Diagnostic and treatment for linear mixed models

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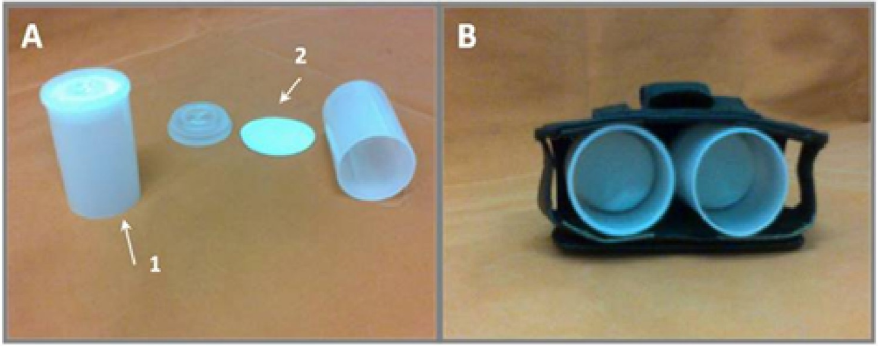
Ozone example

- Ozone concentration: measured with expensive instruments
- Alternative: reflectance in passive filters / calibration curve
- Details in André et al. (2014, Atm. Environ.)



Ozone example

- Experiment LPAE/FMUSP: predict period expected reflectance (latent value) accounting for possible outliers

Period	Reflectance	Period	Reflectance
1	27.0	6	47.9
1	34.0	6	60.4
1	17.4	6	47.3
2	24.8	7	50.4
2	29.9	7	50.7
2	32.1	7	55.9
3	35.4	8	54.9
3	63.2	8	43.2
3	27.4	8	52.1
4	51.2	9	38.8
4	54.5	9	59.9
4	52.2	9	61.1
5	77.7		
5	53.9		
5	48.2		

Ozone example

- Linear mixed model:

$$y_{ij} = \mu + a_i + e_{ij}, \quad i = 1, \ldots, 9,\ j = 1, 2, 3$$

  - $a_i \sim N(0, \sigma_a^2)$ independent
  - $e_{ij} \sim N(0, \sigma^2)$ independent
  - a_i and e_{ij} independent

- Consequently
  - $\mathbb{V}(y_{ij}) = \sigma_a^2 + \sigma^2$
  - $\mathbb{C}\mathrm{ov}(y_{ij}, y_{ik}) = \sigma_a^2$
  - $\mathbb{C}\mathrm{ov}(y_{lj}, y_{ik}) = 0$
  - Intraclass correlation coefficient: $\rho = \sigma_a^2/(\sigma_a^2 + \sigma^2)$

Gaussian LMM (matrix notation)

$$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{b}_i + \mathbf{e}_i \;\; \text{with} \;\; \mathbb{V}(\mathbf{b}_i) = \mathbf{G} \;\; \text{and} \;\; \mathbb{V}(\mathbf{e}_i) = \mathbf{R}_i$$

so that

$$\mathbb{E}(\mathbf{y}_i) = \mathbf{X}_i\boldsymbol{\beta} \;\; \text{and} \;\; \mathbb{V}(\mathbf{y}_i) = \boldsymbol{\Omega}_i = \mathbf{Z}_i\mathbf{G}\mathbf{Z}_i^\top + \mathbf{R}_i$$

Ozone example

$$\mathbf{y}_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \end{pmatrix}, \; \mathbf{X}_i = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; \boldsymbol{\beta} = \mu, \; \mathbf{b}_i = a_i, \; \mathbf{e}_i = \begin{pmatrix} e_{i1} \\ e_{i2} \\ e_{i3} \end{pmatrix}$$

$$\mathbb{V}(a_i) = \sigma_a^2, \; \mathbb{V}(e_{ij}) = \sigma^2, \; \boldsymbol{\Omega}_i = \begin{pmatrix} \sigma_a^2 + \sigma^2 & \sigma_a^2 & \sigma_a^2 \\ \sigma_a^2 & \sigma_a^2 + \sigma^2 & \sigma_a^2 \\ \sigma_a^2 & \sigma_a^2 & \sigma_a^2 + \sigma^2 \end{pmatrix}$$

Gaussian LMM (matrix notation)

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{b} + \mathbf{e}$$

with

$$\begin{aligned}
\mathbf{y} &= (\mathbf{y}_1^\top, \cdots, \mathbf{y}_n^\top)^\top \quad (N \times 1,\ N = \textstyle\sum_{i=1}^n m_i) \\
\mathbf{X} &= (\mathbf{X}_1^\top, \cdots, \mathbf{X}_n^\top)^\top \quad (N \times p) \\
\mathbf{Z} &= \oplus_{i=1}^n \mathbf{Z}_i \quad (N \times nq) \\
\mathbf{b} &= (\mathbf{b}_1^\top, \cdots, \mathbf{b}_n^\top)^\top \quad (nq \times 1) \\
\mathbf{e} &= (\mathbf{e}_1^\top, \cdots, \mathbf{e}_n^\top)^\top \quad (N \times 1) \\
\boldsymbol{\Gamma} &= \mathbf{I}_n \otimes \mathbf{G}(\boldsymbol{\theta}) \quad (nq \times nq) \\
\mathbf{R} &= \oplus_{i=1}^n \mathbf{R}_i(\boldsymbol{\theta}) \quad (N \times N)
\end{aligned}$$

Consequently

$$\mathbb{V}(\mathbf{y}) = \mathbf{V} = \mathbf{Z}\boldsymbol{\Gamma}\mathbf{Z}^\top + \mathbf{R}$$

Maximum likelihood methodology

- (E)BLUE of $\boldsymbol{\beta}$: $\widehat{\boldsymbol{\beta}} = \left(\sum_{i=1}^{n} \mathbf{X}_i^\top \widehat{\boldsymbol{\Omega}}_i^{-1} \mathbf{X}_i\right)^{-1} \sum_{i=1}^{n} \mathbf{X}_i^\top \widehat{\boldsymbol{\Omega}}_i^{-1} \mathbf{y}_i$
- (E)BLUP of
  $\mathbf{b}_i$: $\widehat{\mathbf{b}}_i = \widehat{\mathbf{G}}\mathbf{Z}_i^\top \widehat{\boldsymbol{\Omega}}_i^{-1} [\mathbf{I}_{m_i} - \mathbf{X}_i \left(\mathbf{X}_i^\top \widehat{\boldsymbol{\Omega}}_i^{-1} \mathbf{X}_i\right)^{-1} \mathbf{X}_i^\top \widehat{\boldsymbol{\Omega}}_i^{-1}] \mathbf{y}_i$

