Groups with triality

Alexander N. Grishkov[*]

Departamento de Matemática, Universidade de São Paulo,
Caixa Postal 66281, São Paulo-SP, 05311-970, Brasil,
and Omsk State University, pr. Mira 55-a, 644077, Russia
e-mail: grishkov@ime.usp.br

Andrei V. Zavarnitsine[†]

Sobolev Institute of Mathematics,
pr. Koptyuga 4, Novosibirsk, 630090, Russia
and Departamento de Matemática, Universidade de São Paulo,
Caixa Postal 66281, São Paulo-SP, 05311-970, Brasil
e-mail: zavarn@ime.usp.br

June 8, 2004

Abstract

Groups with triality, which arose in the papers of Glauberman and Doro, are naturally connected with Moufang loops. In this paper, we describe all possible, in a sense, groups with triality associated with a given Moufang loop. We also introduce several universal groups with triality and discuss their properties.

[*]supported by FAPESP, CNPq, and FAPERG, Proc. 03/0516-2, Brazil
[†]supported by FAPESP, Proc. 01/14611-4, Brazil

1 Introduction

With every loop L there are naturally associated several groups: the group $Mlt(L)$ of permutations of L generated by the operators L_x and R_x of left and right multiplication by x in L, the stabilizer $\mathcal{I}(L)$ in $Mlt(L)$ of the identity $1 \in L$, the automorphism group $Aut(L)$, the group of inner automorphisms $Inn(L) = \mathcal{I}(L) \cap Aut(L)$. In the case of Moufang loops Glauberman [5] noted that if L has trivial nucleus then the group $Mlt(L)$ admits a natural action of the symmetric group $S_3 = \langle \sigma, r \mid \sigma^2 = \rho^3 = (\sigma\rho)^2 = 1 \rangle$ so that $L_x^\sigma = R_x^{-1}$, $L_x^\rho = R_x$, $R_x^\rho = L_x^{-1}R_x^{-1}$. Moreover, $\mathcal{I}(L) = \{x \in Mlt(L) \mid x^\sigma = x\}$, $Inn(L) = \{x \in \mathcal{I}(L) \mid x^\rho = x\}$, and the group $Mlt(L)$ in this case satisfies the following identity:

$$(x^{-1}x^\sigma)(x^{-1}x^\sigma)^\rho(x^{-1}x^\sigma)^{\rho^2} = 1 \qquad (1)$$

for all $x \in Mlt(L)$.

Subsequently, Doro [6] called a group G that admits an action of S_3 satisfying (1) a *group with triality* and showed that $G = HM^{\rho^2}$, where $H = \{x \in G \mid x^\sigma = x\}$ and $M = \{x^{-1}x^\sigma \mid x \in G\}$; moreover, $(M^{\rho^2}, \star)$ is a Moufang loop with multiplication $x \star y = z$ iff $xy = hz$ for $h \in H$. He also showed that every Moufang loop can be obtained in this way from a suitable group with triality. This approach made it possible to solve problems about Moufang loops using the well developed theory of groups. For example Liebeck [4] using the classification of finite simple groups proved that every non-associative finite Moufang loop is isomorphic to a Paige loop $M(q)$.

The Moufang loops first appeared in the papers of R. Moufang about projective planes. The above theorem of Doro can be easily (and beautifully) proved [10] using the relation between Moufang loops and projective planes (or 3-nets).

The authors of this paper have used the connection of Moufang loops and groups with triality in the study of subloops of the simple Paige loops $M(q)$. As a consequence, an analog of Lagrange's theorem was proved for finite Moufang loops [12] and the maximal subloops of $M(q)$ were described [13]. In the cited papers we used the fact that for a given group with triality G the set $M = \{x^{-1}x^\sigma \mid x \in G\}$ is a Moufang loop with respect to the multiplication $m.n = m^{-\rho}nm^{-\rho^2}$. This observation gives a (third) simple proof of Doro's theorem (see Theorem 1 of the present paper).

A group with triality corresponding to a given Moufang loop M is not uniquely determined. Doro [6] defined by generators and defining relations a universal group with triality $\mathcal{D}(M)$ such that every other group with triality G corresponding to M and satisfying $[G, S_3] = G$ is a homomorphic image of $\mathcal{D}(M)$.

In turn, Mikheev [7] constructed by an arbitrary Moufang loop M a group with triality $\mathcal{W}(M)$ using the group of pseudoautomorphisms of M. Unfortunately, his paper does not contain proofs which are very non-trivial. In this article, we present a proof of Mikheev's theorem and show that his group with triality $\mathcal{W}(M)$ possesses universal properties dual to those of Doro's group $\mathcal{D}(M)$. In doing so we construct by an arbitrary Moufang loop M a corresponding universal group with triality $\mathcal{U}(M)$ which contains $\mathcal{D}(M)$ and covers $\mathcal{W}(M)$.

There exists another important group with triality $\mathcal{E}(M)$ associated with every Moufang loop M, which is in a sense minimal among all such groups with triality. Moreover, this group $\mathcal{E}(M)$ always covers $Mlt(M)$ and can be viewed as a natural generalization of Glauberman's triality to all Moufang loops. We show that $Mlt(M)$ is a group with triality if and only if it coincides with $\mathcal{E}(M)$.

In the final section we have included some open problems about groups with triality and Moufang loop which in our view are of certain interest and importance.

2 Moufang loops and groups with triality

Introduce some notation. $C_P(Q)$ is the centralizer of Q in P. For x, y in a group G, we put $[x, y] = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$, $x^{-y} = (x^{-1})^y$.

A loop $(Q, .)$ is called a *Moufang loop* if, for all $x, y, z \in Q$, one (hence, any) of the following identities holds:

$$(x.y).(z.x) = (x.(y.z)).x, \qquad ((x.y).x).z = x.(y.(x.z)), \qquad x.(y.(z.y)) = ((x.y).z).y.$$

For basic properties of Moufang loops, see [1]. We use the notation $[\![x, y]\!] = x^{-1}.y^{-1}.x.y$ instead of $[x, y]$ to denote the commutator in Q. By definition, the *nucleus* $Nuc(Q)$ is the set $\{a \in Q \mid a.(x.y) = (a.x).y\}$. The nucleus is a normal subgroup of Q. Also, denote $C(Q) = \{c \in Q \mid xc = cx \ \forall x \in Q\}$.

A group G possessing automorphisms ρ and σ such that $\rho^3 = \sigma^2 = (\rho\sigma)^2 = 1$ is called a *group with triality (relative to ρ and σ)* if the following relation holds for every x in G:

$$[x, \sigma] \cdot [x, \sigma]^{\rho} \cdot [x, \sigma]^{\rho^2} = 1, \qquad (2)$$

where $[x,\sigma]=x^{-1}x^{\sigma}$. Denote $S=\langle\rho,\sigma\rangle$. The relation (2) does not depend on a particular choice of the generators ρ and σ of S (see [6]) and we will thus speak of a group with triality S. If G is a group with triality S then, for $g\in G$, define

$$\xi(g)=g^{-1}g^{\sigma}, \qquad \phi(g)=g^{-\rho}g^{\rho^2}, \qquad \eta(g)=gg^{-\rho\sigma}g^{\rho^2}. \tag{3}$$

Also, put $M=\xi(G)$ and $H=C_G(\sigma)$. Observe that

$$m^{\sigma}=m^{-1}\in M \quad \text{for all} \quad m\in M. \tag{4}$$

Doro showed in [6] that the set M^{ρ^2} is a right transversal of H in G and can be turned to a Moufang loop $(M^{\rho^2},\star)$ by putting

$$m\star n=k \iff mn=hk, \qquad \text{for} \quad h\in H \quad m,n,k\in M^{\rho^2}. \tag{5}$$

In Theorem 1 below we give a new proof of this fact.

