Representations of completely solvable Lie algebras over a ring of polynomials

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1 RÉSUMÉ

Si R est un anneau de polynômes et L est une R-algèbre complétement résoluble, c'est-à-dire qu'il existe une chaine d'idéaux $L = L_0 \subset L_1 \subset ... \subset L_n \subset 0$, avec L_i/L_{i+1} un R-module libre de dimension 1, alors L admet une représentation par matrices triangulaires sur R.

In this article we show that a completely solvable Lie algebra over a ring of polynomials has a triangulable faithful representation. This fact is important in the theory of analytic diassosiative loops. In 1955 Malcev noted [5] that, every Binary Lie algebra B over the field $\mathbf{R}$ of real numbers, some neighborhood $D \subset B$ of zero can be considered as a local analytic diassociative loop with the multiplication $x \star y = H(x, y) = x + y + [x, y]/2 + \ldots$, where $H(x, y)$ stands for the Campbell-Hausdorf series. Recall that by definition an algebra B is Binary Lie if every two elements of B generate a Lie subalgebra. By the Cartan theorem, for every local analytic Lie group, there exists a corresponding global analytic Lie group. An analogy is not valid for local analytic diassociative loops [1]. The problem of existence of a global analytic dissociative loop for a given Binary Lie algebra is not easy. The following can help in solving the problem.

Let B be a Binary Lie algebra over $\mathbf{R}$ with a finite basis $\{b_1, \ldots, b_n\}$ and $B_K = B \otimes_R K$ be corresponding Binary Lie over the polynomial ring $K = \mathbf{R}[x_1, x_2, \ldots, y_1, y_2, \ldots]$. Denote $X = x_1 b_1 + \ldots + x_n b_n$ and $Y = y_1 b_1 + \ldots + y_n b_n$, and let $B(X, Y)$ be a Lie subalgebra generated by X, Y. By the theorem [2], the algebra B is completely solvable, hence, $B(X, Y)$ is so. By the theorem 1 of the present work $B(X, Y)$ has a faithful triangulable representation π. Therefore, there exists a triangular matrix Z such that $exp(\pi(X))exp(\pi(Y)) = exp(Z)$. Suppose that $Z = f_1\pi(b_1) + \ldots + f_n\pi(b_n)$, where $f_1, \ldots, f_n$ are analytic functions on $\mathbf{R}^{2n}$. Then $\mathbf{R}^n$ is a global analytic diassociative loop with a multiplication given by the rule

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = (f_1, \ldots, f_n).$$

Moreover, this loop corresponds to B.

Let R be a polynomial ring, $R = k[x_1, \ldots, x_m]$, where k is a field of characteristic 0, let Q be the field of fractions of R. For every R-module V, we denote by $dim_R V$ the dimension of the Q-module $V \otimes_R Q$. We call an Lie algebra L over R *completely solvable* if L admits some normal series

$$L = L_1 > L_2 > \ldots > L_n = 0,$$

such that $dim_R(L_i/L_{i+1}) = 1, i = 1, \ldots, n-1$.

A module V over the Lie algebra L_R is called *triangulable* if V is free R-module with a free basis $\{v_1, \ldots, v_p\}$, such that $V_i = \{v_i, \ldots, v_p\}_R$ is an L-submodule.

The main result of this section is the following

Theorem 1 *Let $R = k[x_1, \ldots, x_m]$, L be a completely solvable Lie algebra over R, and let L be free as an R-module. Then L has a faithfull triangulable module.*

Proof. Suppose that L is a splitting Lie algebra, i.e. $L = T \oplus N$, where N is the nilradical of L and T is a torus. Moreover, T and N are free R-module. Let $t_1, \ldots, t_q$ be a free basis of the R-module T. First we will prove the statement in this particular case by induction on q.

1. Case $q = 0$. Let $\mathcal{L} = L \otimes_R Q$, a nilpotent Lie algebra over Q. By the Ado theorem [3], $\mathcal{L}$ has a faithfull finite-dimensional representation $\rho : \mathcal{L} \to \quad End_Q V$ such that all elements $\rho(x), x \in \mathcal{L}$ are nilpotent. Fix a Q-basis in V and generators $n_1, \ldots, n_l$ of the R-module $\mathcal{L}$. We can find $a \in R$ such that all coefficients of the matrices $a\rho(n_1), \ldots, a\rho(n_l)$ are in R. We denote $D = diag(1, a, \ldots, a^{n-1})$, where $n = dim_Q V$. If the matrices $\rho(n_1), \ldots, \rho(n_l)$ are upper triangular, then the matrices

$$D^{-1}\rho(n_1)D, \ldots, D^{-1}\rho(n_l)D$$

have coefficients in R. It means that the Lie R-algebra L has faithfull triangable module over R.

