A characterization of Lie algebras of skew-symmetric elements.

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Abstract

A characterization of Lie algebras of skew-symmetric elements of associative algebras with involution is obtained. It is proved that a Lie algebra L is isomorphic to a Lie algebra of skew-symmetric elements of an associative algebra with involution if and only if L admits an additional (Jordan) trilinear operation $\{x, y, z\}$ that satisfies the identities

$$
\begin{aligned}
\{x, y, z\} &= \{z, y, x\}, \\
[[x, y], z] &= \{x, y, z\} - \{y, x, z\}, \\
[\{x, y, z\}, t] &= \{[x, t], y, z\} + \{x, [y, t], z\} + \{x, y, [z, t]\}, \\
\{\{x, y, z\}, t, v\} &= \{\{x, t, v\}, y, z\} - \{x, \{y, v, t\}, z\} + \{x, y, \{z, t, v\}\},
\end{aligned}
$$

where $[x, y]$ stands for the multiplication in L.

Let $(A, *)$ be an associative algebra with an involution $*$. It is well-known that the set $H(A, *) = \{a \in A \,|\, a^* = a\}$ of symmetric elements of A is closed with respect to the Jordan product $a \circ b = 1/2(ab + ba)$ and forms a Jordan algebra with respect to this product, while the set $K(A, *) = \{a \in A \,|\, a^* = -a\}$ of skew-symmetric elements of A forms a Lie algebra with respect to the commutator product $[a, b] = ab - ba$.

In 1957 P.Cohn [2] gave a characterization of Jordan algebras of symmetric elements. He proved that a Jordan algebra J is isomorphic to a Jordan algebra of type $H(A, *)$ for a certain associative algebra with involution $(A, *)$ if and only if J admits an additional quadrilinear operation $[x, y, z, t]$ that satisfies some identities involving the multiplication in J.

Here we prove an analogue of Cohn's result for Lie algebras of skew-symmetric elements. Namely, we prove that a Lie algebra L is isomorphic to a Lie algebra of type $K(A, *)$ for a certain associative algebra with involution $(A, *)$ if and only if L admits a trilinear product $\{x, y, z\}$ such that, with respect to this triple product and the original Lie multiplication $[x, y]$, L forms a so called Lie-Jordan algebra (see [1]).

Let us recall the definition of Lie-Jordan algebra. A vector space L is called a *Lie-Jordan algebra* if L has bilinear operation $[,]$ and trilinear operation $\{,,\}$ such that the following identities hold:

$$\begin{aligned}
\{x, y, z\} &= \{z, y, x\}, \\
[[x, y], z] &= \{x, y, z\} - \{y, x, z\}, \\
[\{x, y, z\}, t] &= \{[x, t], y, z\} + \{x, [y, t], z\} + \{x, y, [z, t]\}, \\
\{\{x, y, z\}, t, v\} &= \{\{x, t, v\}, y, z\} - \{x, \{y, v, t\}, z\} + \{x, y, \{z, t, v\}\}.
\end{aligned}$$

It is easy to see that every Lie-Jordan algebra is a Lie algebra with respect to the operation $[,]$ and a Jordan triple system with respect to the ternary operation $\{,,\}$. Furthermore, every associative algebra A forms a Lie-Jordan algebra A^+ with respect to the commutator product $[x, y]$ and the *triple Jordan product* $\{x, y, z\} = xyz + zyx$. The main result of our previous paper [1] claims that every Lie-Jordan algebra is *special*, that is, isomorphic to a subalgebra of the algebra A^+ for a certain associative algebra A.

Now, if A has an involution $*$ then evidently the space $K(A, *)$ is closed with respect to the both commutator and triple Jordan product and hence is a subalgebra of the Lie-Jordan algebra A^+.

We will prove that, conversely, every Lie-Jordan algebra is isomorphic to an algebra of type $K(A, *)$.

Theorem 1 *Let L be a Lie algebra over a field k of characteristic $\neq 2$. Then L is isomorphic to a Lie algebra $K(A, *)$ for a certain associative algebra with involution $(A, *)$ if and only if L admits a trilinear operation $\{,,\}$ such that L is a Lie-Jordan algebra.*

Proof. Let L be a Lie-Jordan algebra; then an associative algebra $U(L)$ is said to be a *universal enveloping algebra* for L if there exists a homomorphism $\alpha_L : L \longrightarrow U(L)^+$ such that for any associative algebra A and a homomorphism $\beta : L \longrightarrow A^+$ there exists a homomorphism π of associative algebras $\pi : U(L) \longrightarrow A$ such that $\beta = \alpha_L \circ \pi$. In other words, there is a bijection

$$Hom_{\underline{Lie-Jord}}(L, A^+) \longrightarrow Hom_{\underline{Ass}}(U(L), A),$$

which is functorial on the variables L and A.

The existence of a universal enveloping algebra $U(L)$ for a given Lie-Jordan algebra L is obvious. It is isomorphic to the quotient algebra of the tensor algebra $T(L)$ by the ideal I generated by all the elements

$$a \otimes b - b \otimes a - [a, b], \quad a \otimes b \otimes c + c \otimes b \otimes a - \{a, b, c\}, \ a, b, c \in L;$$

with the universal homomorphism $\alpha_L : a \longrightarrow a + I$. By [1, Theorem 1], we have $\ker \alpha_L = 0$, hence we can identify L with its image $\alpha_L(L)$ and assume that L is a subalgebra of $U(L)^+$.

Define on the tensor algebra $T(L)$ an involution $*$ by setting $l^* = -l$ for every $l \in L$. For any $a, b, c \in L$ we have

$$
\begin{aligned}
(a \otimes b - b \otimes a - [a, b])^* &= b^* \otimes a^* - a^* \otimes b^* - ([a, b])^* = b \otimes a - a \otimes b + [a, b] \\
&= -(a \otimes b - b \otimes a - [a, b]) \in I; \\
(a \otimes b \otimes c + c \otimes b \otimes a - \{a, b, c\})^* &= c^* \otimes b^* \otimes a^* + a^* \otimes b^* \otimes c^* - \{a, b, c\}^* \\
&= -(a \otimes b \otimes c + c \otimes b \otimes a - \{a, b, c\}) \in I.
\end{aligned}
$$

Hence the ideal I is invariant with respect to the involution $*$, and so this involution induces an involution on the quotient algebra $U(L) = T(L)/I$. We will denote the induced involution also by $*$.

Let us prove that $L = K(U(L), *)$. It is clear that $L \subseteq K(U(L), *)$. In order to prove the inverse inclusion, consider the structure of $U(L)$. Since L generates $U(L)$, we have

$$U(L) = L + LL + LLL + \cdots .$$

For any $a, b, c \in L$ we have

$$a \otimes b \otimes c - 1/2 \left(\{a, b, c\} + [b, c] \otimes a + b \otimes [a, c] + [a, b] \otimes c \right) \in I,$$

which implies that $U(L) = L + LL$. Moreover, for any $a, b \in L$ we have

$$ab = a \circ b + 1/2[a, b],$$

and since $[a, b] \in L$, this yields that $U(L) = L + L \circ L$. Evidently, $L \circ L \subseteq H(U(L), *)$, so we finally have

$$K(U(L), *) = L, \quad H(U(L), *) = L \circ L.$$

This proves the theorem.

References

[1] A.N.Grishkov, I.P.Shestakov, *Speciality of Lie-Jordan Algebras*, Journal of Algebra 237, (2001) 621-636.

[2] P. M. Cohn, *Two embedding theorems for Jordan algebras*, Proc. London Math. Soc. (3) 9 (1959) 503–524.