 4.2 implies that M is injective if and only if M^N is an injective H/N-supermodule. On the other hand, if M is injective, then so is $M|_N$. □

Every superalgebra A is a $\mathbb{Z}_2$-module, where the generator of $\mathbb{Z}_2$ acts on A as $a\to(-1)^{|a|}a$ for $a\in A$. The *semi-direct product* algebra $A\rtimes\mathbb{Z}_2$ is isomorphic to $A_0\bigoplus A_1$, where each component A_i coincides with A as a vector space for $i\in\mathbb{Z}_2$. Besides, $a_ia'_j=(-1)^{i|a'|}(aa')_{i+j}$ for $a_i\in A_i, a'_j\in A_j$ and $i,j\in\mathbb{Z}_2$. Additionally, if A is a (not necessarily supercommutative) Hopf superalgebra, then $A\rtimes\mathbb{Z}_2$ is a Hopf algebra with the comultiplication

$$\Delta(a_i)=\sum(a_1)_{i+|a_2|}\otimes(a_2)_i,$$

the counit $\epsilon(a_i)=\epsilon_A(a)$ and the antipode $s(a_i)=(-1)^{(i+|a|)|a|}s_A(a)_{i+|a|}$.

Lemma 4.4. *Let H be an infinitesimal supergroup. Then a H-supermodule M is injective if and only if M is projective.*

Proof. By Lemma 1.3, $H-\mathsf{SMod}\simeq_{Dist(H)}\mathsf{SMod}$. Since $Dist(H)\rtimes\mathbb{Z}_2$ is Frobenius by the main theorem from [19] and ${}_{Dist(H)}\mathsf{SMod}\simeq_{Dist(H)\rtimes\mathbb{Z}_2}\mathsf{Mod}$ by Lemma 7.6 from [15], the statement follows. □

5. Frobenius kernels

Assume that K is a perfect field and V is a K-superspace. Denote by $V^{(r)}$ the superspace that coincides with V as a $\mathbb{Z}_2$-graded abelian group, but on which each $a\in K$ acts as $a^{p^{-r}}$ does on V. If A is a (not necessarily (super)commutative) (super)algebra, then $A^{(r)}$ is also a (super)algebra with respect to the same multiplication. Moreover, if A is a Hopf (super)algebra, then $A^{(r)}$ is a Hopf (super)algebra with $\Delta_{A^{(r)}}=\Delta_A, s_{A^{(r)}}=s_A$ and $\epsilon_{A^{(r)}}=\epsilon_A^{p^r}$. From now on we, assume that K is perfect unless stated otherwise.

Let G be an algebraic supergroup. Assume that G is *reduced*, that is G_{res} is an reduced algebraic group. For every $r\geq 1$ there is an exact sequence

$$1\to G_r\to G\to G^{(r)}_{ev}\to 1,$$

where the epimorphism $G\to G^{(r)}_{ev}$ is induced by the Hopf superalgebra embedding $K[G^{(r)}_{ev}]=K[G_{ev}]^{(r)}\to K[G]$ given by the rule $\overline{f}\to f^{p^r}$, where $\overline{f}$ is the residue class

of $f \in K[G]_0$ in $K[G_{ev}] = K[G]_0/K[G]_1^2$ (see §8 of [23]). The subsupergroup G_r is called the *r-th Frobenius kernel* of G. Since $I_{G_r} = \sum_{f \in K[G]_0} K[G]f^{p^r}$, each G_r is a (finite) infinitesimal subsupergroup in G. Lemma 1.3 shows that the category of G_r-supermodules is naturally equivalent to the category of $Dist(G_r)$-supermodules.

Let $G = GL(W)$. A simple G_r-supermodule is isomorphic to the top $L_r(\lambda)$ of $Z_r(\lambda) = Dist(G_r) \otimes_{Dist(B_r^{opp})} K_\lambda$ or to its parity shift $\Pi L_r(\lambda)$ (cf. Theorem 5.4 of [13]). Moreover, $L_r(\lambda) \simeq L_r(\mu)$ if and only if $\lambda - \mu \in p^r X(T)$.

A simple G_r-supermodule $L_r(\lambda)$ is also isomorphic to the socle of $Z_r'(\lambda) = ind_{B_r}^{G_r} K_\lambda$. In fact, there is a natural superscheme isomorphism $B_r \times V_r^{opp} \to G_r$ (see Lemma 14.1 of [25]). By Lemma 8.2 and Remark 8.3 of [25], $Z_r'(\lambda)|_{B_r^{opp}} \simeq K_\lambda \otimes K[V_r^{opp}]$, where T acts on $K[V_r^{opp}]$ by conjugations and V_r^{opp} acts trivially on K_λ. Thus $Z_r'(\lambda)^{V_r^{opp}} \simeq K_\lambda$ as a T_r-(super)module, hence the socle of $Z_r'(\lambda)$ is irreducible. Let $L_r'(\lambda)$ denotes the socle of $Z_r'(\lambda)$. Since $L_r'(\lambda)^{V_r^{opp}} \simeq K_\lambda$, (co)Frobenius reciprocity law implies

$$Hom_{G_r}(Z_r(\lambda), L_r'(\lambda)) \simeq Hom_{B_r^{opp}}(K_\lambda, L_r'(\lambda)) = K;$$

that is $L_r(\lambda) \simeq L_r'(\lambda)$.

Since $Dist(G) = Dist(G_{ev})Dist(G_r)$, one can mimic the proofs of Proposition 4.3 and Proposition 4.4 of [21] to show that a simple G-supermodule $L(\lambda)$ of highest weight $\lambda \in X(T)^+$ is completely reducible as a G_r-supermodule. Furthemore, it is irreducible as G_r-supermodule whenever $\lambda \in X_r(T)^+$, where

$$X_r(T)^+ = \{\lambda \in X(T) | 0 \leq \lambda_i - \lambda_{i+1} < p^r \text{ for } 1 \leq i \leq m+n-1 \text{ such that } i \neq m\}.$$

It is easy to see that for every weight $\lambda \in X(T)$ there are $r \geq 1$ and $\mu \in X_r(T)^+$ such that $\lambda - \mu \in p^r X(T)$. Therefore, $L_r(\lambda) \simeq L(\mu)|_{G_r}$.

6. Frobenius twists

Let G be a reduced algebraic supergroup defined over $\mathbb{F}_p$. In other words, $K[G] = A \otimes_{\mathbb{F}_p} K$, where A is a Hopf $\mathbb{F}_p$-superalgebra. Since $K[G_{ev}] \simeq A/AA_1 \otimes_{\mathbb{F}_p} K$, G_{ev} and G_{res} are also defined over $\mathbb{F}_p$. There is a natural Hopf superalgebra endomorphism $\pi_r : K[G] \to K[G]$ given by $f \otimes b \to f^{p^r} \otimes b$ for $f \in A$ and $b \in K$.

Let M be a left G-supermodule. Then one can define a new representation of G on M, denoted by $M^{[r]}$, by $\tau_{M^{[r]}}(m) = (id_M \otimes \pi_r)\tau_M$ (cf. 9.10, Part I of [11]). We call $M^{[r]}$ the r-th *Frobenius twist* of M.

It is evident that G_r acts on $M^{[r]}$ trivially. Since $G/G_r \simeq G_{ev,r}$, $M^{[r]}$ has the natural structure of a $G_{ev,r}$-supermodule via the Hopf (super)algebra isomorphism $K[G_{ev}]^{(r)} \simeq A_0^{p^r} \otimes_{\mathbb{F}_p} K \subseteq K[G]$ given by $\overline{f} \otimes b \to f^{p^r} \otimes b^{p^r}$, where $\overline{f}$ is the residue class of $f \in A_0$ in A_0/A_1^2 and $b \in K$.

Let M be a G_{res}-module. It has been observed that M is also a G_{ev}-supermodule. The supergroups G_{ev} and $G_{ev}^{(r)}$ are isomorphic to each other (cf. Remark I.9.5 of [11]). More precisely, the corresponding Hopf superalgebra isomorphism $K[G_{ev}] \to K[G_{ev}^{(r)}]$ is defined as $h \otimes b \to h \otimes b^{p^{-r}}$ for $h \in A/AA_1$ and $b \in K$. In particular, M has a natural G-supermodule structure via the epimorphism $G \to G_{ev}^{(r)}$ (see Remark 8.4 later). This structure is defined as

$$\tau'(m) = \sum m_1 \otimes f_2^{p^r} \otimes b_2,$$

where $\tau_M(m) = \sum m_1 \otimes \overline{f_2} \otimes b_2$, $f_2 \in A_0$ and $b_2 \in K$. We call such a G-supermodule the r-th *even Frobenius twist* of M and by abuse of notation we also denote it by $M^{[r]}$.