Ozone example

$$\widehat{\boldsymbol{\beta}} = \overline{y}, \quad \widehat{\mathbf{b}}_i = \widehat{k}(\overline{y}_i - \overline{y}), \quad \widehat{k} = \widehat{\sigma}_a^2 / (\widehat{\sigma}_a^2 + \widehat{\sigma}^2/3)$$

$\widehat{k}$: shrinkage constant

Predicted latent value for period i

$$\widehat{y}_i = \overline{y} + \widehat{k}(\overline{y}_i - \overline{y}) = \widehat{k}\overline{y}_i + (1 - \widehat{k})\overline{y}$$

Diagnostics

- Global influence
  - Leverage analysis [Nobre and Singer (2011, J. Applied Stat.)]
  - Case deletion analysis [Tan et al. (2001, The Statistician)]
- Local influence [Lesaffre and Verbeke (1998, Biometrics)]
- Residual analysis [Nobre and Singer (2007, Biom. Journal)]
- Three types of residuals to accommodate the extra source of variability present in LMM:

i) Marginal residuals, $\widehat{\boldsymbol{\xi}}_i = \mathbf{y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}}$ predictors of marginal errors,
$\boldsymbol{\xi}_i = \mathbf{y}_i - \mathbb{E}[\mathbf{y}_i] = \mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta} = \mathbf{Z}_i\mathbf{b}_i + \mathbf{e}_i$
$\widehat{\xi}_{ij} = y_{ij} - \overline{y}$

ii) Conditional residuals, $\widehat{\mathbf{e}}_i = \mathbf{y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}} - \mathbf{Z}_i\widehat{\mathbf{b}}_i$ predictors of conditional errors $\mathbf{e}_i = \mathbf{y}_i - \mathbb{E}[\mathbf{y}_i|\mathbf{b}_i] = \mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta} - \mathbf{Z}_i\mathbf{b}_i$
$\widehat{e}_{ij} = y_{ij} - \widehat{y}_i$

iii) Random effects residuals, $\mathbf{Z}_i\widehat{\mathbf{b}}_i$, predictors of random effects,
$\mathbf{Z}_i\mathbf{b}_i = \mathbb{E}[\mathbf{y}_i|\mathbf{b}_i] - \mathbb{E}[\mathbf{y}_i] = (\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta} - \mathbf{Z}_i\mathbf{b}_i) - (\mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta})$
$\widehat{b}_{ij} = \widehat{y}_i - \overline{y}$

LMM Residual analysis

- Standardized marginal residual: $\widehat{\boldsymbol{\xi}}^*_{ij} = \widehat{\boldsymbol{\xi}}_{ij}/[diag_{ij}(\widehat{\mathbb{V}}(\widehat{\boldsymbol{\xi}}_i))]^{1/2}$

$$\widehat{\mathbb{V}}(\widehat{\boldsymbol{\xi}}_i) = \widehat{\boldsymbol{\Omega}}_i - \mathbf{X}_i(\mathbf{X}_i\widehat{\boldsymbol{\Omega}}_i^{-1}\mathbf{X}_i)^{-1}\mathbf{X}_i^\top$$

- Standardized conditional residual: $\widehat{\mathbf{e}}^*_{ij} = \widehat{\mathbf{e}}_{ij}/[diag_j(\widehat{\mathbf{R}}_i\widehat{\mathbf{Q}}_i\widehat{\mathbf{R}}_i)]^{1/2}$

$$\widehat{\mathbb{V}}(\widehat{\mathbf{e}_i}) = \widehat{\mathbf{R}}_i\widehat{\mathbf{Q}}_i\widehat{\mathbf{R}}_i \text{ with } \widehat{\mathbf{Q}}_i = \widehat{\boldsymbol{\Omega}}_i^{-1} - \widehat{\boldsymbol{\Omega}}_i^{-1}\mathbf{X}_i\left(\mathbf{X}_i^\top\widehat{\boldsymbol{\Omega}}_i^{-1}\mathbf{X}_i\right)^{-1}\mathbf{X}_i^\top\widehat{\boldsymbol{\Omega}}_i^{-1}$$

- Modified Lesaffre-Verbeke index:

$$\mathcal{V}_i = ||\mathbf{I}_{m_i} - \widehat{\mathbb{V}}(\widehat{\boldsymbol{\xi}}_i)^{-1/2}\widehat{\boldsymbol{\xi}}_i\widehat{\boldsymbol{\xi}}_i^\top\widehat{\mathbb{V}}(\widehat{\boldsymbol{\xi}}_i)^{-1/2}||^2, \quad \mathcal{V}_i^* = \sqrt{\mathcal{V}_i}/m_i$$

- Mahalanobis distance: $\mathcal{M}_i = \widehat{\mathbf{b}}_i^\top[\widehat{\mathbb{V}}(\widehat{\mathbf{b}}_i - \mathbf{b}_i)]^{-1}\widehat{\mathbf{b}}_i$

Confounding

- $\widehat{\mathbf{e}} = \widehat{\mathbf{R}}\widehat{\mathbf{Q}}\mathbf{e} + \widehat{\mathbf{R}}\widehat{\mathbf{Q}}\mathbf{Z}\mathbf{b}$
- Hilden-Minton (1995, PhD thesis, UCLA): ability to check for normality of $\mathbf{e}$, using $\widehat{\mathbf{e}}$, decreases as $\mathbb{V}[\mathbf{RQZ}^\top\mathbf{b}] = \mathbf{RQZGZ}^\top\mathbf{QR}$ increases in relation to $\mathbb{V}[\mathbf{RQe}] = \mathbf{RQRQR}$
- Fraction of confounding for the k-th conditional residual $\widehat{e}_k$

$$0 \leq F_k = \frac{\mathbf{u}_k^\top \mathbf{RQZ\Gamma Z}^\top \mathbf{QR}\mathbf{u}_k}{\mathbf{u}_k^\top \mathbf{RQR}\mathbf{u}_k} = 1 - \frac{\mathbf{u}_k^\top \mathbf{RQRQR}\mathbf{u}_k}{\mathbf{u}_k^\top \mathbf{RQR}\mathbf{u}_k} \leq 1$$