Let G be a group with triality S. S-invariant subgroups of G are called *S-subgroups*. A homomorphism $\varphi: G\to H$ of groups with triality G and H is called an S-homomorphism if $\varphi\alpha=\alpha\varphi$ for all $\alpha\in S$. Denote by $Z_S(G)$ the *S-center* of G, which is by definition the maximal normal S-subgroup of G on which S acts trivially. The following dual properties hold:

Lemma 1 *For every group G with triality S, we have*
(i) $[[G,S],S]=[G,S]$ is the S-subgroup of G generated by M,
(ii) $Z_S(G/Z_S(G))=1$ and $Z_S(G)=C_G([G,S]S)$.

Proof. (i) $[G,S]$ is generated by $[G,\sigma]=M$, $[G,\rho\sigma]=M^{\rho}$, $[G,\sigma\rho]=M^{\rho^2}$, $[G,\rho]$, and $[G,\rho^2]=[G,\rho]^{\sigma}$. Note that $[G,\rho]\subseteq MM^{\rho}$, since $g^{-1}g^{\rho}=(g^{-1}g^{\sigma})((g^{\rho\sigma})^{-1}(g^{\rho\sigma})^{\sigma})^{\rho}\in MM^{\rho}$. Hence $[G,S]$ is generated by M as an S-subgroup.

We also have $[M^{\rho^2},\sigma]=M$, since $m^{-\rho^2}m^{\rho^2\sigma}=m^{-\rho^2}m^{-\rho}=m$ for all $m\in M$ by (4) and (2). Hence the inclusion $[G,S]\subseteq[[G,S],S]$.

(ii) Clearly, $C_G([G,S]S)\subseteq Z_S(G)$. For every $N\trianglelefteq G$ with trivial S-action, we have $[N,G,S]=1$ and $[S,N,G]=1$. By the Three Subgroup Lemma, N centralizes $[G,S]$ and we have the reverse inclusion.

Observe that $C_G(S)\cap M=1$, since $m=m^{\sigma}=m^{-1}$ implies $m^2=1$ by (4) and $m=m^{\rho}$ implies $m^3=1$ by (2), i.e., $m=1$ for every $m\in C_G(S)\cap M$. Now, let N be the full preimage of $Z_S(G/Z_S(G))$ in G. By (i), $[N,S]$ is generated by $[N,\sigma]$ as an

S-subgroup. Since $[N,\sigma] \subseteq Z_S(G) \subseteq C_G(S)$ and $[N,\sigma] \subseteq M$, we have $[N,S] = 1$ as is required. $\square$

Lemma 2 *Let G be a group with triality S, $M = [G,\sigma]$, and $H = C_G(\sigma)$. We have*
(i) $m^{-\rho}nm^{-\rho^2} = n^{-\rho^2}mn^{-\rho} \in M \quad \forall\, m,n \in M$;
(ii) $[m,m^\rho] = [m,m^{\rho^2}] = [m^\rho, m^{\rho^2}] = 1 \quad \forall\, m \in M$;
(iii) $\eta(G) \subseteq H$;
(iv) $[m^{-\rho^2}, n^\rho] = [m^\rho, n^{-\rho^2}] \in H \quad \forall\, m,n \in M$;
(v) $\phi(H) \subseteq M$ *and* $\phi(M) \subseteq H$;
(vi) For every $g \in G$, we have $g = \eta(g)\xi(g)^{\rho^2}$.

Proof. (i) By (2) and (4), we have

$$\xi(m^{\rho^2}n^{\rho^2}) = n^{-\rho^2}m^{-\rho^2}m^{\sigma\rho}n^{\sigma\rho} = n^{-\rho^2}m^{-\rho^2}m^{-\rho}n^{-\rho} = n^{-\rho^2}mn^{-\rho} \in M \qquad (6)$$

for all $m,n \in M$. Moreover, (2) also implies
$n^{-\rho^2}mn^{-\rho}(n^{-\rho^2}mn^{-\rho})^\rho(n^{-\rho^2}mn^{-\rho})^{\rho^2} = n^{-\rho^2}mn^{\rho^2}m^\rho nm^{\rho^2}n^{-1} = 1$.
Conjugating this equality by n, we obtain $n^{-1}n^{-\rho^2}mn^{\rho^2}m^\rho nm^{\rho^2} = n^\rho mn^{\rho^2}m^\rho nm^{\rho^2} = 1$ for all $m,n \in M$. Replacing m by m^{-1}, we have $n^{-\rho^2}mn^{-\rho} = m^{-\rho}nm^{-\rho^2}$.

(ii) For every $m \in M$, we have $mm^\rho m^{\rho^2} = 1$ by (2). Using (4), we obtain $1 = m^\sigma m^{\rho\sigma} m^{\rho^2\sigma} = m^{-1}m^{-\rho^2}m^{-\rho}$. Replacing m by m^{-1}, we have $mm^{\rho^2}m^\rho = mm^\rho m^{\rho^2}$; therefore, $[m^\rho, m^{\rho^2}] = 1$. It remain to act by ρ to obtain the other two relations.

(iii) $\eta(g)^\sigma = g^\sigma g^{-\rho}g^{\sigma\rho} = g(g^{-1}g^\sigma)(g^{-1}g^\sigma)^\rho = g(g^{-1}g^\sigma)^{-\rho^2} = gg^{-\rho\sigma}g^{\rho^2} = \eta(g)$.

(iv) Let $m,n \in M$. Then, by item (i), we have $n^{-\rho^2}m^{-1}n^{-\rho} = m^\rho nm^{\rho^2}$. Applying ρ^2 to this equality, we have $n^{-\rho}m^{-\rho^2}n^{-1} = mn^{\rho^2}m^\rho$. This, together with (2), implies $n^{-\rho}m^{-\rho^2}n^\rho n^{\rho^2} = m^{-\rho^2}m^{-\rho}n^{\rho^2}m^\rho$; hence, $m^{\rho^2}n^{-\rho}m^{-\rho^2}n^\rho = m^{-\rho}n^{\rho^2}m^\rho n^{-\rho^2}$, i.e., $[m^{-\rho^2}, n^\rho] = [m^\rho, n^{-\rho^2}]$. We also have

$$\eta(m^{\rho^2}n^{\rho^2}) = m^{\rho^2}n^{\rho^2}n^{-\sigma}m^{-\sigma}m^{-\rho}n^{-\rho} = m^{\rho^2}n^{-\rho}m^{-\rho^2}n^\rho = [m^{-r^2}, n^\rho] \in H \qquad (7)$$

by (4), (2), and (iii).

(v) Let $h \in H$ and $m \in M$. Then (4) and (ii) imply
$\phi(m)^\sigma = m^{-\rho\sigma}m^{\rho^2\sigma} = m^{\rho^2}m^{-\rho} = m^{-\rho}m^{\rho^2} = \phi(m) \in H$.
$\xi(\phi(h)^{\rho^2}) = \xi(h^{-1}h^\rho) = h^{-\rho}hh^{-\sigma}h^{\sigma\rho^2} = h^{-\rho}h^{\rho^2} = \phi(h) \in M$. $\square$

(vi) This follows from (3).

Theorem 1 *Let G be a group with triality and put $M = \{\, [g,\sigma] \mid g \in G \,\}$ as above. Then the set M is a Moufang loop with respect to the multiplication law*

$$m.n = m^{-\rho} n m^{-\rho^2} = n^{-\rho^2} m n^{-\rho} \qquad \forall \quad m, n \in M. \tag{8}$$

Moreover, this Moufang loop $(M, .)$ is isomorphic to Doro's loop $(M^{\rho^2}, \star)$ with multiplication given by (5).