Remark 6.1. *The concepts of the even Frobenius twist has been introduced in* [13]*, where it has been called just the Frobenius twist (see also* [5]*). Since the concept of the Frobenius twist makes sense for any supermodule, we decided to distinguish these (obviously different) constructions by their names. Observe also that if M is a G_{res}-module, then the r-th Frobenius twist of G_{ev}-supermodule M coincides with its r-th even Frobenius twist, restricted to G_{ev}.*

Lemma 6.2. *If M is a G_{res}-module, then $End_G(M^{[r]}) = End_{G_{res}}(M)$.*

Proof. Consider a basis $m_1, \dots, m_k$ of M. Then

$$\tau_M(m_i) = \sum_{1 \le j \le k} m_j \otimes \overline{f_{ji}} \text{ for } f_{ji} \in A_0 \otimes_{\mathbb{F}_p} K,$$

and

$$\tau_{M^{[r]}}(m_i) = \sum_{1 \le j \le k} m_j \otimes \pi_r(f_{ji}) \text{ for } 1 \le i \le k.$$

An endomorphism $\phi \in End_K(M)$ given by

$$\phi(m_i) = \sum_{1 \le j \le k} \phi_{ji} m_j \text{ for } 1 \le i \le k$$

belongs to $End_G(M^{[r]})$ if and only if $F\Phi = \Phi F$, where

$$F = \begin{pmatrix} \pi_r(f_{11}) & \dots & \pi_r(f_{1k}) \\ \vdots & \vdots & \vdots \\ \pi_r(f_{k1}) & \dots & \pi_r(f_{kk}) \end{pmatrix} \text{ and } \Phi = \begin{pmatrix} \phi_{11} & \dots & \phi_{1k} \\ \vdots & \vdots & \vdots \\ \phi_{k1} & \dots & \phi_{kk} \end{pmatrix}.$$

Since π_r induces an algebra isomorphism $K[G_{res}] = K[G]_0/K[G]_1^2 \to A_0^{p^r} \otimes_{\mathbb{F}_p} K$, the latter condition is equivalent to $\overline{F}\Phi = \Phi\overline{F}$, where

$$F = \begin{pmatrix} \overline{f_{11}} & \dots & \overline{f_{1k}} \\ \vdots & \vdots & \vdots \\ \overline{f_{k1}} & \dots & \overline{f_{kk}} \end{pmatrix}.$$

□

7. Dualities

Let G be an algebraic supergroup and σ be an anti-automorphism of the Hopf superalgebra $K[G]$. If M is a finite-dimensional G-supermodule, then one can define its *σ-dual* $M^{<\sigma>}$ as follows (cf. [22]). Fix a homogeneous basis of M consisting of elements m_i for $1 \le i \le t$. If $\tau_M(m_i) = \sum_{1 \le k \le t} m_k \otimes f_{ki}$, where $f_{ki} \in K[G]$ and $|f_{ki}| = |m_i| + |m_k| \pmod 2$, then the G-supermodule $M^{<\sigma>}$ has a basis consisting of elements $m_i^{<\sigma>}$ such that

$$\tau_{M^{<\sigma>}}(m_i^{<\sigma>}) = \sum_{1 \le k \le t} m_k^{<\sigma>} \otimes (-1)^{|m_k|(|m_i|+|m_k|)} \sigma(f_{ik}).$$

The functor $M \to M^{<\sigma>}$ is a self-duality of the full subcategory of all finite-dimensional supermodules.

Moreover, σ induces an anti-automorphism of $Dist(G)$ by $\phi \mapsto \phi \cdot \sigma = \phi^{<\sigma>}$ for $\phi \in Dits(G)$. In other words, $(\phi\psi)^{<\sigma>} = (-1)^{|\phi||\psi|}\psi^{<\sigma>}\phi^{<\sigma>}$ for every $\phi, \psi \in Dist(G)$.

Lemma 7.1. *$M^{<\sigma>}$ can be identified with the dual superspace M^* on which $Dist(G)$ acts by $(\phi f)(v) = (-1)^{|\phi||f|}f(\phi^{<\sigma>}m)$ for $f \in M^*$ and $m \in M$.*

Proof. Let m_i^* form a dual basis of M^*. The required isomorphism $M^* \to M^{<\sigma>}$ is defined by $m_i^* \mapsto m_i^{<\sigma>}$.

□

Remark 7.2. *If $\sigma = s_{K[G]}$, then $M^{<\sigma_G>}$ coincides with the dual G-supermodule M^*. In particular, $Dist(G)$ acts on M^* by $(\phi f)(m) = (-1)^{|\phi||f|}f(s_{Dist(G)}(\phi)m)$.*

Assume $G = GL(m|n)$. Then the map $t : c_{ij} \mapsto (-1)^{|i|(|i|+|j|)}c_{ji}$ induces an anti-automorphism of the Hopf superalgebra $K[G]$. Furthemore, this anti-automorphism induces a self-duality $M \mapsto M^{<t>}$ of the full subcategory of all finite-dimensional G-supermodules. For example, $H^0(\lambda)^{<t>} = V(\lambda)$ (see [22] for more details).

The related anti-automorphism $\phi \mapsto \phi^{<t>}$ of the superalgebra $Dist(G)$ is defined by $(e_{ij}^{(l)})^{<t>} = e_{ji}^{(l)}$ if e_{ij} is even, and $e_{ij}^{<t>} = (-1)^{|j|(|i|+|j|)}e_{ji}$ otherwise.

8. Good and Weyl filtrations

Let N be a normal subsupergroup of an algebraic supergroup G and M be a finite-dimensional G-supermodule. Denote $M_{Dist(N)}$ by M_N.

Lemma 8.1. *If G is connected, then M_N is a G-subsupermodule.*

Proof. Combine Lemma 1.1 and Lemma 9.4 of [23]. □

Corollary 8.2. *The statement of Lemma 8.1 holds for every G-supermodule M.*

Proof. The supermodule M coincides with a union of its finite-dimensional subsupermodules M_α, where α runs over a directed set I such that $M_\alpha \subseteq M_\beta$ if and only if $\alpha \leq \beta$. We have $M_N = \bigcup_{\alpha \in I}(M_\alpha)_N$. By Lemma 8.1, each $(M_\alpha)_N$ is a G-subsupermodule. Thus M_N is a G-subsupermodule. □

Remark 8.3. *M_N is the smallest subsupermodule of G such that N acts identically on M/M_N.*

Remark 8.4. *Let $\pi : G \to H$ be an epimorphism of algebraic supergroups. Then every H-supermodule M can be regarded as a G-supermodule via π. In* [9] *this G-supermodule is denoted by $\pi_0(M)$. Additionally, if $G = N \rtimes H$, then $\pi_0(M)|_H \simeq M$.*

From now on assume that $G = GL(m|n)$ and $P = Stab_G(V_1)$. Let U denote the kernel of $P \to G_{ev}$. Then $P = U \rtimes G_{ev}$ (cf. Remark 5.2 of [22]). Symmetrically, denote $Stab_G(V_0)$ by P^{opp}. As above, we have an epimorphism $P^{opp} \to G_{ev}$ and its kernel is denoted by U^{opp}. Besides, $P^{opp} = U^{opp} \rtimes G_{ev}$. Both supergroups U and U^{opp} are obviously infinitesimal and unipotent.

A standard supermodule $V(\lambda)$ can be also defined as a universal supermodule of the highest weight λ. In other words, $V(\lambda)$ is generated by a *B^{opp}-primitive vector* of weight λ and if a G-supermodule M is generated by a B^{opp}-primitive vector of weight λ, then there is an epimorphism $V(\lambda) \to M$.

By Remark 8.4, a G_{ev}-(super)module N can be regarded as a P^{opp}-supermodule as well. By Corollary 3.5 of [2], $Dist(G) \otimes_{Dist(P^{opp})} N$ is a G-supermodule.