- Least confounded residual: linear transformation $\mathbf{c}^\top\widehat{\mathbf{e}}$ that minimizes Fraction of Confounding
- Least confounded residuals: homoskedastic, uncorrelated
- Useful for constructing QQ plots to check for normality

Residual diagnostics for Gaussian LMM

Diagnostic for	Residual	Plot
Linearity of effects fixed ($\mathbb{E}[\mathbf{y}] = \mathbf{X}\boldsymbol{\beta}$)	Marginal	$\widehat{\xi}_{ij}^{*}$ *vs* fitted values or explanatory variables
Presence of outlying observations	Marginal	$\widehat{\xi}_{ij}^{*}$ *vs* observation indices
Within-subjects covariance matrix ($\mathbf{V}_i$)	Marginal	$\mathcal{V}_i^{*}$ *vs* unit indices
Presence of outlying observations	Conditional	$\widehat{e}_{ij}^{*}$ *vs* observation indices
Homoskedasticity of conditional errors ($\mathbf{e}_i$)	Conditional	$\widehat{e}_{ij}^{*}$ *vs* predicted values
Normality of conditional errors ($\mathbf{e}_i$)	Conditional	Normal QQ plot for $\mathbf{c}_k^{\top}\widehat{\mathbf{e}}^{*}$
Presence of outlying units	Random effects	$\mathcal{M}_i$ *vs* unit indices
Normality of the random effects ($\mathbf{b}_i$)	Random effects	χ_q^2 QQ plot for $\mathcal{M}_i$

- Diagnostic tools depend on correct specification of covariance structure

Post diagnostic treatment

- Fine tuning of the model based on diagnostic tools
  - Examination of individual profiles [Rocha and Singer (2014), submitted]
  - Plots of correlations *vs* lags [Grady and Helms (1995, SIM)]
  - Use of covariates to model covariance structure [Cúri and Singer (2006), Environ Ecol Stat]
- Elliptically symmetric distributions
  - Useful to accommodate outliers (multivariate t, slash, contaminated normal)
  - Estimation similar but more complicated than gaussian case
  - Local influence [Osorio et al. (2007, CSDA)]
  - Residual analysis?
- Skew-elliptical distributions
  - Fitting is difficult in practice; usually must consider Bayesian methods
  - Local influence [Jara et al. (2008, CSDA)]
  - Residual analysis?
- Robust estimation [Koller (2013), PhD thesis, ETH Zürich]
  - M-methods
  - Limited choices for covariance structure

Computation

Library	Function	Fits	Random effects distribution	$\mathbf{G}$ or $\mathbf{R}_W$ matrix	Error distribution	$\mathbf{R}_i$ matrix
lme4	lmer	LMM	gaussian	unstructured $\mathbf{G}$	gaussian	$\sigma^2\mathbf{I}_{m_i}$
	nlmer	NLMM	gaussian	unstructured $\mathbf{G}$	gaussian	structured
	glmer	GLMM	gaussian	unstructured $\mathbf{G}$	exponential family	NA
nlme	lme	LMM	gaussian	structured $\mathbf{G}$	gaussian	structured
	nlme	NLMM	gaussian	structured $\mathbf{G}$	gaussian	structured
	gls	LM	NA	NA	gaussian	structured
gee	gee	GEE-based model	NS	structured $\mathbf{R}_W$	exponential family or NS	NA
geepack	geeglm	GEE-based model	NS	structured $\mathbf{R}_W$	exponential family or NS	NA
heavy	heavyLme	ES-LMM	elliptically symmetric	unstructured $\mathbf{G}$	elliptically symmetric	NA
robustlmm	rlmer	Robust LMM	symmetric	diagonal or unstructured $\mathbf{G}$	symmetric	$\sigma^2\mathbf{I}_{m_i}$

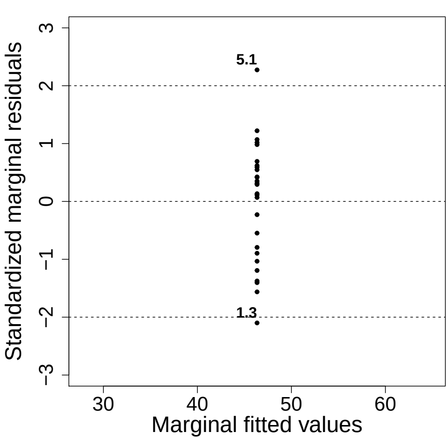
NA: not applicable
NS: not specified

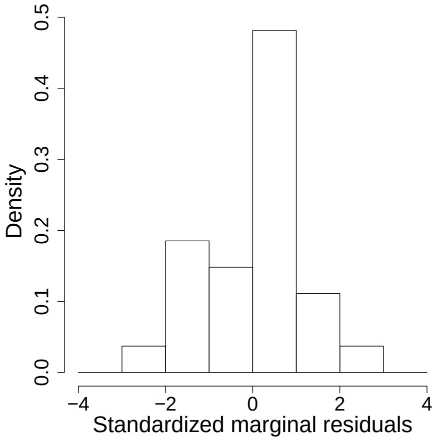
- Some functions for diagnostic available only from authors
- Difficult to use in more complicated problems
- First version of functions for residual diagnostic based on lme4 and nlme available from www.ime.usp.br/~jmsinger/lmmdiagnostics.zip

Ozone example - standard model (A)