Proof. First, note that (8) correctly defines an operation on M in view of (i) of Lemma 2. Also, the identity of G is the identity of M and taking inverses or powers of elements is the same whether considered in G or M, which follows from (2) and (ii) of Lemma 2. We need to prove the Moufang identity $((m.n).m).k = m.(n.(m.k))$. For $m, n, k \in M$, we have

$((m.n).m).k = ((m^{-\rho} n m^{-\rho^2}).m).k = ((m^{-\rho} n m^{-\rho^2})^{-\rho} m (m^{-\rho} n m^{-\rho^2})^{-\rho^2}).k =$
$(mn^{-\rho}(m^{\rho^2} m m^{\rho}) n^{-\rho^2} m).k = (mnm).k = (mnm)^{-\rho} k (mnm)^{-\rho^2}$,

where we have used (2) and (i–ii) of Lemma 2. On the other hand,

$m.(n.(m.k)) = m.(n.(m^{-\rho} k m^{-\rho^2})) =$
$m.(n^{-\rho} m^{-\rho} k m^{-\rho^2} n^{-\rho^2}) = (mnm)^{-\rho} k (mnm)^{-\rho^2}$. Hence, M is a Moufang loop.

By (vi) of Lemma 2, for every $g \in G$, we have $g = \eta(g)\xi(g)^{\rho^2}$; hence, $G = HM^{\rho^2}$. Moreover, M^{ρ^2} is a right transversal of H in G. Indeed, if $Hm^{\rho^2} = Hn^{\rho^2}$ for $m, n \in M$ then $(m^{\rho^2} n^{-r^2})^{\sigma} = m^{\rho^2} n^{-r^2}$, which implies $m^{-\rho^2} m^{-\rho} = n^{-\rho^2} n^{-\rho}$. Hence, $m = n$ by (2). Furthermore, for $m, n \in M$, we have

$$m^{\rho^2} n^{\rho^2} = \eta(m^{\rho^2} n^{\rho^2}) \xi(m^{\rho^2} n^{\rho^2})^{\rho^2} = [m^{-\rho^2}, n^{\rho}](m^{-\rho} n m^{-\rho^2})^{\rho^2}$$

by (6) and (7). Hence, $m^{\rho^2} \star n^{\rho^2} = (m^{-\rho} n m^{-\rho^2})^{\rho^2}$ and $(M, .) \cong (M^{\rho^2}, \star)$, where the isomorphism is the map $m \mapsto m^{\rho^2}$. $\Box$

We henceforth denote by $\mathcal{M}(G)$ the Moufang loop $(M, .)$ constructed as in the above lemma from a given group with triality G. Conversely, given an arbitrary Moufang loop Q there exist groups with triality G such that $\mathcal{M}(G) \cong Q$. One such group $\mathcal{D}(Q)$ was constructed by Doro [6] and is defined in terms of abstract generators $P_{(x)}$, $L_{(x)}$, $R_{(x)}$ indexed by elements of Q as follows:

$$\begin{aligned} \mathcal{D}(Q) \;\; = \{\, & P_{(x)}, L_{(x)}, R_{(x)},\ x \in Q \mid P_{(1)} = L_{(1)} = R_{(1)} = 1,\ P_{(x)} L_{(x)} R_{(x)} = 1, \\ & P_{(x)} P_{(y)} P_{(x)} = P_{(x.y.x)},\ L_{(x)} L_{(y)} L_{(x)} = L_{(x.y.x)},\ R_{(x)} R_{(y)} R_{(x)} = R_{(x.y.x)}, \\ & P_{(y^{-1}.x)} = L_{(y)} P_{(x)} R_{(y)},\ L_{(y^{-1}.x)} = R_{(y)} L_{(x)} P_{(y)},\ R_{(y^{-1}.x)} = P_{(y)} R_{(x)} L_{(y)}, \\ & P_{(x.y^{-1})} = R_{(y)} P_{(x)} L_{(y)},\ L_{(x.y^{-1})} = P_{(y)} L_{(x)} R_{(y)},\ R_{(x.y^{-1})} = L_{(y)} R_{(x)} P_{(y)} \,\} \end{aligned} \tag{9}$$

with the action of ρ and σ given by

$$\begin{array}{ccccccccc} & & P_{(x)} & \overset{\rho}{\longmapsto} & L_{(x)} & \overset{\rho}{\longmapsto} & R_{(x)} & \overset{\rho}{\longmapsto} & P_{(x)}, \\ P_{(x)} \overset{\sigma}{\longmapsto} P^{-1}_{(x)}, & & L_{(x)} & \overset{\sigma}{\longmapsto} & R^{-1}_{(x)}, & & R_{(x)} & \overset{\sigma}{\longmapsto} & L^{-1}_{(x)}. \end{array} \tag{10}$$

This group satisfies $[\mathcal{D}(Q), S] = \mathcal{D}(Q)$ and $\mathcal{M}(\mathcal{D}(Q)) \cong Q$. Moreover, $\mathcal{D}(Q)$ is a universal projective object in the following sense: if G is any group with triality such that $\mathcal{M}(G) \cong Q$ and $G = [G, S]$ then there exists an S-epimorphism $\tau : \mathcal{D}(Q) \to G$ defined by

$$P_{(x)} \overset{\tau}{\longmapsto} x, \qquad L_{(x)} \overset{\tau}{\longmapsto} x^{\rho}, \qquad R_{(x)} \overset{\tau}{\longmapsto} x^{\rho^2}, \tag{11}$$

where Q is identified with $\mathcal{M}(G) \subseteq G$ (see [6]).

A *pseudoautomorphism* of a Moufang loop $(Q, .)$ is a bijection $A : Q \to Q$ with the property that there exists an element $a \in Q$ such that

$$xA.(yA.a) = (x.y)A.a \qquad \text{for all} \quad x, y \in Q.$$

This element a is called a *right companion* of A. In general, a right companion of A is not unique. Similarly, an element b such that

$$(b.xA).yA = b.(x.y)A \qquad \text{for all} \quad x, y \in Q$$

is a *left companion* of A. The following properties are well known:

Lemma 3 *Let A be a pseudoautomorphism of a Moufang loop Q with right companion a then*

(i) $x^n A = (xA)^n$ for all $x \in Q$, $n \in \mathbb{Z}$,

(ii) a^{-1} is a left companion of A,

(iii) $(x.y.x)A = xA.yA.xA$ for all $x, y \in Q$,

(iv) all the right companions of A form the coset $Na = aN$, where $N = Nuc(Q)$.

Proof. See [9, 3]. □

The set of pairs (A, a), where A is a pseudoautomorphism of Q with right companion a is a group with respect to the operation

$$(A, a)(B, b) = (AB, aB \cdot b).$$

This group is denoted by $PsAut(Q)$.

With arbitrary elements x, y of a Moufang loop Q, there are associated the bijections L_x, R_x, T_x, P_x $L_{x,y}$, $R_{x,y}$ of Q defined as follows:

$$yL_x = x.y, \qquad yR_x = y.x, \qquad T_x = L_x^{-1}R_x, \qquad P_x = L_x^{-1}R_x^{-1}, \\ L_{x,y} = L_xL_yL_{yx}^{-1}, \qquad R_{x,y} = R_xR_yR_{xy}^{-1}. \tag{12}$$

In particular, we have $zL_{x,y} = (y.x)^{-1}.(y.(x.z))$ and $zR_{x,y} = ((z.x).y).(x.y)^{-1}$. It is known (see [1]) that (T_x, x^{-3}) and $(R_{x,y}, [\![x, y]\!])$ belong to $PsAut(Q)$. We give a new proof of these facts below (see Lemma 4) using groups with triality. Denote by $PsInn(Q)$ the subgroup of $PsAut(Q)$ generated by the elements (T_x, x^{-3}) and $(R_{x,y}, [\![x, y]\!])$ for all x, y in Q.

Some properties of Moufang loops can be proven using their "enveloping" group with triality:

Lemma 4 *Let M be any Moufang loop and let G be a group with triality $S = \langle\rho, \sigma\rangle$ such that $\mathcal{M}(G) = M$. Put $H = C_G(\sigma)$. Then we have:*

(i) $[\![m, n]\!] = [n^{-1}, m^{\rho^2}][m, n^{-\rho}]$ for all $m, n \in M$.