Lemma 8.5. *For every dominant weight λ, the Weyl supermodule $V(\lambda)$ is isomorphic to $Dist(G) \otimes_{Dist(P^{opp})} V_{ev}(\lambda)$.*

Proof. Denote a G-supermodule $Dist(G) \otimes_{Dist(P^{opp})} V_{ev}(\lambda)$ by M. The formal character of M coincides with the formal character of $V(\lambda)$ (cf. Proposition 5.9 and Theorem 5.4 of [22]). Since M is generated by a primitive vector of weight λ, it is an epimorphic image of $V(\lambda)$. Therefore $M \simeq V(\lambda)$. □

Proposition 8.6. *For each G-supermodule M and each G_{ev}-supemodule N there is the following spectral sequence*

$$E_2^{n,m} = Ext^n_{G_{ev}}(N, H^m(U^{opp}, M)) \Rightarrow Ext^{n+m}_G(Dist(G) \otimes_{Dist(P^{opp})} N, M).$$

Proof. Using (co)Frobenius reciprocity law we have

$$Hom_G(Dist(G) \otimes_{Dist(P^{opp})} N, M) \simeq Hom_{P^{opp}}(N, M|_{P^{opp}}).$$

Let $I^\bullet$ be an injective resolution of G-supermodule M. Since $P^{opp}_{ev} = G_{ev}$, G/P^{opp} is an affine superscheme, hence P^{opp} is exact (even faithfully exact) in G (see also Proposition 5.1 of [22]). Thus $I^\bullet|_{P^{opp}}$ is an injective resolution of P_{opp}-supermodule $M|_{P^{opp}}$ such that the complexes $Hom_G(Dist(G) \otimes_{Dist(P^{opp})} N, I^\bullet)$ and $Hom_{P^{opp}}(N, I^\bullet|_{P^{opp}})$ are isomorphic to each other. Then

$$Ext^i_G(Dist(G) \otimes_{Dist(P^{opp})} N, M) \simeq Ext^i_{P^{opp}}(N, M|_{P^{opp}}).$$

The statement now follows from Proposition 3.1 (2) of [18]. □

Remark 8.7. *If $M|_{U^{opp}}$ is an injective U^{opp}-supermodule, then the above spectral sequence degenerates and there is an isomorphism*

$$Ext^n_{G_{ev}}(N, M^{U^{opp}}) \simeq Ext^n_G(Dist(G) \otimes_{Dist(P^{opp})} N, M)$$

for all $n \geq 0$.

By Lemma 5.1 of [22], for every P-supermodule M there is an isomorphism of U^{opp}-supermodules $M \otimes K[U^{opp}] \to ind^G_P M$. Moreover, it is also an isomorphism of G_{ev}-(super)modules, where G_{ev} acts on $K[U^{opp}]$ by conjugations. In particular, $ind^G_P M$ is an injective U^{opp}-supermodule and the G_{ev}-supermodule $(ind^G_P M)^{U^{opp}}$ is isomorphic to $M|_{G_{ev}}$.

For example, Lemma 5.2 from [22] states $H^0(\lambda) \simeq ind^G_P H^0_{ev}(\lambda)$. Thus $H^0(\lambda)$ is an injective U^{opp}-supermodule and $H^0(\lambda)^{U^{opp}} \simeq H^0_{ev}(\lambda)$. These elementary observations inspire the following theorem.

Let Γ be a finitely generated ideal of (dominant) weights and M be a Γ-restricted G-supermodule (cf. [15]).

Theorem 8.8. *M has a decreasing good filtration if and only if G_{ev}-supermodule (or G_{res}-module) $M^{U^{opp}}$ has a decreasing good filtration and $H^1(U^{opp}, M) = 0$.*

Proof. Apply Proposition 8.6 to $N = V_{ev}(\lambda)$ and obtain the five-term exact sequence (cf. p.50 of [11]):

$$0 \to Ext^1_{G_{ev}}(V_{ev}(\lambda), M^{U^{opp}}) \to Ext^1_G(V(\lambda), M) \to Hom_{G_{ev}}(V_{ev}(\lambda), H^1(U^{opp}, M)) \to$$
$$\to Ext^2_{G_{ev}}(V_{ev}(\lambda), M^{U^{opp}}) \to Ext^2_G(V(\lambda), M).$$

By Theorem 4.9 and Lemma 4.19 of [15], M has a decreasing good filtration if and only if $Ext^1_G(V(\lambda), M) = 0$ for every (dominant) weight λ. Furthemore, in this case $Ext^i_G(V(\lambda), M) = 0$ for every $i \geq 1$. Thus $Ext^1_{G_{ev}}(V_{ev}(\lambda), M^{U^{opp}}) = 0$,

and $M^{U^{opp}}$ has a decreasing good filtration as a G_{ev}-supermodule, provided M has a decreasing good filtration as a G-supermodule. The latter condition implies $Ext^2_{G_{ev}}(V_{ev}(\lambda), M^{U^{opp}}) = 0$ and $Hom_{G_{ev}}(V_{ev}(\lambda), H^1(U^{opp}, M)) = 0$ for every λ. On the other hand, if $H^1(U^{opp}, M) \neq 0$, then there is a weight λ such that $L_{ev}(\lambda)$ is a direct summand of $soc(H^1(U^{opp}, M))$. Thus $Hom_{G_{ev}}(V_{ev}(\lambda), H^1(U^{opp}, M)) \neq 0$, which is a contradiction. Conversely, if M satisfies the conditions of this theorem, then the first and the third member of the above exact sequence are zeroes, hence $Ext^1_G(M, V(\lambda)) = 0$. □

Remark 8.9. *By Lemmas 4.1 and 4.2, $M|_{U^{opp}}$ is injective and isomorphic to $K[U^{opp}]^{\oplus I} \bigoplus \Pi K[U^{opp}]^{\oplus J}$ for (possibly infinite) index sets I and J. If M is finite-dimensional, then the cardinalities $I^\sharp$ and $J^\sharp$ of I and J are given as*

$$I^\sharp = \sum_{\lambda,(M:H^0(\lambda^0))\neq 0} \dim H^0_{ev}(\lambda) \text{ and } J^\sharp = \sum_{\lambda,(M:H^0(\lambda^1))\neq 0} \dim H^0_{ev}(\lambda).$$

In fact, if M has a good filtration

$$0 \subseteq M_1 \subseteq \ldots \subseteq M_s = M$$

such that $M_i/M_{i-1} \simeq H^0(\lambda_i^{a_i})$, where $a_i = 0, 1$ and $1 \leq i \leq s$, then $M^{U^{opp}}$ has a good filtration

$$0 \subseteq M_1^{U^{opp}} \subseteq \ldots \subseteq M_s^{U^{opp}} = M^{U^{opp}}$$

such that $M_i^{U^{opp}}/M_{i-1}^{U^{opp}} \simeq H^0_{ev}(\lambda_i^{a_i})$, where $1 \leq i \leq s$. Besides, $M \simeq \bigoplus_{1\leq i\leq s} M_i/M_{i-1}$ as U^{opp}-supermodule.

Observe that the anti-automorphism $\phi \to \phi^{<t>}$ of $Dist(G)$ induces an anti-isomorphism between subsuperalgebras $Dist(U)$ and $Dist(U^{opp})$. Moreover, if M is a $Dist(U)$-supermodule, then the dual superspace M^* has a natural structure of a $Dist(U^{opp})$-supermodule given by

$$(\phi f)(m) = (-1)^{|\phi||f|}(\phi^{<t>} m), \text{ where } \phi \in Dist(U^{opp}), f \in V^* \text{ and } m \in M.$$

Symmetrically, if M is a $Dist(U^{opp})$-supermodule, then M^* has a structure of $Dist(U)$-supermodule via the same rule. We denote M^* by $M^{<t>}$, no matter over which superalgebra the supermodule M is defined. The following lemma is now obvious.

Lemma 8.10. *The functor $M \mapsto M^{<t>}$ is an anti-equivalence between the categories of finite-dimensional U-supermodules and U^{opp}-supermodules.*

Proof. Since $Dist(U) = K[U]^*$ and $Dist(U^{opp}) = K[U^{opp}]^*$, the result follows from Lemma 1.3. □

Observe that if M is a G-supermodule, then $M^{<t>}|_U = (M|_{U^{opp}})^{<t>}$, and symmetrically, $M^{<t>}|_{U^{opp}} = (M|_U)^{<t>}$.