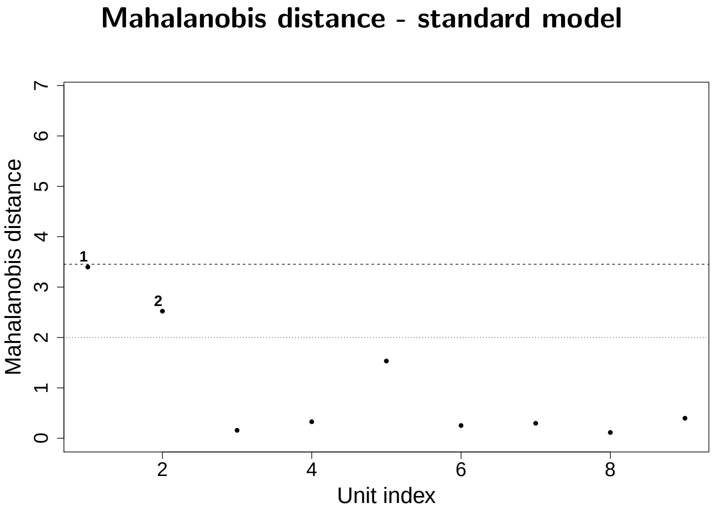
Results (standard model): $\widehat{\mu} = 46.4$, $\widehat{\sigma}_a^2 = 100.4$, $\widehat{\sigma}^2 = 104.8$, $\widehat{k} = 0.75$

Standardized marginal residuals - standard model

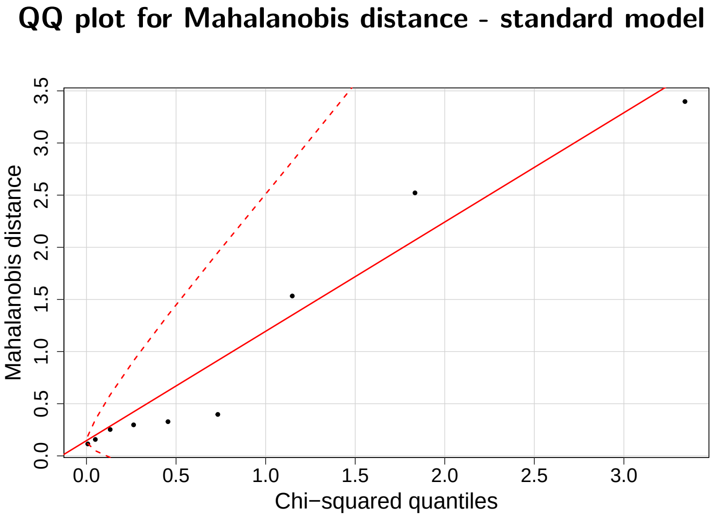




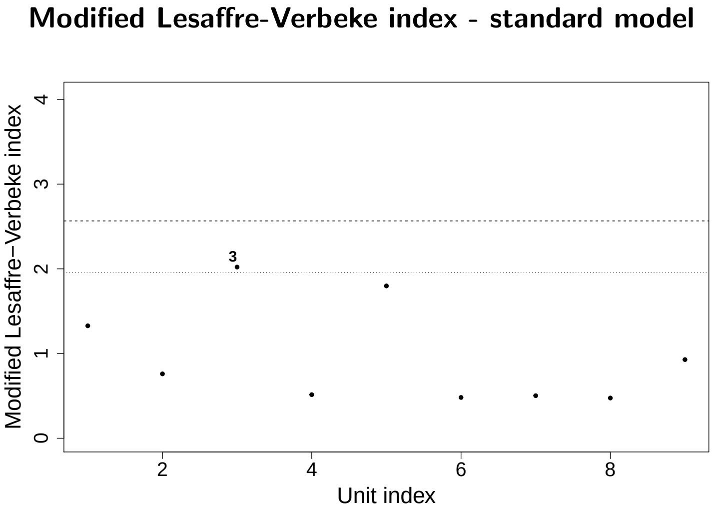
Ozone example - standard model (B)



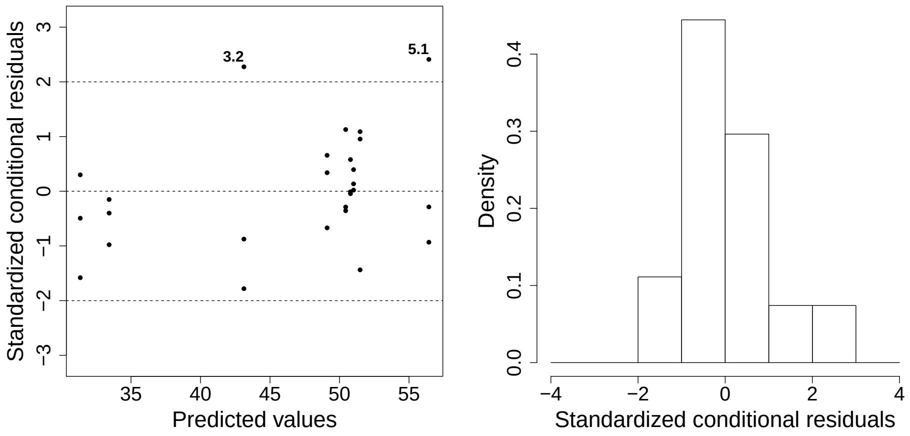
Ozone example standard model (C)



Ozone example standard model (D)

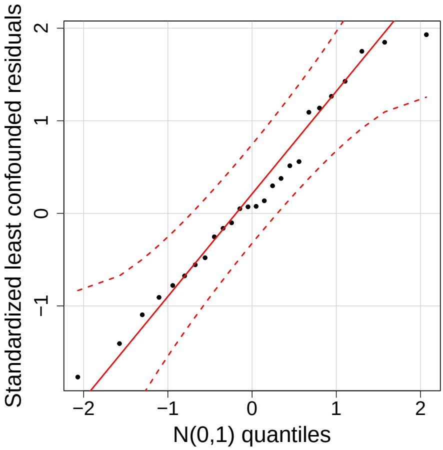


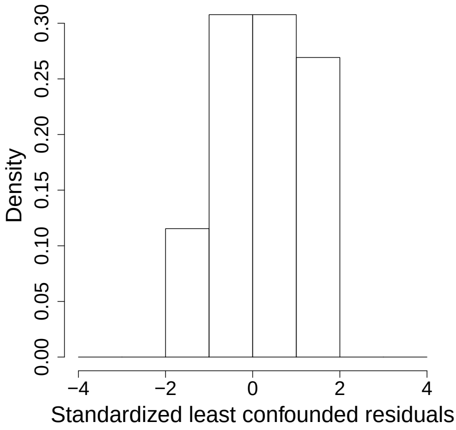
Standardized conditional residuals - standard model