2. Case $q > 0$. Denote $t = t_q$ and $L_0 = T_0 \oplus N$, where $T_0 = \{t_1, \ldots, t_{q-1}\}_R$. Now it is suffices to prove the following

Lemma 1 . *Let L be a completely solvable Lie R-algebra and let $L = Rt \oplus L_0$, where L_0 is an ideal of L. Then, for every representation $\rho_0 : L_0 \to End_R(V_0)$, where V_0 is a free R-module, there exists an extension $\rho : L \to End_R(V)$ such that V is a free R-module and $V = V_0 \oplus V_1$, V_1 is an R-submodule of V, $\rho|_{V_0} = \rho_0$. Moreover, if V_0 is a triangulable L_0-module, then V is triangulable.*

Proof. Let W be an L-module induced by V_0, $W = \sum_{i=0}^{\infty} \oplus V_0 \otimes t^i$. The lemma will be proved if we construct a unitary polynomial $f(x) \in R[x]$ such that $V_0 f(t) L \cap V_0 = 0$ with all the roots of $f(x)$ are in R. We will construct $f(x)$ by induction on $dim V_0$. If $dim V_0 = 1$, we can take $f(x) = x^2$. In the case of $dim V_0 > 0$, we need the following

Lemma 2 *Let $s(x)$ and $p(x)$ be unitary polynomials, $s(x), p(x) \in R[x]$. Then there exists an unitary polynomial $f(x) \in R[x]$ such that, in the ring $R[x, y]$,*

$$f(x + y) = f(y) + s(x)r_1(x, y) + xp(x)r_2(x, y) \tag{1}$$

for some $r_1(x, y), r_2(x, y) \in R[x, y]$. Moreover, if all the roots of $s(x)$ and $p(x)$ are in R then all the roots of $f(x)$ are in R.

Proof. Let R_1 be an extension of R such that $s(x) = (x-\alpha_1)\cdot\ldots\cdot(x-\alpha_n)$ and $p(x) = (x-\beta_1)\cdot\ldots\cdot(x-\beta_m)$ for some $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m \in R_1$. We define

$$f(x) = p^n(x)\prod_{i,j}(x-\alpha_i-\beta_j).$$

Note that $f(x) \in R[x]$ if $s(x), p(x) \in R[x]$. We will prove (1) by induction on n. For $n = 1$,

$$f(x+y) - f(x) =$$

$$p(x+y)\prod_j(x+y-\alpha_1-\beta_j) - p(y)\prod_j(y-\alpha_1-\beta_j) =$$

$$p(x+y)\prod_j[(x-\alpha_1)+(y-\beta_j)] - p(y)\prod_j(y-\alpha_1-\beta_j) =$$

$$p(x+y)\prod_j(y-\beta_j) + (x-\alpha_1)r_1'(x,y) - p(y)\prod_j(y-\alpha_1-\beta_j) =$$

$$(x-\alpha_1)r_1'(x,y) + p(y)[\prod_j(x+y-\beta_j) - \prod_j(y-\alpha_1-\beta_j)].$$

Since

$$\prod_j(x+y-\beta_j) = \quad xr_3(x,y)+p(y),$$

$$\prod_j(y-\alpha_1-\beta_j) = \quad \prod_j[(x-\alpha_1)+(y-\beta_j)-x] =$$

$$(x-\alpha_1)r_4(x,y)+ \quad xr_5(x,y)+p(y),$$

we obtain

$$[\prod_j(x+y-\beta_j) - \prod_j(y-\alpha_1-\beta_j)] = x(r_3-r_5)-(x-\alpha_1)r_4.$$

The equality (1) is proved for $n = 1$.

Let $n > 1$. Denote $s_1(x) = \prod_i^{n-1}(x-\alpha_i)$. By induction,

$$f(x+y) - f(y) = p^n(x+y)\prod_{i,j}(x+y-\alpha_i-\beta_j) - f(y) =$$

$$\{p^{n-1}(x+y)\prod_{i,j}(x+y-\alpha_i-\beta_j)\}\{p(x+y)\prod_j(x+y-\alpha_n-\beta_j)\}-$$

$$f(y) = \{s_1(x)r_1'(x,y) + xp(y)r_2'(x,y) + p^{n-1}(y)\prod_{i<n,j}(y-\alpha_i-\beta_j)\}\cdot$$

$$\cdot\{(x-\alpha_n)r_3'(x,y) + xp(y)r_4'(x,y) + p(y)\prod_j(y-\alpha_n-\beta_j)\}-$$

$$p^n(y)\prod_{i,j}(y-\alpha_i-\beta_j) = s(x)r_1'(x,y)r_3'(x,y) + xp(y)r_2(x,y).$$