Proposition 8.11. *Let M be a finite-dimensional U-supermodule. Then for every $k \geq 0$ there is a natural isomorphism of superspaces $H_k(U, M)^{<t>} \simeq H^k(U^{opp}, M^{<t>})$. Moreover, if M is a G-supermodule, then it is an isomorphism of G_{ev}-supermodules.*

Proof. An element f belongs to $(M^{<t>})^{U^{opp}}$ if and only if for every $\phi \in Dist(U^{opp})^+$ one has $\phi f = 0$; meaning that for every $m \in M$ one has $f(\phi^{<t>} m) = 0$. The last condition is equivalent to $f \in (M/M_U)^{<t>}$. Since the functor $V \mapsto V^{<t>}$ maps projective resolutions to injective resolutions, both statements follow. □

Theorem 8.12. *A finite-dimensional G-supermodule M has a Weyl filtration if and only if G_{ev}-supermodule (or G_{res}-module) M/M_U has a Weyl filtration and $H_1(U, M) = 0$.*

Proof. Apply Theorem 8.8 to the G-supermodule $M^{<t>}$ and refer to Proposition 8.11. □

Remark 8.13. *By Lemmas 4.1 and 4.2, $M|_U$ is projective and isomorphic to $Dist(U)^{\oplus I} \bigoplus \Pi Dist(U)^{\oplus J}$. As in Remark 8.9, if M has a Weyl filtration*

$$0 \subseteq M_1 \subseteq \ldots \subseteq M_s = M$$

such that $M_i/M_{i-1} \simeq V(\lambda_i^{a_i})$ for $a_i = 0, 1$ and $1 \leq i \leq s$, then M/M_U has a Weyl filtration

$$0 \subseteq M_1/(M_1)_U \subseteq \ldots \subseteq M_s/(M_s)_U = M/M_U$$

such that $M_i/(M_i)_U/M_{i-1}/(M_{i-1})_U \simeq (M_i/M_{i-1})/(M_i/M_{i-1})_U = V_{ev}(\lambda_i^{a_i})$ for $1 \leq i \leq s$. Additionally, $M \simeq \bigoplus_{1\leq i\leq s} M_i/M_{i-1}$ as an U-supermodule. Therefore

$$I^\sharp = \sum_{\lambda,(M:V(\lambda^0))\neq 0} \dim V_{ev}(\lambda) \text{ and } J^\sharp = \sum_{\lambda,(M:V(\lambda^1))\neq 0} \dim V_{ev}(\lambda).$$

Corresponding to a decreasing chain of finitely generated ideals

$$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \ldots$$

such that $\Gamma \setminus \Gamma_k$ is finite for every $k \geq 0$ and $\bigcap_{k\geq 0} \Gamma_k = \emptyset$, there is an increasing chain of finite-dimensional subsupermodules

$$0 \subseteq O^{\Gamma_1}(M) \subseteq O^{\Gamma_2}(M) \subseteq \ldots.$$

By Theorem 4.11 of [15], M has an increasing Weyl filtration if and only if each $M_k = O^{\Gamma_k}(M)$ has an increasing Weyl filtration if and only if each $M_k^{<t>}$ has a good filtration.

Corollary 8.14. *M has an increasing Weyl filtration if and only if for every $k \geq 1$ $H_1(U, M_k) = 0$ and $M_k/(M_k)_U$ has a Weyl filtration as a G_{ev}-supermodule (or as a G_{res}-module).*

Analogously as in Remark 8.13, we derive that $M \simeq \bigoplus_{k\geq 1} M_k/M_{k-1}$ is a projective U-supermodule. Moreover, M/M_U has an increasing Weyl filtration consisting of submodules $M_k/(M_k)_U$.

Question 8.15. *Assume that M is projective as U-supermodule and G_{ev}-supermodule M/M_U has an increasing Weyl filtration. Does it imply that M has an increasing Weyl filtration as a G-supermodule?*

We have proved the following theorem.

Theorem 8.16. *A restricted G-supermodule M is tilting if and only if the following two conditions are satisfied.*

1. *The G_{ev}-supermodule $M^{U^{opp}}$ has a decreasing good filtration and $M|_{U^{opp}}$ is injective;*
2. *For any decreasing chain of finitely generated ideals*

   $$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \ldots$$

   such that $\Gamma \setminus \Gamma_k$ is finite for every $k \geq 0$ and $\bigcap_{k\geq 0} \Gamma_k = \emptyset$, each $M_k = O^{\Gamma_k}(M)$ is projective as U-supermodule and $M_k/(M_k)_U$ has a Weyl filtration as a G_{ev}-supermodule.

9. Representations of G_rT and G_rB

Following [11], we denote $ind_B^{G_rB}K_\lambda^a$ and $Dist(G_rB^{opp})\otimes_{Dist(B^{opp})}K_\lambda^a$ by $\hat{Z}_r'(\lambda^a)$ and $\hat{Z}_r(\lambda^a)$ respectively. Without a loss of generality, one can assume that $a=0$.

By Lemma 14.2 of [25] there are superscheme isomorphisms

$$B\times V_r^{opp}\simeq BG_r=G_rB \text{ and } B^{opp}\times V_r\simeq B^{opp}G_r=G_rB^{opp}.$$

Thus $Dist(G_rB^{opp})$ is a free $Dist(B^{opp})$-supermodule (see Lemma 5.2 of [13]). Combining this and Theorem 10.1 of [25], we obtain that $\hat{Z}_r'(\lambda)|_{G_r}\simeq Z_r'(\lambda)$ and $\hat{Z}_r(\lambda)|_{G_r}\simeq Z_r(\lambda)$. Analogously, there are natural superscheme isomorphims

$$B_rT\times V_r^{opp}\simeq G_rT \text{ and } B_r^{opp}T\times V_r\simeq G_rT$$

such that $\hat{Z}_r'(\lambda)|_{G_rT}\simeq ind_{B_rT}^{G_rT}K_\lambda$ and $\hat{Z}_r(\lambda)|_{G_rT}\simeq Dist(G_rT)\otimes_{Dist(B_r^{opp}T)}K_\lambda$.

As in the fifth section, we see that $\hat{Z}_r'(\lambda)|_{B_r^{opp}T}\simeq K_\lambda\otimes K[V_r^{opp}]$ and $\hat{Z}_r'(\lambda)^{V_r^{opp}}=\hat{Z}_r'(\lambda)_\lambda\simeq K_\lambda$. Thus the socle of $\hat{Z}_r'(\lambda)$ is irreducible and generated by a $B_r^{opp}T$-primitive vector of weight λ. Observe also that $\hat{Z}_r'(\lambda)_\mu\neq 0$ implies $\mu\leq\lambda$.

Analogously, $\hat{Z}_r(\lambda)|_{B_rT}\simeq Dist(V_r)\otimes K_\lambda$. Since the B_rT-top of $\hat{Z}_r(\lambda)$ is isomorphic to $\hat{Z}_r(\lambda)/\hat{Z}_r(\lambda)_{V_r}\simeq K_\lambda$, the G_rB^{opp}-top of $\hat{Z}_r(\lambda)$ is also irreducible. Moreover, $\hat{Z}_r(\lambda)$ is generated by a B^{opp}-primitive vector of weight λ and $\hat{Z}_r(\lambda)_\mu\neq 0$ implies $\mu\leq\lambda$.

Denote by $\hat{L}_r'(\lambda)$ and $\hat{L}_r(\lambda)$ the modules $soc_{G_rB}\hat{Z}_r'(\lambda)$ and $\hat{Z}_r(\lambda)/rad_{G_rB^{opp}}\hat{Z}_r(\lambda)$, respectively.

Lemma 9.1. *The supermodules $\hat{Z}_r'(\lambda)$ (and $\hat{Z}_r(\lambda)$, respectively) are couniversal (and universal, respectively) objects in the category $G_rB-\mathsf{SMod}$ (and $G_rB^{opp}-\mathsf{SMod}$, respectively). In particular, each irreducible G_rB-supermodule is isomorphic to exactly one $\hat{L}_r'(\lambda^a)$ and each irreducible G_rB^{opp}-supermodule is isomorphic to exactly one $\hat{L}_r(\lambda^a)$.*

Proof. The statement related to $\hat{Z}_r(\lambda)$ is obvious. Let M be a G_rB-supermodule that is cogenerated by a $B_r^{opp}T$-primitive vector v of weight λ and $M_\mu\neq 0$ implies $\mu\leq\lambda$. Consider a vector $m\in M^{V_r^{opp}}\setminus 0$ of weight μ. Set $M'=Dist(G_rB)m=Dist(V)m$ (this is a G_rB-subsupermodule generated by m; see Lemma 9.4 of [23]). Then $M'_\mu=Km$ and $M'_\pi\neq 0$ implies $\pi\leq\mu$. Since $v\in M'$, it follows that $\mu=\lambda$ and $M^{V_r^{opp}}=M_\lambda=Kv$. In particular, $soc_{G_rB}M=\hat{L}_r(\lambda^{|v|})$.