Ozone example - standard model (F)

Standardized least confounded conditional residuals - standard model





Ozone example - heteroskedastic model 1 (A)

- Suggested (heteroskedastic) model

$$y_{ij} = \mu + a_i + e_{ij} \text{ with } e_{ij} \sim N(0, \sigma_i^2)$$

- For parsimony: $\sigma_i^2 = \tau^2, i = 3, 5$, $\sigma_i^2 = \sigma^2$, otherwise
- Shrinkage constant: $k_i = \sigma_a^2/(\sigma_a^2 + \sigma_i^2/3)$
- Predicted latent value for period i

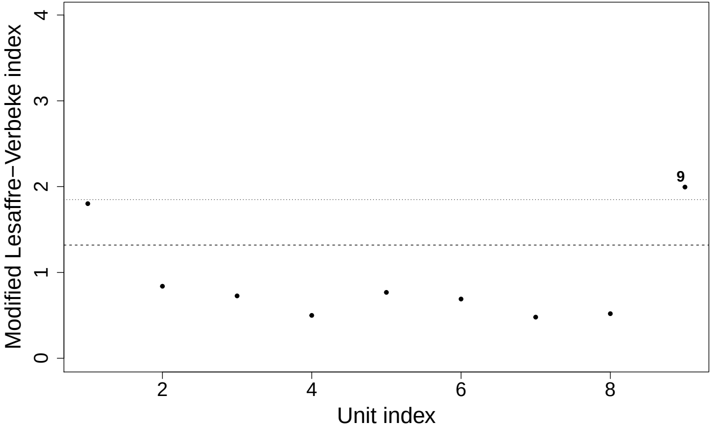
$$\widehat{y}_i = \widehat{\mu} + \widehat{k}_i(\overline{y}_i - \widehat{\mu}), \quad \widehat{\mu} = \sum_{i=1}^{9}(w_i/\sum_{i=1}^{9} w_i)\overline{y}_i, \quad w_i = (\sigma_a^2 + \sigma_i^2)^{-1}$$

Results

- Heteroskedastic model 1: $\widehat{\mu} = 45.9$, $\widehat{\sigma}_a^2 = 114.3$, $\widehat{\sigma}^2 = 49.6$, $\widehat{\tau}^2 = 274.0$, $\widehat{k}_{i\neq 3,5} = 0.87$, $\widehat{k}_{i=3,5} = 0.56$
- Homoskedastic model: $\widehat{\mu} = 46.4$, $\widehat{\sigma}_a^2 = 100.4$, $\widehat{\sigma}^2 = 104.8$, $\widehat{k} = 0.75$

Ozone example - heteroskedastic model 1 (B)

Modified Lesaffre-Verbeke index - heteroskedastic model 1



Ozone example - heteroskedastic model 2 (A)

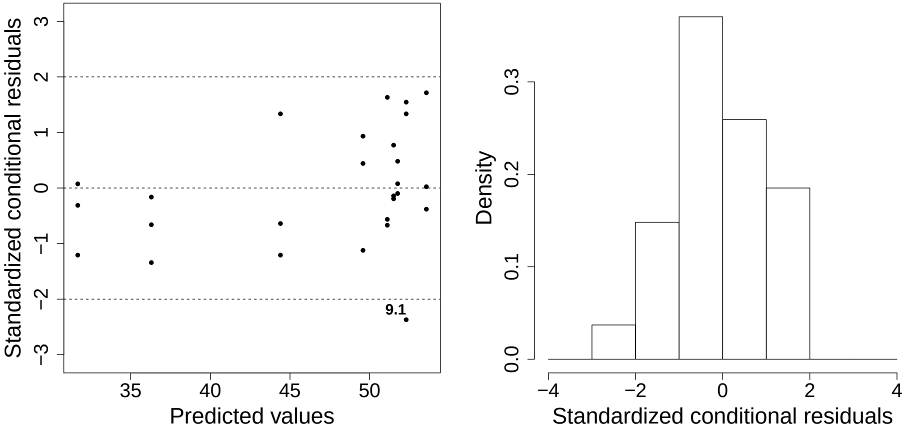
- Suggested (heteroskedastic) model 2

$$y_{ij} = \mu + a_i + e_{ij} \text{ with } e_{ij} \sim N(0, \sigma_i^2)$$

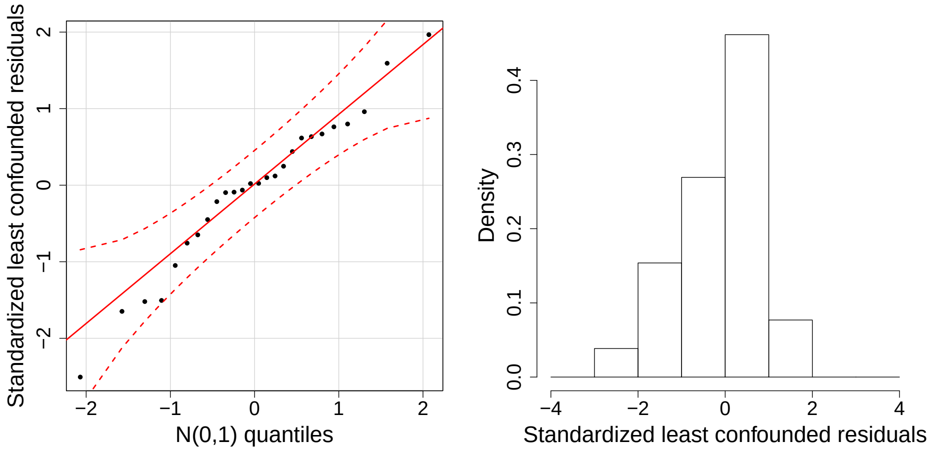
  - $\sigma_i^2 = \tau^2, \quad i = 3, 5,$
  - $\sigma_i^2 = \nu^2, \quad i = 1, 9$
  - $\sigma_i^2 = \sigma^2, \quad i = 2, 4, 6, 7, 8.$
- Results:
  - $\widehat{\mu} = 46.2,$
  - $\widehat{\sigma}_a^2 = 103.8,$
  - $\widehat{\sigma}^2 = 123.6$, $\widehat{\tau}^2 = 270.0$, $\widehat{\nu}^2 = 23.3$
  - $\widehat{k}_{i=1,9} = 0.72,$
  - $\widehat{k}_{i=3,5} = 0.54,$
  - $\widehat{k}_{i=2,4,6,7,8} = 0.93$
- Homoskedastic model: $\widehat{\mu} = 46.4$, $\widehat{\sigma}_a^2 = 100.4$, $\widehat{\sigma}^2 = 104.8$, $\widehat{k} = 0.75$