(ii) For every $h \in H$, the mapping $T_h : M \to M$ defined by $mT_h = h^{-1}mh$ is a pseudoautomorphism of M with right companion $\phi(h)$. Moreover, the mapping

$$h \mapsto (T_h, \phi(h)) \tag{13}$$

is a homomorphism from H to $PsAut(M)$ whose kernel is the S-center $Z_S(G)$.

(iii) Let $m \in M$ and $T_m = L_m^{-1}R_m$. Then $nT_m = m^{-1}.n.m$ is a pseudoautomorphism of M with right companion m^{-3}. Moreover, $T_m = T_h$ where $h = \phi(m) \in H$.

(iv) Let $m, n \in M$. Then $R_{m,n}$ is a pseudoautomorphism of M with right companion $[\![m, n]\!]$. Moreover, $R_{m,n} = T_h$ where $h = [m^\rho, n^{-\rho^2}] = [m^{-\rho^2}, n^\rho] \in H$. Similarly, $L_{m,n}$ is a pseudoautomorphism with right companion $[\![m^{-1}, n^{-1}]\!]$ and $L_{m,n} = T_h$ where $h = [m^{-\rho}, n^{\rho^2}] = [m^{\rho^2}, n^{-\rho}] \in H$.

(v) $R_{x,y} = L_{x^{-1},y^{-1}} = R_{y,x}^{-1}$ for all $x, y \in M$

Proof. (i) Let $m, n \in M$. By (iv) of Lemma 2, we have
$[m^\rho, n^{-\rho^2}] = [m^{-\rho^2}, n^\rho]$. Acting by ρ^2 on both sides, we obtain
$[m, n^{-\rho}] = [m^{-\rho}, n]$. Hence, (2) implies
$[n^{-1}, m^{\rho^2}][m, n^{-\rho}] = [n^{-1}, m^{\rho^2}][m^{-\rho}, n] = nm^{-\rho^2}n^{-1}(m^{\rho^2}m^\rho)n^{-1}m^{-\rho}n =$
$nm^{-\rho^2}n^{-1}m^{-1}n^{-1}m^{-\rho}n$. On the other hand,

$[\![m,n]\!] = (m^{-1}.n^{-1}).(m.n) = (n^{\rho^2}m^{-1}n^{\rho}).(n^{-\rho^2}mn^{-\rho}) =$
$(n^{-\rho^2}mn^{-\rho})^{-\rho^2}(n^{\rho^2}m^{-1}n^{\rho})(n^{-\rho^2}mn^{-\rho})^{-\rho} = nm^{-\rho^2}(n^{\rho}n^{\rho^2})m^{-1}(n^{\rho}n^{\rho^2})m^{-\rho}n =$
$nm^{-\rho^2}n^{-1}m^{-1}n^{-1}m^{-\rho}n.$

(ii) By (v) of Lemma 2, $\phi(h) \in M$ and $MT_h = M$ for every $h \in H$. For $m \in M$ we have

$$mT_h.\phi(h) = (h^{-1}mh).\phi(h) = h^{-\rho}hh^{-1}mhh^{-1}h^{\rho^2} = h^{-\rho}mh^{\rho^2}. \tag{14}$$

Hence, for $m,n \in M$, $(m.n)T_h.\phi(h) = h^{-\rho}m^{-\rho}nm^{-\rho^2}h^{\rho^2}$. On the other hand $(mT_h).(nT_h.\phi(h)) = (h^{-1}mh).(h^{-\rho}nh^{\rho^2}) = h^{-\rho}m^{-\rho}h^{\rho}h^{-\rho}nh^{\rho^2}h^{-\rho^2}m^{-\rho^2}h^{\rho^2} =$ $h^{-\rho}m^{-\rho}nm^{-\rho^2}h^{\rho^2}$. It follows that $(T_h, \phi(h)) \in PsAut(M)$. By (14), we also have $(T_h, \phi(h))(T_k, \phi(k)) = (T_hT_k, \phi(h)T_k.\phi(k)) = (T_{hk}, k^{-\rho}h^{-\rho}h^{\rho^2}k^{\rho^2}) = (T_{hk}, \phi(hk))$ for all $h,k \in H$. Consequently, (13) is a homomorphism of H to $PsAut(M)$. Obviously, $h \in H$ is in its kernel iff h centralizes both M and S, i.e. $h \in Z_S(G)$.

(iii) For $m,n \in M$, we have
$nT_m = m^{-1}.n.m = (m^{\rho}nm^{\rho^2}).m = m^{-\rho^2}m^{\rho}nm^{\rho^2}m^{-\rho} =$
$(m^{-\rho}m^{\rho^2})^{-1}n(m^{-\rho}m^{\rho^2}) = nT_{\phi(m)}$ by (ii) of Lemma 2.
Now, by item (ii), $T_{\phi(m)}$ is a pseudoautomorphism with right companion

$$\phi(\phi(m)) = (m^{-\rho}m^{\rho^2})^{-\rho}(m^{-\rho}m^{\rho^2})^{\rho^2} = m^{-1}m^{\rho^2}m^{-1}m^{\rho} = m^{-3}. \tag{15}$$

(iv) For $m,n,k \in M$ we have
$kR_{m,n} = ((k.m).n).(m.n)^{-1} = ((m^{-\rho^2}km^{-\rho}).n).(n^{-\rho^2}mn^{-\rho})^{-1} =$
$(n^{-\rho^2}mn^{-\rho})^{\rho^2}(n^{-\rho^2}m^{-\rho^2}km^{-\rho}n^{-\rho})(n^{-\rho^2}mn^{-\rho})^{\rho} = n^{-\rho}m^{\rho^2}n^{\rho}m^{-\rho^2}km^{-\rho}n^{\rho^2}m^{\rho}n^{-\rho^2} =$
$[n^{\rho}, m^{-\rho^2}]k[m^{\rho}, n^{-\rho^2}] = kT_h$ by (iv) of Lemma 2, where $h = [m^{\rho}, n^{-\rho^2}]$. Therefore, by item (ii), $R_{m,n}$ is a pseudoautomorphism with right companion

$$\phi([m^{\rho}, n^{-\rho^2}]) = [n^{-1}, m^{\rho^2}][m, n^{-\rho}] = [\![m,n]\!] \tag{16}$$

by item (i). The assertion about $L_{m,n}$ may be proved similarly.

(v) The equality $R_{x,y} = L_{x^{-1},y^{-1}}$ follows from item (iv). Since $R_{y,x} = T_h$, where $h = [y^{\rho}, x^{-\rho^2}]$, we have $R_{y,x}^{-1} = T_{h^{-1}}$. However, $h^{-1} = [x^{-\rho^2}, y^{\rho}] = [x^{\rho}, y^{-\rho^2}]$ by (iv) of Lemma 2. Therefore, $R_{y,x}^{-1} = R_{x,y}$ by item (iv). □

Note that the transformation T_x of M is defined for $x \in M$ and $x \in H$. In the former case T_x acts by conjugation in M and in the latter by conjugation in G. Henceforth it should be clear from the context which of the two actions is considered.