Furthermore, $M|_B$ has a factor that is isomorphic to K_λ (cf. Remark 5.3 of [22]), hence $Hom_{G_rB}(M,\hat{Z}_r'(\lambda))\simeq Hom_B(M,K_\lambda)=K$ and M is embedded into $\hat{Z}_r'(\lambda)$. □

We leave for the reader to verify that the socle of $\hat{Z}_r'(\lambda)|_{G_rT}$ and the top of $\hat{Z}_r(\lambda)|_{G_rT}$ are irreducible G_rT-supermodules. Furthemore, $\hat{Z}_r'(\lambda)|_{G_rT}$ and $\hat{Z}_r(\lambda)|_{G_rT}$, respectively are couniversal and universal objects, respectively, in $G_rT-\mathsf{SMod}$. Therefore every irreducible G_rT-supermodule is isomorphic to the socle of exactly one $\hat{Z}_r'(\lambda^a)|_{G_rT}$ and to the top of exactly one $\hat{Z}_r(\lambda^a)|_{G_rT}$, respectively.

The anti-automorphism $\phi\mapsto\phi^{<t>}$ maps $Dist(G_rB)$ onto $Dist(G_rB^{opp})$. In particular, the functor $M\mapsto M^{<t>}$ induces a duality between the full subcategory of $G_rB-\mathsf{SMod}$, consisting of all finite-dimensional supermodules, and the same kind subcategory in $G_rB^{opp}-\mathsf{SMod}$. Since $M^{<t>}|_T\simeq M|_T$ for every T-supermodule M (cf. Lemma 5.4 of [22]), this implies $\hat{Z}_r'(\lambda^a)^{<t>}\simeq\hat{Z}_r(\lambda^a)$ and $\hat{L}_r'(\lambda^a)^{<t>}\simeq$

$\hat{L}_r(\lambda^a)$. Moreover, $\phi \mapsto \phi^{<t>}$ induces an anti-automorphism of $Dist(G_rT)$, hence it induces a self-duality of the full subcategory of $G_rT - \mathsf{SMod}$, consisting of all finite-dimensional supermodules, such that $(\hat{Z}'_r(\lambda^a)|_{G_rT})^{<t>} \simeq \hat{Z}_r(\lambda^a)|_{G_rT}$.

The following lemma generalizes Proposition II.9.6 from [11].

Lemma 9.2. *The socle of $\hat{Z}'_r(\lambda)|_{G_rT}$ is isomorphic to the top of $\hat{Z}_r(\lambda)|_{G_rT}$. If $\tilde{L}_r(\lambda)$ denotes this irreducible G_rT-supermodule, then $\tilde{L}_r(\lambda)^{<t>} \simeq \tilde{L}_r(\lambda)$.*

Also,

$$soc_{G_r}\hat{Z}'_r(\lambda) = soc_{G_rT}\hat{Z}'_r(\lambda) = soc_{G_rB}\hat{Z}'_r(\lambda)$$

and

$$rad_{G_r}\hat{Z}_r(\lambda) = rad_{G_rT}\hat{Z}_r(\lambda) = rad_{G_rB^{opp}}\hat{Z}_r(\lambda).$$

Proof. Since $(soc_{G_rT}\hat{Z}'_r(\lambda))^{V_r^{opp}} \simeq K_\lambda$, the first statement follows by the same arguments as in the fifth section.

Let v denotes a generator of $\hat{Z}'_r(\lambda)_\lambda$. Then

$$soc_{G_rT}\hat{Z}'_r(\lambda) = Dist(V_r)v = soc_{G_rT}\hat{Z}'_r(\lambda) \subseteq Dist(V)v = soc_{B_rT}\hat{Z}'_r(\lambda).$$

On the other hand, $Dist(G_rB) = Dist(B_{ev})Dist(G_r)$. Thus $soc_{G_rB}\hat{Z}'_r(\lambda)$ is completely reducible as G_r-supermodule, and therefore it is contained in $soc_{G_rT}\hat{Z}'_r(\lambda)$. Application of the duality $M \mapsto M^{<t>}$ implies the last statement. □

10. Steinberg supermodules

The positive even roots of G are $\epsilon_i - \epsilon_j$, where $1 \leq i < j \leq m$ or $m+1 \leq i < j \leq m+n$ and the positive odd roots of G are $\epsilon_i - \epsilon_j$ for $1 \leq i \leq m < j \leq m+n$. Let ρ_0 denote the half of the sum of all positive even roots of G, ρ_1 denote the half of the sum of all positive odd roots of G, $\rho = \rho_0 - \rho_1$, and $\rho_{s,t}$ denotes a weight $s\sum_{1\leq i\leq m}\epsilon_i + t\sum_{m+1\leq i\leq m+n}\epsilon_i$ for $s,t \in \mathbb{Z}$. Then $\rho_1 = \frac{1}{2}\rho_{n,-m}$.

Lemma 10.1. *For every dominant weight λ and for every positive integer r there is an isomorphism*

$$H^0((p^r-1)\rho_0 + \rho_{s,t}) \otimes H^0_{ev}(\lambda)^{[r]} \simeq H^0((p^r-1)\rho_0 + \rho_{s,t} + p^r\lambda).$$

Proof. The module $H^0_{ev}(\lambda)^{[r]}$, considered as a G-supermodule, is an even Frobenius twist of $H^0_{ev}(\lambda)$. The tensor identity implies

$$H^0((p^r-1)\rho_0 + \rho_{s,t}) \otimes H^0_{ev}(\lambda)^{[r]} \simeq ind_P^G(H^0_{ev}((p^r-1)\rho_0 + \rho_{s,t}) \otimes H^0_{ev}(\lambda)^{[r]}).$$

The simple G_{ev}-module $L_{ev}(\rho_{s,t})$ is one-dimensional. In fact, it is isomorphic to $\det(C_{00})^s \otimes \det(C_{11})^t$. Using Proposition II.3.19 of [11] and the tensor identity we obtain

$$H^0_{ev}((p^r-1)\rho_0 + \rho_{s,t}) \otimes H^0_{ev}(\lambda)^{[r]} \simeq H^0_{ev}((p^r-1)\rho_0) \otimes H^0_{ev}(\lambda)^{[r]} \otimes L_{ev}(\rho_{s,t}) \simeq$$
$$H^0_{ev}((p^r-1)\rho_0 + p^r\lambda) \otimes L_{ev}(\rho_{s,t}) \simeq H^0_{ev}((p^r-1)\rho_0 + p^r\lambda + \rho_{s,t}).$$

□

Remark 10.2. *Application of the functor $M \to M^{<t>}$ gives*

$$V((p^r-1)\rho_0 + \rho_{s,t}) \otimes V_{ev}(\lambda)^{[r]} \simeq V((p^r-1)\rho_0 + \rho_{s,t} + p^r\lambda).$$

Lemma 10.3. *The supermodule $H^0((p^r-1)\rho_0 + \rho_{s,t})$ is irreducible if and only if $p \not| (\frac{(p^r+1)}{2}m + \frac{(p^r-1)}{2}n + s + t)$.*

Proof. Combining Theorem 1 of [14] and Remark II.3.19 from [11] we see that $H^0((p^r-1)\rho_0)+\rho_{s,t})$ is irreducible if and only if $p \not| ((p^r-1)\rho_0)+\rho_{s,t}+\rho,\alpha)$ for every positive odd root α. If $\alpha=\epsilon_i-\epsilon_j$ and $1\leq i\leq m<j\leq m+n$, then

$$((p^r-1)\rho_0)+\rho_{s,t}+\rho,\alpha)=\frac{(p^r-1)}{2}(m+n)+s+t+m+p^r(m-i-j-1)$$

and the lemma follows. □

An irreducible G-supermodule from Lemma 10.3 is called an r-th *Steinberg supermodule*. It is obvious that an r-th Steinberg supermodule remains irreducible as a G_r-supermodule.

Let $T(\lambda)$ denotes the indecomposable tilting G-supermodule of the highest weight λ (cf. [15]). Let $T_{ev}(\lambda)$ denotes the indecomposable tilting G_{ev}-supermodule (or G_{res}-module) of the highest weight λ. The following proposition generalizes Proposition (2.1) from [8] to general linear supergroups.