Ozone example heteroskedastic model 2 (B)

Standardized conditional residuals - heteroskedastic model 2



Standardized least confounded conditional residuals - heteroskedastic model 2



Ozone example - data revisited

Period	Reflectance	Period	Reflectance
1	27.0	6	47.9
1	34.0	6	60.4
1	17.4	6	47.3
2	24.8	7	50.4
2	29.9	7	50.7
2	32.1	7	55.9
3	35.4	8	54.9
3	63.2	8	43.2
3	27.4	8	52.1
4	51.2	9	38.8
4	54.5	9	59.9
4	52.2	9	61.1
5	77.7		
5	53.9		
5	48.2		

Ozone example – Latent value predictions

Period	Sample mean	Homoskedastic LMM	Heteroskedastic LMM (3 variances)	t (df=21.8)	Robust
1	26.1	31.4	31.9	31.8	29.2
2	28.9	33.4	30.2	33.8	31.6
3	42.0	43.1	44.0	43.3	39.4
4	52.6	51.0	52.2	50.9	51.5
5	59.9	56.4	53.5	56.3	54.4
6	51.9	50.4	51.5	50.4	50.9
7	52.3	50.8	51.9	50.8	51.3
8	50.1	49.1	49.8	49.1	49.4
9	53.3	51.5	51.3	51.4	52.7
Mean	46.4	46.4	46.2	46.6	45.7

Shrinkage towards mean (sample mean-latent value prediction)

Period	Homoskedastic LMM	Heteroskedastic LMM (3 variances)	t(df=21.8)	Robust
3	-1.1	-2.0	-1.3	2.6
5	3.5	6.4	3.6	5.5

Boston house price example

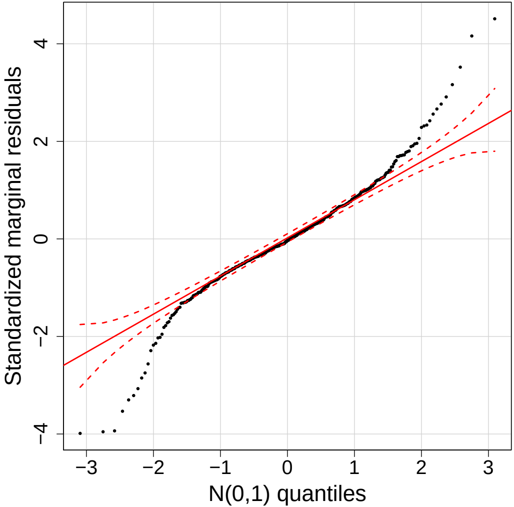
- Data originally from Harrison and Rubinfeld (1978, J Environ Econ & Manag)
- Used as example of heavy-tailed data Belsley et al. (1980, Wiley), Longford (1993, Oxford)
- Objective: Willingness to pay for better air quality based on the analysis of housing market
- Data on 14 variables obtained from 506 SMSA arising form 92 towns
- Belsley et al. (1980) fitted standard linear model via OLS

$$y_i = \text{explanatory variables} + e_i,\ e_i \sim N(0, \sigma^2)$$

$i = 1, \ldots, 506$ and suggested that error distribution should have heavier tails than the gaussian distribution

Boston example: Belsley et al. model

Standardized conditional residuals - Belsley et al. model



Boston example: Longford model

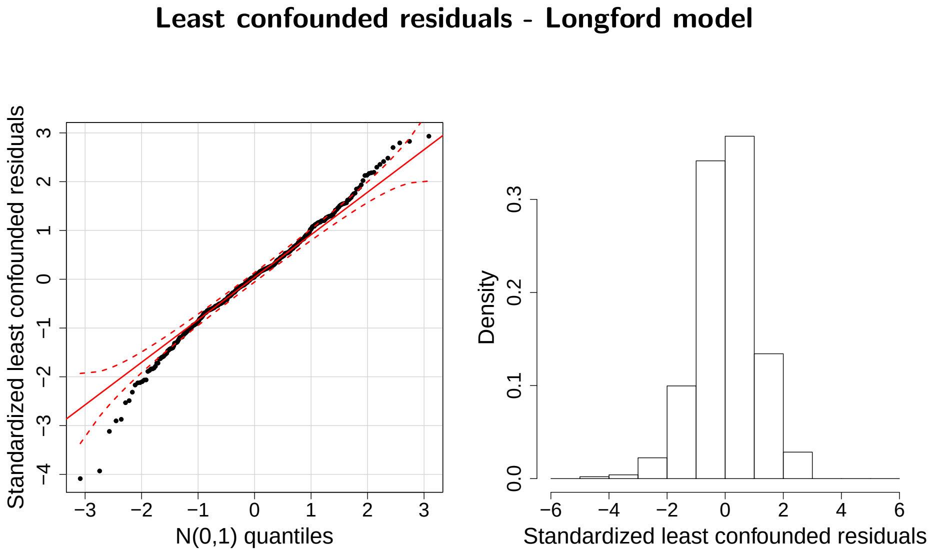
- Observations in the same town should be considered as clusters
- Re-analyzed data introducing random effect for towns

$$
\begin{aligned}
y_{ij} &= \text{explanatory variables} + a_i + e_{ij} \\
a_i &\sim N(0, \tau^2) \\
e_{ij} &\sim N(0, \sigma^2)
\end{aligned}
$$