Let $(Q,.)$ be a Moufang loop. Following Mikheev [7], define a binary operation on the Cartesian product $\mathrm{PsAut}(Q)\times Q$ by the rule

$$[(A,a),x][(B,b),y]=[(A,a)(B,b)(C,c),xB.y], \tag{17}$$

where

$$(C,c)=(R_{b,xB},[\![b,xB]\!])(R_{xB,y},[\![xB,y]\!]).$$

Denote by $\mathcal{W}(Q)$ the groupoid $\mathrm{PsAut}(Q)\times Q$ with the binary operation (17). It was announced in [7] that $\mathcal{W}(Q)$ is in fact a group with triality (the *extended group of pseudoautomorphisms* of Q) with the action of the triality automorphisms ρ and σ given by

$$\begin{aligned}[(A,a),x] &\stackrel{\rho}{\longmapsto} [(A,a),a][(T_x,x^{-3}),x^{-2}],\\ [(A,a),x] &\stackrel{\sigma}{\longmapsto} [(A,a)(T_x,x^{-3}),x^{-1}],\end{aligned} \tag{18}$$

and that the loop $\mathcal{M}(\mathcal{W}(Q))$ is isomorphic to Q. We present a proof of these facts and then establish some properties of $\mathcal{W}(Q)$. However a direct verification of associativity of the product (17) is technically intractable. We chose an alternative way of showing this. The following abstract result will be needed:

Lemma 5 *Let P be a group and Q a groupoid. Let H and M be subsets of P such that $P=HM$. Then a map $\psi:P\to Q$ is a homomorphism if and only if*

$$\begin{aligned}&(i)\quad \psi(mn)=\psi(m)\psi(n), && (ii)\quad \psi(hk)=\psi(h)\psi(k),\\ &(iii)\quad \psi(mh)=\psi(m)\psi(h), && (iv)\quad \psi(hm)=\psi(h)\psi(m),\end{aligned}$$

for all $h,k\in H$, $m,n\in M$. In particular, $Im\,\psi$ is a group, whenever (i–iv) hold.

Proof. Let $h,k\in H$, $m,n\in M$. There exist $k_1\in H$, $m_1\in M$ such that $mk=k_1m_1$. We have

$$\psi(hmkn)=\psi(hk_1m_1n)\stackrel{(iv)}{=}\psi(hk_1)\psi(m_1n)\stackrel{(i),(ii)}{=}\psi(h)\psi(k_1)\psi(m_1)\psi(n)\stackrel{(iv)}{=}$$
$$\psi(h)\psi(k_1m_1)\psi(n)=\psi(h)\psi(mk)\psi(n)\stackrel{(iii)}{=}\psi(h)\psi(m)\psi(k)\psi(n)\stackrel{(iv)}{=}\psi(hm)\psi(kn).$$

□

Theorem 2 *Let G be a group with triality $S=\langle\rho,\sigma\rangle$ and $H=C_G(\sigma)$. Let $M=\mathcal{M}(G)$ be the corresponding Moufang loop and let $\mathcal{W}(M)=M\times PsAut(M)$ be the above groupoid. Then the map $\tau:G\to\mathcal{W}(M)$ defined by*

$$\tau(g) = [(T_{\eta(g)}, \phi(\eta(g))), \xi(g)] \qquad \text{for all} \quad g \in G$$

is an S-homomorphism, where $\eta(g) = gg^{-\rho\sigma}g^{\rho^2} \in H$, $\phi(h) = h^{-\rho}h^{\rho^2}$, and the action of S on $\mathcal{W}(M)$ is given by (18). In particular, $Im\,\tau$ is a group with triality. Moreover, $\mathcal{M}(Im\,\tau) \cong M$ and $Ker\,\tau = Z_S(G)$.

Proof. First, we show that the map τ satisfies the conditions (i)–(iv) of Lemma 5, with respect to the decomposition $G = HM^{\rho^2}$. This will imply that $Im\,\tau$ is a group.

(i) Let $m, n \in M$, then $m.n = m^{-\rho}nm^{-\rho^2}$. By definition
$\tau(m^{\rho^2}) = [(T_{\eta(m^{\rho^2})}, \phi(\eta(m^{\rho^2}))), \xi(m^{\rho^2})]$. We have
$\xi(m^{\rho^2}) = m^{-\rho^2}m^{\rho^2\sigma} = m^{-\rho^2}m^{-\rho} = m$,
$\eta(m^{\rho^2}) = m^{\rho^2}m^{-\sigma}m^{\rho} = m^{\rho^2}mm^{\rho} = 1$, and $\phi(\eta(m^{\rho^2})) = 1$. Hence,

$$\tau(m^{\rho^2}) = [(1,1), m]. \tag{19}$$

Similarly, $\tau(n^{\rho^2}) = [(1,1), n]$. Thus, (17) implies
$\tau(m^{\rho^2})\tau(n^{\rho^2}) = [(1,1), m][(1,1), n] = [(R_{m,n}, [\![m,n]\!]), m.n]$.
On the other hand, (6) and (7) imply $\tau(m^{\rho^2}n^{\rho^2}) = [(T_h, \phi(h)), m.n]$, where
$h = [m^{-\rho^2}, n^{\rho}] \in H$; and, by (16) and (iv) of Lemma 2, we have
$\phi(h) = \phi([m^{\rho}, n^{-\rho^2}]) = [\![m,n]\!]$. Hence,
$[(R_{m,n}, [\![m,n]\!]), m.n] = [(T_h, \phi(h)), m.n]$ by (iv) of Lemma 4

(ii) If $h, k \in H$ then $\eta(h) = hh^{-\rho^2}h^{\rho^2} = h$ and $\xi(h) = 1$. Hence,

$$\tau(h) = [(T_h, \phi(h)), 1]. \tag{20}$$

Similarly, $\tau(k) = [(T_k, \phi(k)), 1]$ and $\tau(hk) = [(T_{hk}, \phi(hk)), 1]$. By (17), we have
$\tau(h)\tau(k) = [(T_hT_k, \phi(h)^k.\phi(k), 1]$. However, $T_hT_k = T_{hk}$ and
$\phi(h)^k.\phi(k) = (k^{-1}h^{-\rho}h^{\rho^2}k).(k_{-\rho}k^{\rho^2}) = (k^{-\rho}k^{\rho^2})^{-\rho^2}k^{-1}\phi(h)k(k^{-\rho}k^{\rho^2})^{-\rho} =$
$k^{-\rho}kk^{-1}\phi(h)kk^{-1}k^{\rho^2} = k^{-\rho}\phi(h)k^{\rho^2}$. On the other hand,
$\phi(hk) = (hk)^{-\rho}(hk)^{\rho^2} = k^{-\rho}\phi(h)k^{\rho^2}$.

(iii) Let $m \in M$ and $h \in H$. Then $\tau(hm^{\rho^2}) = [(T_h, \phi(h)), m]$,
since $\eta(hm^{\rho^2}) = h$ and $\xi(hm^{\rho^2}) = m$. On the other hand, we have
$\tau(h)\tau(m^{\rho^2}) = [(T_h, \phi(h)), 1][(1,1), m] = [(T_h, \phi(h)), m]$ by (19) and (17).

(vi) For $m \in M$ and $h \in H$, we have
$\xi(m^{\rho^2}h) = h^{-1}m^{-\rho^2}m^{\rho^2\sigma}h = h^{-1}m^{-\rho^2}m^{-\rho}h = h^{-1}mh$.