□

Now we return to the proof of Lemma 1. As L_0 is triangulable L_0-module, then there exists an L_0-submodule $V_1 < V_0$ such that $V_0 = V_1 \oplus Rv, v \in V_0$. By induction, there exists a polynomial $p(x)$ such that the induced L-module $W_1 = \sum_{i=0}^{\infty} \oplus V_1 t^i$ contains a submodule $V_2 \supset V_1 p(t)$ and $V_2 \cap V_1 = 0$. Moreover, W_1/V_2 is a triangulable L-module. Denote by $s(x)$ a minimal polynomial such that $Ls(ad(t)) = 0$. By induction, all the roots of $p(x)$ are in R and all the roots of $s(x)$ are in R, since L is completely solvable. By Lemma 2, to given $s(x)$ and $p(x)$, we can associate a polynomial $f(x)$ such that equality (1) holds. We will prove that $W_0 = V_2 + Wf(t)$ is an L-submodule what we need. As $W_0 t \subseteq W_0$, we have to prove that $Wf(t)a \subseteq W_0$ for any $a \in L_0$. Denote $a^i = [a, \underbrace{t, \ldots, t}_{i}], v \in W$. From ([3], p.50), it follows that

$$vf(t)a = \sum_{i=0}^{\infty} va^i f^{(i)}(t)/i!. \tag{2}$$

By Lemma 2, in the commutative ring $R[x, y]$ we get

$$\begin{aligned} &\textstyle\sum_{i=0}^{\infty} x^i f^{(i)}(y)/i! = f(x+y) = \\ &f(y) + s(x)r_1(x,y) + xp(y)r_2(x,y) = \\ &f(y) + s(x)\textstyle\sum_{i=0}^{k_1} r_{1i}(x)y^i + \sum_{i=1}^{k_2} r_{2i}(x)p(y)y^i, \end{aligned} \tag{3}$$

where

$$r_1(x,y) = \sum_{i=0}^{k_1} r_{1i}(x)y^i, r_2(x,y) = \sum_{i=0}^{k_2} r_{2i}(x)y^i.$$

Denote $s_i(x) = s(x)r_{1i}(x)$. From (3), we derive

$$\sum_{i=0}^{\infty} x^i f^{(i)}(y)/i! = f(y) + \sum_{i=0}^{k_1} s_i(x)y^i + \sum_{i=1}^{k_2} r_{2i}(x)p(y)y^i. \tag{4}$$

Let Ass be a free associative algebra over R with free generators $\{y, x_1, x_2, \ldots\}$. Then (4) yields in Ass

$$\sum_{i=0}^{\infty} x_i f^{(i)}(y)/i! = f(y) + \sum_{i=0}^{k_1} s_i^{\sigma}(x)y^i + \sum_{i=1}^{k_2} r_{2i}^{\sigma}(x)p(y)y^i, \tag{5}$$

where σ is a linear map: $R[x] \to Ass$,

$$g(x)^\sigma = \sum_{i=0}^{k} \alpha_i x_i, \quad \text{if} \quad g(x) = \sum_{i=0}^{k} \alpha_i x^i.$$

Finally, (5) implies

$$vf(t)a = vaf(t) + \sum_{i=0}^{k_1} [a, s_i(adt)]t^i + \sum_{i=1}^{k_2} [a, r_{2i}(adt)]p(t)t^i \in W_0,$$

since $[a, r_{2i}(adt)] \in L_0$,$[L, s(adt)] = 0$ and $s_i(x) = s(x)l_i(x)$. We constructed an L-module $U = W/W_0$, which is free as R-module since $f(x)$ is unitary polynomial. The L-module U is triangulable since all the roots of $f(x)$ are in R.

□

With this lemma we can complete the inductive step and prove theorem in the of L splitting.

Suppose now that L is not splitting. Recall that Q is the field of fraction of R. A Lie Q-algebra $L_Q = L \otimes_R Q$ is embedded to a splitting Q-algebra $\mathcal{L} = \mathcal{T} \oplus \mathcal{N}$. Here, $\mathcal{N} = \sum_{\alpha \in \tau} \oplus \mathcal{N}_\alpha$, where $\tau \subset Q^\star$ and $\mathcal{N}_\alpha = \{n \in \mathcal{N} | nt = \alpha(t)n, \forall t \in \mathcal{T}\}$ (see [4]). Let $\{g_1, \ldots, g_n\} \subseteq L$ be free basis of L. In $\mathcal{L}$, we have $g_i = t_i + n_i, t_i \in \mathcal{T}, n_i \in \mathcal{N}, i = 1, \ldots, n$. Note that $g_1, \ldots, g_n \in T = \{t \in \mathcal{T} \mid \alpha(t) \in R, \forall \alpha \in \tau\}$ and T is a free R-module. In $\mathcal{L}$, we have $n_i = \sum_{\alpha \in \tau} n_{i\alpha}, n_{i\alpha} \in \mathcal{N}_\alpha, i = 1, \ldots, n$. Let N_1 be an R-algebra Lie with generators $n_{j\alpha} | \alpha \in \tau, i = 1, \ldots, n\}$. It is obvious that N_1 is a finite-dimensional R-module and $N_1 T \subset N_1$. Hence $L_1 = T \oplus N_1$ is a splitting complitely solvable R-algebra and L is subalgebra of L_1.

□

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