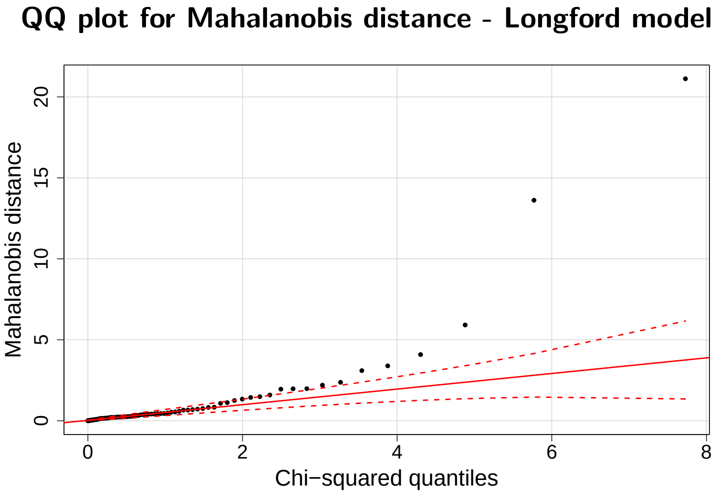
a_i and e_{ij} independent $i = 1, \ldots, 92, j = 1, \ldots m_i$

- Concluded that heavy-tailedness of error distribution is attenuated

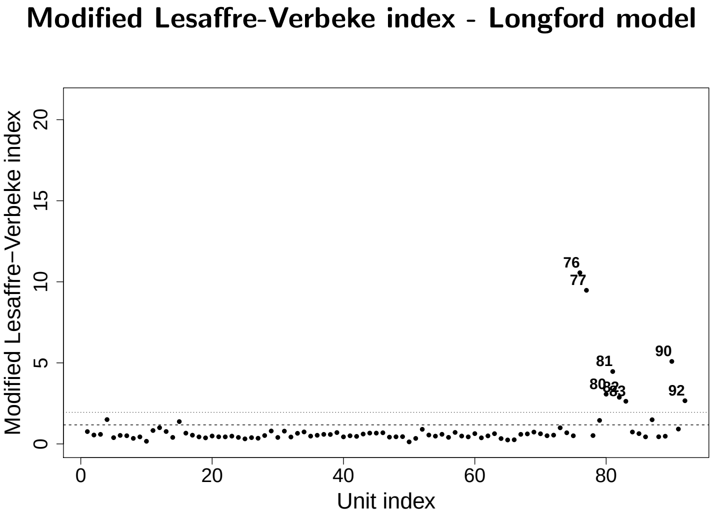
Boston example: Longford model



Boston example: Longford model



Boston example: Longford model diagnostics



Boston example: final model

- Diagnostics re-examined after fitting alternative models
- Final model

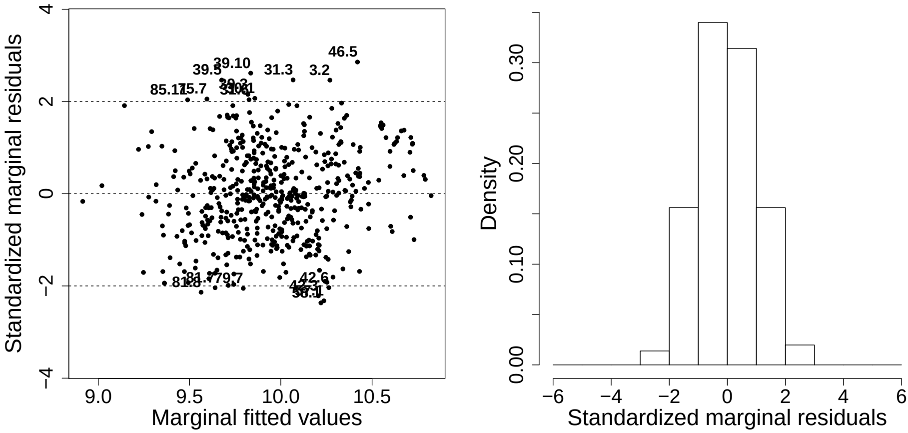
$$
\begin{aligned}
y_{ij} &= \text{explanatory variables} + a_i + e_{ij}\\
a_i &\sim N(0,\tau^2)\\
\mathbf{e}_i &\sim N_{m_i}(\mathbf{0},\mathbf{R}_i),\ \mathbf{R}_i \text{ compound symmetry}\\
\mathbf{R}_i &= \mathbf{R}_i^*, i = 4, 15, \ldots, 91, 92:\ \text{selected towns}\\
\mathbf{R}_i &= \mathbf{R}^*,\ \text{otherwise}
\end{aligned}
$$

a_i and e_i independent $i = 1, \ldots, 92, j = 1, \ldots m_i$

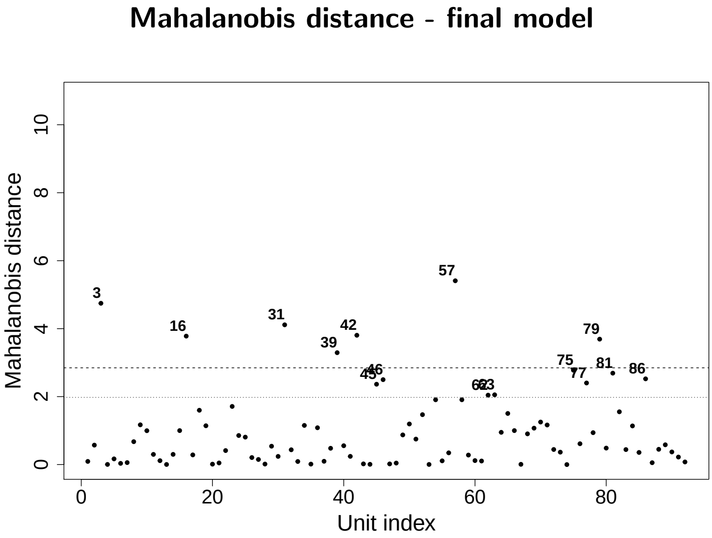
- Compound symmetry to allow for possible within-unit spatial correlation
- Different within-unit covariance matrix for selected SMSAs