Denote $n = h^{-1}mh = mT_h$ and $l = \phi(h)$. Then we also have
$\eta(m^{\rho^2}h) = m^{\rho^2}h(m^{\rho^2}h)^{-\sigma\rho^2}(m^{\rho^2}h)^{\rho^2} =$
$m^{\rho^2}hh^{-\rho^2}mm^{\rho}h^{\rho^2} = m^{\rho^2}hh^{-\rho^2}m^{-\rho^2}h^{\rho^2} = m^{\rho^2}hn^{-\rho^2}$.
Hence, $\tau(m^{\rho^2}h) = [(T_{m^{\rho^2}hn^{-\rho^2}}, \phi(m^{\rho^2}hn^{-\rho^2})), n]$.
On the other hand, by (17) and (iv) of Lemma 4, we have
$\tau(m^{\rho^2})\tau(h) = [(1,1),m][(T_h,l),1] = [(T_h,l)(R_{l,n}, [\![l,n]\!]), n] =$
$[(T_hT_{[l^\rho,n^{-\rho^2}]}, lR_{l,n}.[\![l,n]\!]), n] = [(T_{h[l^\rho,n^{-\rho^2}]}, l.[\![l,n]\!]), n] =$
$[(T_{h[l^\rho,n^{-\rho^2}]}, n^{-1}.l.n), n]$. However,
$h[l^\rho, n^{-\rho^2}] = h(h^{-\rho}h^{\rho^2})^{-\rho}(h^{-1}mh)^{\rho^2}(h^{-\rho}h^{\rho^2})^{\rho}n^{-\rho^2} =$
$hh^{-1}h^{\rho^2}h^{-\rho^2}m^{\rho^2}h^{\rho^2}h^{-\rho^2}hn^{-\rho^2} = m^{\rho^2}hn^{-\rho^2}$. Hence,
$\phi(h[l^\rho, n^{-\rho^2}]) = \phi(m^{\rho^2}hn^{-\rho^2})$. Furthermore,
$\phi(h[l^\rho, n^{-\rho^2}]) = [l^\rho, n^{-\rho^2}]^{-\rho}h^{-\rho}h^{\rho^2}[l, n^{-\rho}] =$
$[n^{-1}, l^{\rho^2}]l[l, n^{-\rho}] = nl^{-\rho^2}n^{-1}l^{\rho^2}lk^{-1}n^{\rho}ln^{-\rho} =$
$nl^{-\rho^2}n^{-1}l^{\rho^2}n^{\rho}n^{\rho^2}(n^{-\rho^2}ln^{-\rho}) = nl^{-\rho^2}n^{-1}l^{\rho^2}n^{-1}(l.n) =$
$nl^{-\rho^2}n^{-1}n^{\rho}(n^{-\rho^2}ln^{-\rho})^{\rho^2}(l.n) = (n^{-\rho^2}ln^{-\rho})^{-\rho^2}n^{-1}(l.n)^{\rho^2}(l.n) =$
$(l.n)^{-\rho^2}n^{-1}(l.n)^{-\rho} = n^{-1}.l.n$. Therefore, $\tau(m^{\rho^2})\tau(h) = \tau(m^{\rho^2}h)$.

Hence, τ is a group homomorphism. We now show that $\tau\rho = \rho\tau$ and $\tau\sigma = \sigma\tau$, where the action of ρ and σ is defined by (18). For all $h \in H$ and $m \in M$, we have
$\tau(\sigma(hm^{\rho^2})) = \tau(hm^{-\rho}) = \tau(h)\tau(m^{-\rho}) = [(T_h, \phi(h)), 1][(T_m, m^{-3}), m^{-1}] =$
$[(T_h, \phi(h))(T_m, m^{-3}), m^{-1}]$, where we have used (20), the relations
$\eta(m^{-\rho}) = m^{-\rho}m^{-\rho}m^{-1} = m^{-\rho}m^{-\rho^2} = \phi(m)$,
$\xi(m^{-\rho}) = m^{\rho}m^{-\rho\sigma} = m^{\rho}m^{\rho^2} = m^{-1}$,
as well as (15), (17), and (iii) of Lemma 4 from which it follows that
$\tau(m^{-\rho}) = [(T_m, m^{-3}), m^{-1}]$. On the other hand, we have by (18)
$\sigma(\tau(hm^{\rho^2})) = \sigma([(T_h, \phi(h), m)]) =$
$[(T_h, \phi(h))(T_m, m^{-3}), m^{-1}]$. Therefore, $\tau\sigma = \sigma\tau$. We also have
$\tau(\rho(hm^{\rho^2})) = \tau(h^\rho)\tau(m) = [(T_h, \phi(h)), \phi(h)][(T_{\phi(m)}, m^{-3}), m^{-2}]$, since
$\eta(h^\rho) = h^\rho h^{-\rho^2\sigma}h = h^\rho h^{-\rho}h = h$,
$\xi(h^\rho) = h^{-\rho}h^{\rho\sigma} = h^{-\rho}h^{\rho^2} = \phi(h)$, and
$\eta(m) = mm^{-\sigma\rho^2}m^{\rho^2} = (mm^{\rho^2})m^{\rho^2} = m^{-\rho}m^{\rho^2} = \phi(m)$,
$\xi(m) = m^{-1}m^{\sigma} = m^{-2}$. On the other hand, (18) implies
$\rho(\tau(hm^{\rho^2})) = \rho([(T_h, \phi(h)), m]) = [(T_h, \phi(h)), \phi(h)][(T_m, m^{-3}), m^{-2}]$. However,
$T_m = T_{\phi(m)}$ by (iii) of Lemma (4). Hence, τ is an S-homomorphism and $Im\,\tau$ is a group with triality. Moreover, $\mathcal{M}(Im\,\tau) = \tau(M) \cong M$.

It remains to find $Ker\,\tau$. We have

$[(T_{\eta(g)}, \phi(\eta(g))), \xi(g)] = 1$ if and only if $\xi(g) = 1$ and $(T_{\eta(g)}, \phi(\eta(g))) = (1,1)$, i.e.,
$g \in H$ and $(T_g, \phi(g)) = (1,1)$. However,
$\phi(g) = g^{-\rho} g^{\rho^2} = 1$ if and only if ρ centralizes g and
$T_g = 1$ if and only if g centralizes M. Hence, we have
$Ker\,\tau = C_G(\sigma) \cap C_G(\rho) \cap C_G(M) = C_G(S) \cap C_G(M \cup M^\rho) = C_G(S[G,S]) = Z_S(G)$
by (i) of Lemma 1. □

Our aim is to show that $\mathcal{W}(M)$ is an epimorphic image of some group with triality under the S-homomorphism τ. Starting with an arbitrary Moufang loop $(Q, .)$ consider the action of $PsAut(Q)$ on $\mathcal{D}(Q)$ defined on the generators as follows

$$\begin{aligned}(P_{(x)})^{(A,a)} &= P_{(xA)},\\ (L_{(x)})^{(A,a)} &= R_{(a)} L_{(xA)} R_{(a^{-1})},\\ (R_{(x)})^{(A,a)} &= L_{(a^{-1})} R_{(xA)} L_{(a)}\end{aligned} \tag{21}$$

for all $x \in Q$ and $(A,a) \in PsAut(Q)$.

Lemma 6 *The relations (21) define an action of $PsAut(Q)$ by automorphisms on $\mathcal{D}(Q)$.*

