ULRICH IDEALS AND MODULES – A BASIC THEORY

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My lecture is based on the joint work [GOTWY] and reports a generalization of Ulrich modules (and ideals) in a given Cohen-Macaulay local ring A. The purpose is to explore their structure and give some applications. I will talk about the basic theory of Ulrich ideals and modules.

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and $d = \dim A \geq 0$. We assume that the residue class field $A/\mathfrak{m}$ of A is infinite. Let M be a finitely generated A-module. In [BHU] B. Ulrich and other authors gave structure theorems of Maximally Generated Maximal Cohen-Macaulay modules, i.e., those Cohen-Macaulay A-modules M such that $\dim_A M = d$ and $\mathrm{e}^0_{\mathfrak{m}}(M) = \mu_A(M)$, where $\mathrm{e}^0_{\mathfrak{m}}(M)$ (resp. $\mu_A(M)$) denotes the multiplicity of M with respect to $\mathfrak{m}$ (resp. the number of elements in a minimal system of generators for M). Let us call these modules MGMCM or, simply, Ulrich modules ([HK]).

We shall generalize this notion in the following way.

Definition 1. Let I be an $\mathfrak{m}$-primary ideal in A and let M be a finitely generated A–module. Then M is called a Ulrich A–module with respect to I, if

(1) M is a Cohen-Macaulay A-module with $\dim_A M = d$,
(2) $\mathrm{e}^0_I(M) = \ell_A(M/IM)$, and
(3) M/IM is A/I-free,

where $\ell_A(*)$ stands for the length.

If the ideal I contains a parameter ideal Q as a reduction, condition (2) is equivalent to saying that $IM = QM$, provided M is a Cohen-Macaulay A-module with $\dim_A M = d$. Remember that Ulrich modules with respect to the maximal ideal $\mathfrak{m}$ are exactly Ulrich modules in the sense of [HK].

We define Ulrich ideals as follows.

Definition 2. Let I be an $\mathfrak{m}$–primary ideal in A. Then we say that I is a Ulrich ideal of A, if I/I^2 is A/I-free, I is not a parameter ideal of A, but contains a minimal reduction Q such that $I^2 = QI$.

When $I = \mathfrak{m}$, this condition is equivalent to saying that our Cohen-Macaulay local ring A is not a RLR, possessing maximal embedding dimension in the sense of J. Sally [S].

In my lecture I will report some basic structure theorems of Ulrich modules and ideals, including the following.

Theorem 3. *The following three conditions are equivalent, where* $\mathrm{Syz}^i_A(A/I)$ $(i \geq 0)$ *stands for the* i^{th} *syzygy module of the* A*–module* A/I *in a minimal free resolution.*

(1) *I is a Ulrich ideal of A.*
(2) *$\mathrm{Syz}_A^i(A/I)$ is a Ulrich A-module with respect to I for all $i \geq d$.*
(3) *There exists an exact sequence*

$$0 \to X \to F \to Y \to 0$$

of A-modules such that
 (a) *F is a finitely generated free A-module,*
 (b) *$X \subseteq \mathfrak{m}F$, and*
 (c) *both X and Y are Ulrich A-modules with respect to I.*

When $d > 0$, one can add the following.
(4) *$\mu_A(I) > d$, I/I^2 is A/I-free, and $\mathrm{Syz}_A^i(A/I)$ is a Ulrich A-module with respect to I for some $i \geq d$.*

It seems very interesting to explore how many Ulrich ideals are contained in a given Cohen–Macaulay local ring. For example, let $k[[t]]$ be the formal power series ring over a field k and let

$$A = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \subseteq k[[t]]$$

be a numerical semigroup ring, where $0 < a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ such that $\mathrm{GCD}(a_1, a_2, \ldots, a_\ell) = 1$. Let $\mathcal{X}_A^g$ be the set of Ulrich ideals in A which are generated by monomials in t. Then the set $\mathcal{X}_A^g$ is finite, because every Ulrich ideal $I$of A contains the conductor $A : \overline{A}$. Just for example, we get the following.

(1) $\mathcal{X}^g_{k[[t^3,t^4,t^5]]} = \{\mathfrak{m}\}$.
(2) $\mathcal{X}^g_{k[[t^4,t^5,t^6]]} = \{(t^4, t^6)\}$.
(3) $\mathcal{X}^g_{k[[t^a,t^{a+1},\ldots,t^{2a-2}]]} = \emptyset$, if $a \geq 5$.
(4) Let $1 < a < b$ be integers such that $\mathrm{GCD}(a, b) = 1$. Then $\mathcal{X}^g_{k[[t^a,t^b]]} \neq \emptyset$ if and only if a or b is even.
(5) Let $A = k[[t^4, t^6, t^{4\ell-1}]]$ ($\ell \geq 2$). Then $\sharp \mathcal{X}_A^g = 2\ell - 2$.

Let $\mathcal{X}_A = \{I \mid I \text{ is a Ulrich ideal of } A\}$. We then have the following answer also.

Theorem 4. *Suppose that A is of finite C-M representation type. Then $\mathcal{X}_A$ is a finite set.*

When A is of finite C-M representation type and $\dim A = 1$, I will give a complete list of Ulrich ideals in A.

References

[BHU] J. Brennan, J. Herzog, and B. Ulrich, *Maximally generated Cohen-Macaulay modules*, Math. Scand. **61**, 1987, 181–203.

[GOTWY] S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe, and K.-i. Yoshida, *Ulrich ideals and modules*, Preprint 2012.

[HK] J. Herzog and M. Kühl, *Maximal Cohen-Macaulay modules over Gorenstein rings and Bourbaki sequences. Commutative Algebra and Combinatorics*, Adv. Stud. Pure Math., **11**, 1987, 65–92.

[S] J. Sally, *Cohen–Macaulay local rings of maximal embedding dimension*, J. Algebra, **56** (1979), 168–183.