Proposition 10.4. *If $H^0((p^r-1)\rho_0+\rho_{s,t})$ is a Steinberg supermodule, then*

$$H^0((p^r-1)\rho_0+\rho_{s,t})\otimes T_{ev}(\lambda)^{[r]}\simeq T((p^r-1)\rho_0+\rho_{s,t}+p^r\lambda).$$

Proof. Denote $(p^r-1)\rho_0+\rho_{s,t}$ by π. Since $H^0(\pi)=L(\pi)=V(\pi)$, Lemma 10.1 and Remark 10.2 imply that $T=H^0(\pi)\otimes T_{ev}(\lambda)^{[r]}$ has both good and Weyl filtrations. Moreover, $\pi+p^r\lambda$ is the unique highest weight of T. It remains to prove that T is indecomposable (cf. Theorem 4.17 and Theorem 4.19 of [15]; see also Theorem 1.1 of [8]).

We have $End_G(T)\simeq (T^*\otimes T)^G=(T^*\otimes T)^{Dist(G)}$. The action of $Dist(G)$ on $End_K(T)$ can be described as $(\psi\cdot\phi)(v)=(-1)^{|\phi||\psi|}\psi\phi(s_{Dist(G)}(\psi)v))$ for $\psi\in Dist(G)$ and $\phi\in End_K(T)$. We will follow the idea from Lemma in §2 of [7]. Since G_r acts on $T_{ev}(\lambda)^{[r]}$ trivially, there is an isomorphism of superalgebras

$$End_{G_r}(T)\simeq End_{G_r}(L_r((\pi))\otimes End_K(T_{ev}(\lambda)^{[r]}).$$

Additionally, we have $L_r(\pi)^{V_r^{opp}}=Z'_r(\pi)^{V_r^{opp}}=K_\mu$, hence $End_{G_r}(L_r(\pi))=Kid_{L(\pi)}$.

Finally,

$$End_G(T)=End_{G_r}(T)^G\simeq Kid_{L(\pi)}\otimes End_G(T_{ev}(\lambda)^{[r]})\simeq End_G(T_{ev}(\lambda)^{[r]}).$$

Using Lemma 6.2 we derive that $End_G(T_{ev}(\lambda)^{[r]})\simeq End_{G_{res}}(T_{ev}(\lambda))$ is a local purely even superalgebra, hence T is indecomposable. □

Lemma 10.5. *Let $H^0((p^r-1)\rho_0+\rho_{s,t})$ be a Steinberg supermodule. Then*

$$H^0((p^r-1)\rho_0+\rho_{s,t})|_{G_rB}\simeq\hat{L}'_r((p^r-1)\rho_0+\rho_{s,t})$$

and

$$H^0((p^r-1)\rho_0+\rho_{s,t})|_{G_rB^{opp}}\simeq\hat{L}_r((p^r-1)\rho_0+\rho_{s,t}).$$

Thus

$$H^0((p^r-1)\rho_0+\rho_{s,t})|_{G_rT}\simeq\tilde{L}_r((p^r-1)\rho_0+\rho_{s,t}).$$

Proof. Denote $(p^r-1)\rho_0+\rho_{s,t}$ by π, again. Lemma 9.1 implies that $H^0(\pi)|_{G_rB}$ is embedded into $\hat{Z}'_r(\pi)$. On the other hand, $\dim H^0(\pi)=\dim K[U^{opp}]\dim H^0_{ev}(\pi)=\dim K[V_r^{opp}]=\dim\hat{Z}'_r(\pi)$ (cf. II.9.16 of [11]). Thus $H^0(\pi)|_{G_rB}\simeq\hat{Z}'_r(\pi)$ and by Lemma 9.2 we obtain $\hat{Z}'_r(\pi)=\hat{L}'_r(\pi)$.

Since $H^0(\pi)^{<t>} = L(\pi)^{<t>} \simeq L(\pi) = H^0(\pi)$, the second statement follows. The last statement follows by Lemma 9.2. □

The proof of the following lemma can be modified from the proof of Proposition II.2.14 in [11].

Lemma 10.6. *If $\lambda \not< \mu$, then*

$$Ext^1_{G_rT}(\tilde{L}_r(\lambda), \tilde{L}_r(\mu)) \simeq Hom_{G_rT}(rad_{G_rT}\hat{Z}_r(\lambda), \tilde{L}_r(\mu)).$$

For any finite-dimensional G_rT-supermodules M_1 and M_2 there is a (superspace) isomorphisms

$$Ext^n_{G_rT}(M_1, M_2) \simeq Ext^n_{G_rT}(M_2^{<t>}, M_1^{<t>}), n \geq 0.$$

We will only sketch a proof of this statement. By Proposition 3.2 from [22] (see also Lemma I.4.4 of [11]), there is a superspace isomorphism

$$Ext^n_{G_rT}(M_1, M_2) \simeq Ext^n_{G_rT}(K, M_1^* \otimes M_2).$$

We leave it for the reader to verify that the G_rT-supermodule $M_1^* \otimes M_2$ is isomorphic to the G_rT-supermodule $(M_2^{<t>})^* \otimes M_1^{<t>}$.

Lemma 10.7. *Every Steinberg supermodule is both injective and projective as a G_r-supermodule and as a G_rT-supermodule.*

Proof. Combining the above observation with Lemma 10.6, one can easily superize Proposition II.10.2 from [11]. □

Lemma 10.8. *Let H be an algebraic supergroup and N be a normal subsupergroup of H. For every H-supermodules M_1 and M_2 such that $M_2^N = M_2$ there is an isomorphism of H/N-supermodules $Ext^\bullet_N(M_1, M_2) \simeq Ext^\bullet_N(M_1, K) \otimes M_2$.*

Proof. Let $K \to I^\bullet$ be an injective resolution of the trivial H-supermodule. Then $M_2 \to I^\bullet \otimes M_2$ is an injective resolution of M_2 as a H-supermodule and hence, also as a N-supermodule. Since $Hom_N(M_1, I^\bullet \otimes M_2)$ and $Hom_N(M_1, I^\bullet) \otimes M_2$ are isomorphic to each other as complexes of H/N-supermodules, the claim follows. □

Lemma 10.9. *If $H^0((p^r-1)\rho_0 + \rho_{s,t})$ is a Steinberg supermodule, then for every G_r-supermodule M we have $Ext^1_U(M, H^0_{ev}((p^r-1)\rho_0 + \rho_{s,t}))^{G_{ev,r}} = 0$.*

Proof. Denote the weight $(p^r-1)\rho_0 + \rho_{s,t}$ by π, one more time. Observe that $H^0(\pi)|_{G_r} \simeq Z'_r(\pi) \simeq ind^{G_r}_{P_r} H^0_{ev}(\pi)$, where $M' = H^0_{ev}(\pi)|_{G_{ev,r}}$ is regarded as a P_r-supermodule via the epimorphism $P_r \to G_{ev,r}$.

As it has been already observed, P_r is a faithfully exact subsupergroup of G_r and the functor $ind^{G_r}_{P_r}$ is faithfully exact. Thus for every G_r-supermodule M there is a natural isomorphism

$$Ext^1_{G_r}(M, H^0(\pi)) \simeq Ext^1_{P_r}(M, M').$$

The functor $Hom_{P_r}(M, ?)$ is isomorphic to $Hom_U(M, ?)^{P_r/U}$. Since the functor $Hom_U(M, ?)$ from $P_r - \mathsf{SMod}$ to $P_r/U - \mathsf{SMod}$ is left exact and takes injectives to injectives, there is a spectral sequence

$$E_2^{n,m} = H^n(P_r/U, Ext^m_U(M, N)) \Rightarrow Ext^{n+m}_{P_r}(M, N), N \in P_r - \mathsf{SMod}.$$

Lemma 10.8 combined with Proposition II.10.2 of [11] imply that $Ext^m_U(M,M') \simeq Ext^m_U(M,K)\otimes M'$ is an injective $G_{ev,r} = P_r/U$-supermodule. In particular, $E_2^{n,m} = 0$ for every $n \geq 1, m \geq 0$. Using the five-term exact sequence we obtain

$$Ext^1_{P_r}(M,M') \simeq Ext^1_U(M,M')^{P_r/U}.$$

Lemma 10.7 concludes the proof. □

Let $I_{ev}(\lambda)$ be the injective envelope of an irreducible G_{res}-module of the highest weight λ. We have already seen that $I_{ev}(\lambda)$ is also an injective envelope of the purely even irreducible G_{ev}-supermodule of the highest weight λ. Denote by $I(\lambda)$ the injective envelope of the simple G-module of the highest weight λ.