Proof. We need to show that the action of $(A,a) \in PsAut(Q)$ preserves all the defining relations in (9). For all $x, y \in Q$, using these relations and Lemma 3, we have
$(P_{(x)})^{(A,a)}(L_{(x)})^{(A,a)}(R_{(x)})^{(A,a)} = P_{(xA)} R_{(a)} L_{(xA)} (R_{(a^{-1})} L_{(a^{-1})}) R_{(xA)} L_{(a)} =$
$P_{(xA)} R_{(a)} (L_{(xA)} P_{(a)} R_{(xA)}) L_{(a)} = P_{(xA)} (R_{(a)} P_{(x^{-1}A.a)} L_{(a)}) = P_{(xA)} P_{(x^{-1}A)} = 1.$
$(L_{(x)})^{(A,a)}(L_{(y)})^{(A,a)}(L_{(x)})^{(A,a)} = R_{(a)} L_{(xA)} (R_{(a^{-1})} R_{(a)}) L_{(yA)} (R_{(a^{-1})} R_{(a)}) L_{(xA)} R_{(a^{-1})} =$
$R_{(a)} (L_{(xA)} L_{(yA)} L_{(xA)}) R_{(a^{-1})} = R_{(a)} L_{(xA.yA.xA)} R_{(a^{-1})} =$
$R_{(a)} L_{((x.y.x)A)} R_{(a^{-1})} = (L_{(x.y.x)})^{(A,a)}.$
$(L_{(y)})^{(A,a)}(P_{(x)})^{(A,a)}(R_{(y)})^{(A,a)} = R_{(a)} L_{(yA)} (R_{(a^{-1})} P_{(xA)} L_{(a^{-1})}) R_{(yA)} L_{(a)} =$
$R_{(a)} (L_{(yA)} P_{(xA.a)} R_{(yA)}) L_{(a)} = R_{(a)} P_{(y^{-1}A.(xA.a))} L_{(a)} = R_{(a)} P_{((y^{-1}.x)A.a)} L_{(a)} =$
$P_{((y^{-1}.x)A)} = (P_{(y^{-1}x)})^{(A,a)}.$
$(L_{(y)})^{(A,a)}(R_{(x)})^{(A,a)}(P_{(y)})^{(A,a)} = R_{(a)} L_{(yA)} (R_{(a^{-1})} L_{(a^{-1})}) R_{(xA)} L_{(a)} P_{(yA)} =$
$R_{(a)} L_{(yA)} (P_{(a)} R_{(xA)} L_{(a)}) P_{(yA)} = R_{(a)} (L_{(yA)} R_{(a^{-1}.xA)} P_{(yA)}) = R_{(a)} R_{((a^{-1}.xA).y^{-1}A)} =$
$R_{(a)} R_{(a^{-1}.(x.y^{-1})A)} = (R_{(a)} P_{(a)}) R_{((x.y^{-1})A)} L_{(a)} = L_{(a^{-1})} R_{((x.y^{-1})A)} L_{(a)} = (R_{(x.y^{-1})})^{(A,a)}.$
The remaining identities can be proved similarly and are therefore omitted. Hence (A,a) induces an endomorphism of $\mathcal{D}(M)$. We also have for $(A,a), (B,b) \in PsAut(Q)$
$((P_{(x)})^{(A,a)})^{(B,b)} = (P_{(xA)})^{(B,b)} = P_{(xAB)} = (P_{(x)})^{(AB,aB.b)} = (P_{(x)})^{(A,a)(B,b)},$
$((L_{(x)})^{(A,a)})^{(B,b)} = (R_{(a)} L_{(xA)} R_{(a^{-1})})^{(B,b)} =$

$L_{(b^{-1})}R_{(aB)}(L_{(b)}R_{(b)})L_{(xAB)}(R_{(b^{-1})}L_{(b^{-1})})R_{(a^{-1}B)}L_{(b)} =$
$(L_{(b^{-1})}R_{(aB)}P_{(b^{-1})})L_{(xAB)}(P_{(b)}R_{(a^{-1}B)}L_{(b)}) = R_{(aB.b)}L_{(xAB)}R_{(b^{-1}.a^{-1}B)} =$
$(L_{(x)})^{(AB,aB.b)} = (L_{(x)})^{(A,a)(B,b)}$.

The identity for $R_{(x)}$ is proved similarly. Therefore, we have a group action of $PsAut(Q)$ on $\mathcal{D}(Q)$ by automorphisms. $\square$

Denote by $\mathcal{U}(Q)$ the semidirect product $PsAut(Q)\mathcal{D}(Q)$ and extend the action of ρ and σ on $\mathcal{U}(Q)$ as follows:

$$\begin{aligned} (A,a)D &\stackrel{\rho}{\longmapsto} (A,a)R_{(a)}D^{\rho}, \\ (A,a)D &\stackrel{\sigma}{\longmapsto} (A,a)D^{\sigma}. \end{aligned} \tag{22}$$

for all $(A,a) \in PsAut(Q)$ and $D \in \mathcal{D}(Q)$.

Lemma 7 *The group $\mathcal{U}(Q)$ is a group with triality $S = \langle \rho, \sigma \rangle$ given by (22). Moreover, $\mathcal{M}(\mathcal{U}(Q)) \cong Q$ and the mapping $\tau : \mathcal{U}(Q) \to \mathcal{W}(Q)$ defined in Theorem 2 is an S-epimorphism.*

Proof. First, show that ρ and σ are automorphisms. Given $(A,a), (B,b) \in PsAut(Q)$ and $D, E \in \mathcal{D}(Q)$, we have

$((A,a)D)^{\rho}((B,b)E)^{\rho} = (A,a)R_{(a)}D^{\rho}(B,b)R_{(b)}E^{\rho} =$
$(A,a)(B,b)(R_{(a)})^{(B,b)}D^{\rho(B,b)}R_{(b)}E^{\rho}$. On the other hand,
$((A,a)D(B,b)E)^{\rho} = ((A,a)(B,b)D^{(B,b)}E)^{\rho} = (A,a)(B,b)R_{(aB.b)}D^{(B,b)\rho}E^{\rho}$. However,
$R_{(aB.b)}D^{(B,b)\rho} = L_{(b^{-1})}R(aB)P_{(b^{-1})}D^{\rho}(B,b)^{\rho} = (L_{(b^{-1})}R(aB)L_{(b)})R_{(b)}D^{\rho}(B,b)R_{(b)} =$
$(R_{(a)})^{(B,b)}D^{\rho(B,b)}R_{(b)}$. Hence, ρ is an automorphism. We also have
$((A,a)D)^{\sigma}((B,b)E)^{\sigma} = (A,a)D^{\sigma}(B,b)E^{\sigma} = (A,a)(B,b)D^{\sigma(B,b)}E^{\sigma} =$
$(A,a)(B,b)D^{(B,b)\sigma}E^{\sigma} = ((A,a)(B,b)D^{(B,b)}E)^{\sigma} = ((A,a)D(B,b)E)^{\sigma}$.
Hence, σ is an automorphism. Now, obviously, $\sigma^2 = 1$. Furthermore,
$((A,a)D)^{\rho^3} = ((A,a)R_{(a)}D^{\rho})^{\rho^2} = ((A,a)R_{(a)}P_{(a)}D^{\rho^2})^{\rho} =$
$(A,a)R_{(a)}P_{(a)}L_{(a)}D = (A,a)D$, and
$((A,a)D)^{(\rho\sigma)^2} = ((A,a)L_{(a^{-1})}D^{\rho\sigma})^{\rho\sigma} = (A,a)L_{(a^{-1})}L_{(a)}D = (A,a)D$. Note that
$[(A,a)D, \sigma] = (1,1)[D,\sigma]$, which implies that $\mathcal{U}(Q)$ is a group with triality $\langle \rho, \sigma \rangle$, since $\mathcal{D}(Q)$ is. Moreover, $\mathcal{M}(\mathcal{U}(Q)) = \mathcal{M}(\mathcal{D}(Q)) \cong Q$. The latter isomorphism is the map $\mathcal{M}(\mathcal{U}(Q)) \ni P_{(x)} \mapsto x \in Q$. Hence we may identify Q with its image in $\mathcal{U}(Q)$. Under this identification, the action of a pseudoautomorphism A with companion a on Q corresponds to the mapping $T_{(A,a)}$ (the conjugation by (A,a) in $\mathcal{U}(q)$), which is a pseudoautomorphism of Q with companion $P_{(a)}$ in view of the first relation in (21).