Boston example: final model

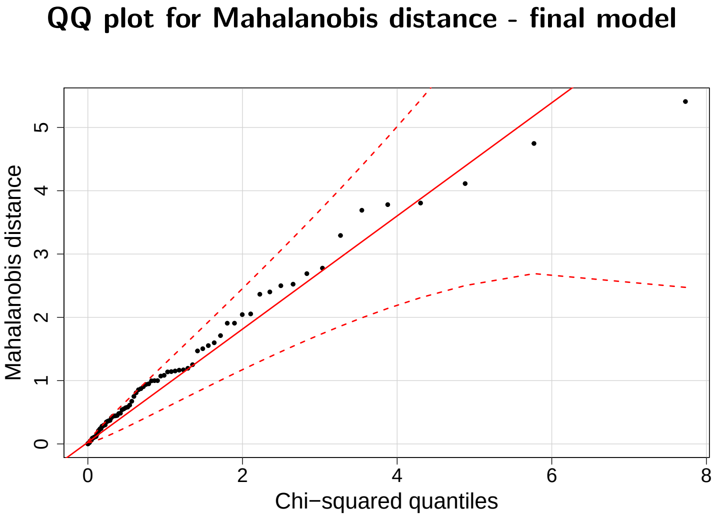
Standardized marginal residuals - final model



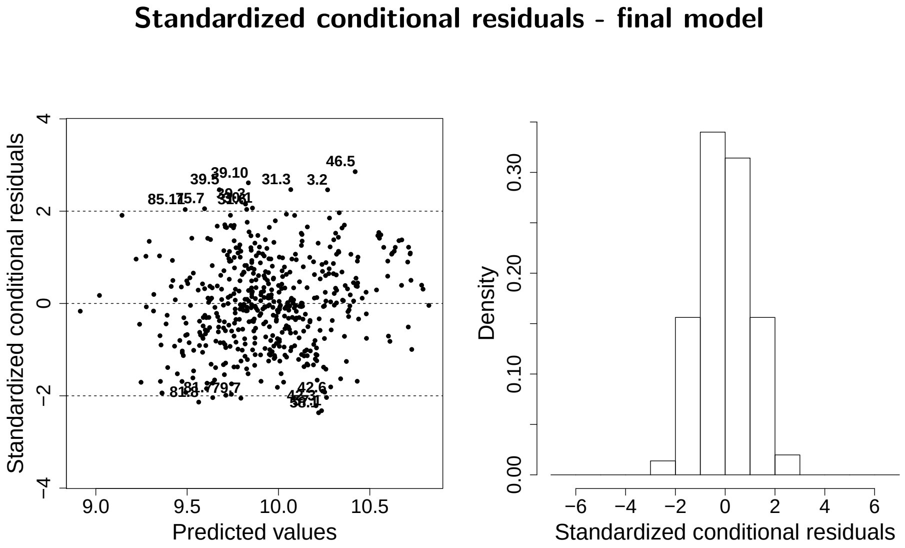
Boston example: final model



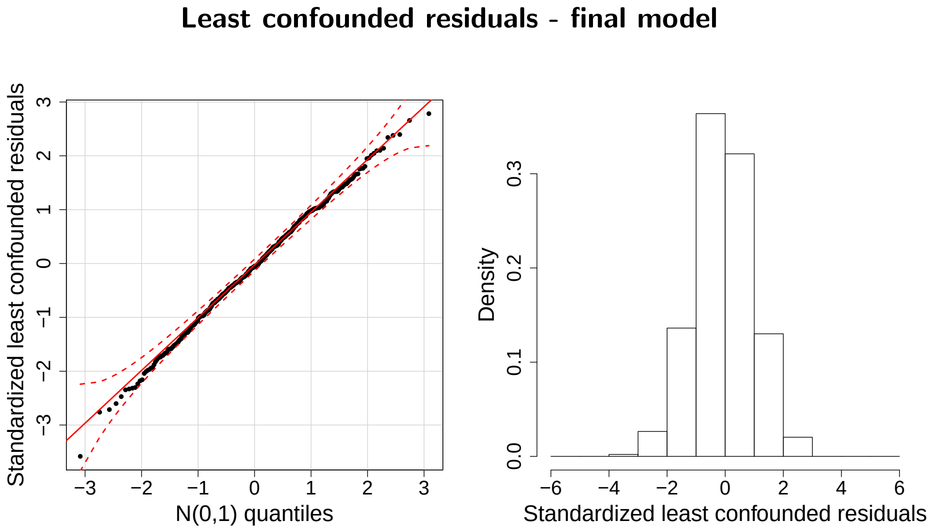
Boston example: final model



Boston example: final model



Boston example: final model



Boston example: estimates of fixed effects

	Belsley		Gaussian LMM				t -LMM		Robust	
			Longford		Final		df=2.19		LMM	
Var	Estim	SE	Estim	SE	Estim	SE	Estim	SE	Estim	SE
(Int)	9.756	0.150	9.672	0.214	9.669	0.125	9.682	0.250	9.612	0.158
crim	-0.012	0.001	-0.007	0.001	-0.006	0.002	-0.006	0.002	-0.006	0.001
zn	0.000	0.001	0.000	0.001	0.000	0.000	0.001	0.001	0.001	0.001
indus	0.000	0.002	0.002	0.005	0.003	0.003	0.003	0.006	0.004	0.003
chasyes	0.091	0.033	-0.014	0.029	0.017	0.018	0.016	0.042	-0.004	0.022
nox	-0.006	0.001	-0.006	0.001	-0.005	0.001	-0.005	0.003	-0.005	0.001
rm	0.006	0.001	0.009	0.001	0.018	0.001	0.018	0.002	0.014	0.001
age	0.000	0.001	-0.001	0.001	-0.002	0.000	-0.002	0.001	-0.002	0.000
dis	-0.191	0.033	-0.125	0.047	-0.116	0.029	-0.117	0.058	-0.109	0.035
rad	0.096	0.019	0.098	0.030	0.062	0.018	0.073	0.027	0.078	0.022
tax	-0.000	0.000	-0.000	0.000	-0.000	0.000	-0.000	0.000	-0.000	0.000
ptratio	-0.031	0.005	-0.030	0.010	-0.024	0.006	-0.025	0.011	-0.026	0.008
blacks	0.364	0.103	0.582	0.101	0.657	0.088	0.639	0.186	0.659	0.076
lstat	-0.371	0.025	-0.282	0.024	-0.113	0.016	-0.099	0.032	-0.195	0.018