Theorem 10.10. *If $H^0((p^r-1)\rho_0+\rho_{s,t})$ is a Steinberg supermodule, then*

$$H^0((p^r-1)\rho_0+\rho_{s,t})\otimes I_{ev}(\lambda)^{[r]} \simeq I((p^r-1)\rho_0+\rho_{s,t}+p^r\lambda).$$

Proof. We have again $G_{ev} = P_{ev}$, which implies that G/P is an affine superscheme and the functor $ind_P^G?$ is (faithfully) exact. For every $\mu \in X(T)^+$ we have

$$Ext^1_G(L(\mu), H^0(\pi)\otimes I_{ev}(\lambda)^{[r]}) \simeq Ext^1_P(L(\mu), H^0_{ev}(\pi)\otimes I_{ev}(\lambda)^{[r]}),$$

where $\pi = (p^r-1)\rho_0+\rho_{s,t}$. Since

$$H^0_{ev}(\pi)\otimes I_{ev}(\lambda)^{[r]} \simeq I_{ev}(\pi+p^r\lambda)$$

(see II.10.5.2 (1) of [11]), arguing as in Lemma 10.9 we obtain

$$Ext^1_P(L(\mu), H^0_{ev}(\pi)\otimes I_{ev}(\lambda)^{[r]}) \simeq Ext^1_U(L(\mu), H^0_{ev}(\pi)\otimes I_{ev}(\lambda)^{[r]}))^{P/U} \simeq$$

$$(Ext^1_U(L(\mu),K)\otimes H^0_{ev}(\pi)\otimes I_{ev}(\lambda)^{[r]})^{G_{ev}} \subseteq (Ext^1_U(L(\mu),K)\otimes H^0_{ev}(\pi)\otimes I_{ev}(\lambda)^{[r]})^{G_{ev,r}}$$

$$= (Ext^1_U(L(\mu), H^0_{ev}(\pi))^{G_{ev,r}})^{\bigoplus \dim I_{ev}(\lambda)^{[r]}} = 0.$$

Therefore $H^0(\pi)\otimes I_{ev}(\lambda)^{[r]}$ is an injective G-supermodule. As in the proof of Proposition 10.4, we obtain that $H^0(\pi)\otimes I_{ev}(\lambda)^{[r]}$ is indecomposable. Since this supermodule contains a subsupermodule $H^0(\pi+p^r\lambda)$, its socle coincides with $L(\pi+p^r\lambda)$. □

Remark 10.11. *Let $(p^r-1)\rho_0+\rho_{s,t}$ be a highest weight of some Steinberg supermodule. Define a map $\theta_r : X(T)^+ \to X(T)^+$ by $\theta_r(\lambda) = (p^r-1)\rho_0+\rho_{s,t}+p^r\lambda$. Theorem 10.10 implies that if $\mathfrak{B}$ is a block of $GL(m|n)$-supermodules, then $\theta_r(\mathfrak{B})$ is a block as well. The proof can be modified from the proof of Corollary 2.3,* [9].

11. Examples

Consider two G-supermodules M and M' over an algebraic supergroup G. The right $K[G]$-supecormodule structure on $M\otimes M'$ is given by $\tau_{M\otimes M'}(m\otimes m') = \sum(-1)^{|f_2||m_1|}m_1\otimes m_1'\otimes f_2g_2$, where $\tau_M(m) = \sum m_1\otimes f_2$ and $\tau_{M'}(m') = \sum m_1'\otimes g_2$. The following lemma is evident.

Lemma 11.1. *$Dist(G)$ acts on $M\otimes M'$ by the rule*

$$\phi\cdot(m\otimes m') = \sum(-1)^{|f_2||m_1|+|\phi|(|m_1|+|m_1'|)+|\phi_2||f_2|}m_1\otimes m_1'\phi_1(f_2)\phi_2(g_2).$$

In particular, if ϕ is a primitive element, then

$$\phi\cdot(m\otimes m') = \phi\cdot m\otimes m' + (-1)^{|\phi||m|}m\otimes\phi\cdot m'.$$

As above, let $G = GL(m|n)$. From now on we assume that $m, n \geq 1$.

The superalgebra $Dist(U^{opp})$ is generated by the primitive elements (matrix units) e_{ij} for $1 \leq i \leq m < j \leq m+n$. They act on basis vectors w_k of W as $e_{ij} \cdot w_k = \delta_{jk} w_i$ for $1 \leq k \leq m+n$.

Consider the k-th (super) exterior power $\Lambda^k(W) = \bigoplus_{0 \leq i \leq \min\{m,k\}} \Lambda^i(W_0) \otimes S^{k-i}(W_1)$. Basis vectors of $\Lambda^k(W)$ are of the form

$$w_{i_1} \dots w_{i_s} w_{m+1}^{\beta_1} \dots w_{m+n}^{\beta_n}, \text{ where } 1 \leq i_1 < \dots < i_s \leq m \text{ and } \beta_1, \dots, \beta_n \geq 0.$$

We will denote $w_{i_1} \dots w_{i_s}$ by w_+^I, where $I = \{i_1, \dots, i_s\}$, and $w_{m+1}^{\beta_1} \dots w_{m+n}^{\beta_n}$ by w_-^β, where $\beta = (\beta_1, \dots, \beta_n)$. Also denote $\sum_{1 \leq t \leq n} \beta_t$ by $|\beta|$.

Lemma 11.2. *There is a decomposition*

$$\Lambda^k(W)^{U^{opp}} = \Big(\bigoplus_{0 \leq i \leq \min\{k, m-1\}, p|(k-i)} \Lambda^i(W_0) \otimes S^{\frac{k-i}{p}}(W_1^p) \Big) \bigoplus \Lambda^m(W_0) \otimes S^{k-m}(W_1).$$

Here $S^{\frac{k-i}{p}}(W_1^p)$ is a subspace of $S^k(W_1)$ spanned by the elements $w_-^{p\gamma}$, where $|\gamma| = \frac{k-i}{p}$. The last summand appears only if $k \geq m$.

Proof. If $1 \leq i \leq m < j \leq m+n$, then

$$e_{ij} \cdot (w_+^I w_-^\beta) = \beta_{j-m} w_+^I w_i w_{m+1}^{\beta_1} \dots w_j^{\beta_{j-m}-1} \dots w_{m+n}^{\beta_n}.$$

The decomposition in the statement of the lemma follows. □

Corollary 11.3. *The G_{ev}-supermodule $\Lambda^k(W)^{U^{opp}}$ has a good filtration if and only if $n = 1$ or $k = ps + r$, where $s \geq 0$ and $m \leq r < p$.*

Lemma 11.4. *If $n = 1$, then $\Lambda^k(W)$ has a good filtration if and only if $k = ps + r$, where $s \geq 0$ and $m \leq r < p$. In this case $\Lambda^k(W) \simeq H^0((1^m|k-m)^a) = L((1^m|k-m)^a) = V((1^m|k-m)^a)$, where $a = k - m \pmod 2$, is a tilting supermodule.*

Proof. Assume that $\Lambda^k(W)$ has a good filtration. By Lemma 4.1, $\dim \Lambda^k(W)$ is a multiple of $\dim K[U^{opp}] = 2^m$. Thus $k \geq m$, $\dim \Lambda^k(W) = \dim \Lambda^m(W) = 2^m$ and $\Lambda^k(W)|_{U^{opp}} \simeq \Pi^a K[U^{opp}]$ for $a = 0, 1$.

Since $K[U^{opp}]^{U^{opp}} = K$ and $\Lambda^k(V)$ has a unique highest weight $(1^m|k-m)$, Lemma 11.2 and Remark 8.9 imply that $k = ps + r$, where $s \geq 0, m \leq r < p$ and $\Lambda^k(W) \simeq H^0((1^m|k-m)^a)$ for $a = k - m \pmod 2$. Moreover, by Theorem 1 of [14], $H^0((1^m|k-m))$ is irreducible, hence $H^0((1^m|k-m)^a) = L((1^m|k-m)^a) = V((1^m|k-m)^a)$ is tilting. In fact, $((1^m|k-m) + \rho, \epsilon_i - \epsilon_{m+1}) = k + 1 - i$ is not divided by p for any i.

Conversely, let $L(\pi^b)$ be a simple subsupermodule of $\Lambda^k(W)$, where $k = ps + r, s \geq 0$ and $m \leq r < p$. Then its highest weight vector is annihilated by $Dist(B^{opp})^+$. Since there is only one such vector $w_1 \dots w_m w_{m+1}^{k-m}$ of weight $\pi = (1^m|k-m)$, we get $L((1^m|k-m)^a) \subseteq \Lambda^m(W)$. As it has already been observed, $L((1^m|k-m)) = H^0((1^m|k-m))$ and this supermodule has the dimension 2^m. Thus $\Lambda^k(W) \simeq H^0((1^m|k-m)^a)$. □

Since U^{opp} is abelian, any subsupergroup of U^{opp} is normal. Let N be a subsupergroup of U^{opp} that is defined by the equations $c_{i,m+1} = 0$ for $1 \leq i \leq m$. In

other words, an element $g \in U^{opp}$ belongs to N if and only if $gw_{m+1} = w_{m+1}$. Let W' denote the subsuperspace $\sum_{1\leq i\leq m+n, i\neq m+1} Kw_i$ of W. Then

$$\Lambda^k(W)|_N \simeq \bigoplus_{0\leq t\leq k} \Lambda^{k-t}(W').$$

If $n > 1$ and $\Lambda^k(W)$ is an injective U^{opp}-supermodule, then every $\Lambda^s(W')$ for $0 \leq s \leq k$, is injective as a N-supermodule by Lemma 4.3. On the other hand, every (finite-dimensional) injective N-supermodule has a dimension that is divided by $2^{m(n-1)} \geq 2$. Thus $\Lambda^0(W') = K$ can not be injective as a N-supermodule. This contradiction shows that the only case when $\Lambda^k(W)$ is injective as a U^{opp}-supermodule is $n = 1$. The following proposition is now evident.

Proposition 11.5. *$\Lambda^k(W)$ has a good filtration if and only if $n = 1$ and $k = ps+r$, where $s \geq 0$ and $m \leq r < p$.*

In the previous notation, $e_{ji} \cdot (w_+^I w_-^\beta) \neq 0$ if and only if $i \in I$. In this case $e_{ji} \cdot (w_+^I w_-^\beta) = \pm w_+^{I\setminus i} w_-^{\beta'}$, where $\beta'_j = \beta_j + 1$ and $\beta'_k = \beta_k$ for $k \neq j$. Thus $Dist(U)^+ \cdot \Lambda^k(W)$ is spanned by all $w_+^I w_-^\beta$ with $I^\sharp < m$ and $|\beta| \geq 1$. Therefore, $\Lambda^k(W)/\Lambda^k(W)_U \simeq \Lambda^m(W_0) \otimes S^{k-m}(W_1)$ provided $k > m$, and $\Lambda^k(W)/\Lambda^k(W)_U \simeq \Lambda^k(W_0)$ otherwise.

Proposition 11.6. *The G-supermodule $\Lambda^k(W)$ has a Weyl filtration if and only if $n = 1$ and $k = ps + r$, where $s \geq 0$ and $m \leq r < p$.*

Proof. Assume that $\Lambda^k(W)$ has a Weyl filtration. Then $\Lambda^k(W)$ is a projective U-supermodule, hence injective by Lemma 4.4. Let N be a subsupergroup of U that is defined by the equations $c_{m+n,i} = 0$ for $1 \leq i \leq m$. Then

$$\Lambda^k(W)|_N \simeq \bigoplus_{0\leq t\leq k} \Lambda^{k-t}(W'),$$

where $W' = \sum_{1\leq i\leq m+n-1} Kw_i$. Arguing as before, $n = 1$ and $k \geq m$ is the only case when $\Lambda^k(W)$ can be an injective, hence projective U-supermodule. Remark 8.13 implies that $\Lambda^k(W)$ has to be isomorphic to $V((1^m|k-m)^a)$ for $a = k - m \pmod 2$. Furthermore, since the vector $w_1 \dots w_m w_{m+1}^{k-m}$ is annihilated by $Dist(B^{opp})^+$, it implies that $L((1^m|k-m)^a) \subseteq \Lambda^k(W) \simeq V((1^m|k-m)^a)$. On the other hand, $L((1^m|k-m)^a)$ is the top of $V((1^m|k-m)^a)$. Since $V((1^m|k-m)^a)$ is indecomposable, $L((1^m|k-m)^a) = V((1^m|k-m)^a) = H^0((1^m|k-m)^a)$. Theorem 1 from [14] implies that $k = ps + r$, where $s \geq 0$ and $m \leq r < p$. Lemma 11.4 concludes the proof. □

Any basis element of $S^k(W)$ has the form $w_+^\gamma w_-^J = w_1^{\gamma_1} \dots w_m^{\gamma_m} w_{j_1} \dots w_{j_s}$, where $\gamma_1, \dots, \gamma_m \geq 0$ and $m + 1 \leq j_1 < \dots < j_s \leq m + n$. As we have seen earlier, $e_{ij} \cdot (w_+^\gamma w_-^J) \neq 0$ if and only if there is an index t such that $j = j_t$. In this case

$$e_{ij} \cdot (w_+^\gamma w_-^J) = (-1)^{t-1} w_1^{\gamma_1} \dots w_i^{\gamma_i+1} \dots w_m^{\gamma_m} w_-^{J\setminus\{j_t\}}.$$

Analogously, we have

$$e_{ji} \cdot (w_+^\gamma w_-^J) = \gamma_i w_1^{\gamma_1} \dots w_i^{\gamma_i-1} \dots w_m^{\gamma_m} w_j w_-^J.$$

In particular, $S^k(W)^{U^{opp}} = S^k(W_0) = H^0_{ev}((k|0^n))$.

If $N = Stab_{U^{opp}}(W')$, where $W' = \sum_{1\leq i\leq m+n, i\neq m} Kw_i$, then

$$S^k(W)|_N \simeq \bigoplus_{0\leq t\leq k} S^{k-t}(W').$$

Again, if $S^k(W)$ is an injective U^{opp}-supermodule, then $m = 1$ and $k \geq n$. Analogously, if $S^k(W)$ is a projective U-supermodule, then it is also injective. Working with the subsupergroup of U defined by the equations $c_{jm} = 0$ for $m+1 \leq j \leq m+n$, we derive that this is possible only if $m = 1$ and $k \geq n$.

Proposition 11.7. *$S^k(W)$ has a good filtration if and only if $m = 1$ and $n \leq k < p$. In this case $S^k(W) = H^0((k|0^n)) = L((k|0^n)) = V((k|0^n))$ is tilting.*

Proof. If $S^k(W)$ has a good filtration, then $m = 1$ and $k \geq n$. Remark 8.9 implies that $S^k(W) \simeq H^0((k|0^n))$. Let $k = ps + r$, where $s \geq 0$ and $0 \leq r < p$. Assume that $s > 0$. Then the subsuperspace

$$W' = \sum_{J^\sharp = r-t, 0\leq t\leq r} Kw_1^{ps+t} w_-^J$$

is a proper subsupermodule in $S^k(W)$. But the highest weight vector w_1^k does not belong to W', which is a contradiction. Thus $s = 0$ and $H^0((k|0^n))$ is irreducible by Theorem 1 of [14]. Conversely, the vector w_1^k of the unique highest weight $(k|0^n)$ generates a simple subsupermodule of $S^k(W)$ that is isomorphic to $L((k|0^n))$. If $m = 1$ and $n \leq k < p$, then $H^0((k|0^n)) = L((k|0^n))$ has the dimension 2^n, and the proof is concluded. □

Proposition 11.8. *$S^k(W)$ has a Weyl filtration if and only if $m = 1$ and $n \leq k < p$.*

Proof. The conditions $m = 1$ and $k \geq n$ are necessary for $S^k(W)$ to have a Weyl filtration. Furthermore, we have

$$S^k(W)_U = \sum_{p\not|(t+1), 0\leq t\leq k-1, k=t+J^\sharp} Kw_1^t w_-^J.$$

This implies

$$S^k(W)/S^k(W)_U \simeq S^k(W_0) \bigoplus \Big(\sum_{1\leq l\leq [\frac{k+1}{p}], pl-1+J^\sharp = k, J^\sharp \geq 1} Kw_1^{pl-1} w_-^J \Big).$$

As above, $S^k(W)|_U \simeq \Pi^b Dist(U)$, where $b = 0, 1$. Therefore $\dim S^k(W)/S^k(W)_U = 1$, hence $k < p$ and $S^k(W) \simeq V((k|0^n)) = L((k|0^n)) = H^0((k|0^n))$ is tilting. The converse statement follows by Proposition 11.7. □

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