It is now easy to see that the mapping $\tau : \mathcal{U}(Q) \to \mathcal{W}(Q)$ defined in Theorem 2 is surjective. Indeed, the elements $[(1,1),x]$ of $\mathcal{W}(Q)$ always lie in $Im\,\tau$ and we show that so do the elements $[(A,a),1]$ for arbitrary $(A,a) \in PsAut(Q)$. By the above identification, it is sufficient to show that the element $[(T_{(A,a)}, P_{(a)}),1]$ lies in $Im\,\tau$. However, this is exactly the image under τ of (A,a) viewed as an element of $\mathcal{U}(Q)$. Indeed,

$\xi((A,a)) = 1$,

$\eta((A,a)) = (A,a)\xi((A,a))^{-\rho^2} = (A,a)$,

$\phi((A,a)) = [(A,a),\rho]^\rho = R_{(a)}^\rho = P_{(a)}$; hence,

$\tau((A,a)) = [(T_{(A,a)}, P_{(a)}),1]$. □

The important properties of Mikheev's group $\mathcal{W}(Q)$ dual to those of Doro's group $\mathcal{D}(Q)$ are explained in the following assertion:

Corollary 1 *For every Moufang loop Q, the set $\mathcal{W}(Q)$ with multiplication (17) is a group with triality such that $\mathcal{M}(\mathcal{W}(Q)) \cong Q$ and $Z_S(\mathcal{W}(Q)) = 1$. Moreover, $\mathcal{W}(Q)$ is a universal injective object in the following sense: if G is any group with triality such that $\mathcal{M}(G) \cong Q$ and $Z_S(G) = 1$ then there exists an S-monomorphism $\tau : G \to \mathcal{W}(Q)$.*

Proof. This is a consequence of Theorem 2 and Lemmas 7 and 1. □

Introduce yet another group with triality associated with any Moufang loop Q. Denote by $\mathcal{E}(Q)$ the image of $\mathcal{D}(Q)$ in $\mathcal{W}(Q)$ under τ from Theorem 2. We have $\mathcal{M}(\mathcal{E}(Q)) \cong Q$. It is easy to see that $\mathcal{E}(Q)$ is the set $PsInn(Q) \times Q$ with multiplication (17) (the *extended group of inner pseudoautomorphisms*). In particular, $\mathcal{E}(Q)$ is generated by the elements of $\mathcal{W}(Q)$ of the form

$$[(T_m, m^{-3}),1], \qquad [(R_{m,n}, [\![m,n]\!]),1], \qquad [(1,1),m]$$

for all $m,n \in Q$. Moreover, this group satisfies $Z_S(\mathcal{E}(Q)) = 1$ and $[\mathcal{E}(Q),S] = \mathcal{E}(Q)$. Hence, $\mathcal{E}(Q)$ is the absolutely minimal group with triality corresponding to Q in the sense that it has neither proper S-subgroups nor S-factor groups G satisfying $\mathcal{M}(G) = Q$. Observe that the above homomorphism $\tau : \mathcal{D}(Q) \to \mathcal{E}(Q)$ coincides with the homomorphism (11). We also have

$$\mathcal{E}(Q) \cong \mathcal{D}(Q)/Z_S(\mathcal{D}(Q)) \cong [\mathcal{W}(Q),S].$$

Furthermore, the centralizer $C_S(\mathcal{E}(Q))$ coincides with $Inn(Q)$ since it consists of the elements of form $[(A,1),1]$, where A is an (inner) pseudoautomorphism with companion 1, i.e. an automorphism.

By definition, the *multiplication group* $Mlt(Q)$ of a Moufang loop Q is the group of permutations of Q generated by L_x and R_x for all $x \in Q$, and the *inner mapping group* $\mathcal{I}(Q)$ is the subgroup of $Mlt(Q)$ generated by T_x and $R_{x,y}$ for all x, y in Q. It is known (see [1]) that $\mathcal{I}(Q) = \{A \in Mlt(Q) \mid 1A = 1\}$. Glauberman [5] noted that the multiplication group of a Moufang loop Q with $Nuc(Q) = 1$ is a group with triality with respect to the action

$$\begin{gathered} P_x \overset{\rho}{\longmapsto} L_x \overset{\rho}{\longmapsto} R_x \overset{\rho}{\longmapsto} P_x, \\ P_x \overset{\sigma}{\longmapsto} P_x^{-1}, \qquad L_x \overset{\sigma}{\longmapsto} R_x^{-1}, \qquad R_x \overset{\sigma}{\longmapsto} L_x^{-1}. \end{gathered} \tag{23}$$

Phillips [8] remarks that not every Moufang loop multiplication group has triality and discusses the question: for what other Moufang loops Q does the group $Mlt(Q)$ admit Glauberman's triality (23)? We show that the group $\mathcal{E}(Q)$ is a natural generalization of the triality on $Mlt(Q)$ to all Moufang loops and that $Mlt(Q)$ has triality if and only if it coincides with $\mathcal{E}(Q)$. We will need the following fact:

Lemma 8 *Let Q be a Moufang loop. The subgroup $H = C_{\mathcal{D}(Q)}(\sigma)$ of $\mathcal{D}(Q)$ is generated by the elements $T_{(x)} = L_{(x^{-1})}R_{(x)}$ and $R_{(x,y)} = R_{(x)}R_{(y)}R_{(y^{-1}.x^{-1})}$ for all $x, y \in Q$.*

Proof. First note that there exists a natural epimorphism $\mu : \mathcal{D}(Q) \to Mlt(Q)$, which act on the generators by

$$P_{(x)} \overset{\mu}{\longmapsto} P_x, \qquad R_{(x)} \overset{\mu}{\longmapsto} R_x, \qquad L_{(x)} \overset{\mu}{\longmapsto} L_x \tag{24}$$

because all the relations corresponding to those in (9) hold in $Mlt(Q)$ as well. We have $T^{\sigma}_{(x)} = L^{\sigma}_{(x^{-1})}R^{\sigma}_{(x)} = R_{(x)}L_{(x^{-1})} = T_{(x)}$, since $R_{(x)}$ and $L_{(x)}$ commute. Also, $R^{\sigma}_{(x,y)} = L_{(x^{-1})}L_{(y^{-1})}L_{(x.y)} = R_{(x,y)}$, which follows from $R_{(y)}P_{(x.y)}L_{(y)} = P_{(x)}$. Hence, $T_{(x)}, R_{(x,y)} \in H$.

Every $W \in \mathcal{D}(Q)$ can be expressed as a word in $R_{(x)}$, $L_{(x)}$. Denote by $l(W)$ the minimal length of such an expression. Prove the assertion by induction on $l(W)$, where $W \in H$. If $l(W) = 0$, the claim holds. Suppose $l(W) = 1$, i.e., W is $L_{(x)}$ or $R_{(x)}$ for some $x \in Q$. In either case, we have $L_{(x)} = R_{(x^{-1})}$, which implies $L_x = R_{x^{-1}}$ by the above homomorphism (24). Hence, $x^2 = 1$. In particular, $W^2 = 1$ and $L_{(x)} = R_{(x)}$. Acting by ρ on both sides (see (10)), we obtain $R_{(x)} = P_{(x)}$. Hence, $1 = P_{(x)}L_{(x)}R_{(x)} = W^3$ and $W = 1$, a contradiction; i.e., H does not contain words of length 1. Suppose $l(W) = n \geqslant 2$. Then $W = A_{(x)}B_{(y)}W_0$ for some $x, y \in Q$, where $A, B \in \{L, R\}$ and $l(W_0) = n - 2$. If $A = B = L$, we have

$$W = L_{(x)}L_{(y)}L_{(x^{-1}.y^{-1})}W_1 = R_{(x^{-1},y^{-1})}W_1,$$

where $W_1 = L_{(y.x)}W_0$. Hence, $l(W_1) \leqslant n-1$, $W_1 \in H$, and the claim holds by induction. If $A = L$ and $B = R$, we have

$$W = L_{(x)}R_{(x^{-1})}R_{(x)}R_{(y)}R_{(y^{-1}.x^{-1})}W_1 = T_{(x^{-1})}R_{(x,y)}W_1,$$

where $W_1 = R_{(y^{-1}.x^{-1})}W_0$ and $l(W_1) \leqslant n-1$. Again, the claim holds by induction. The remaining two cases are considered similarly. $\Box$

Lemma 9 *The mapping $\lambda : \mathcal{E}(Q) \to Mlt(Q)$ defined by*

$$[(A,a),x] \overset{\lambda}{\longmapsto} AR_x$$

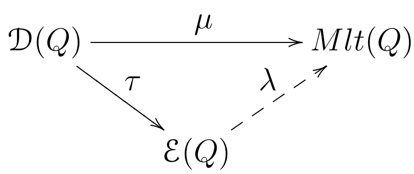
is an epimorphism. Moreover, λ is an S-epimorphism if and only if $Mlt(Q)$ is a group with triality (23).

Proof. We could use the decomposition $\mathcal{E}(Q) = PsInn(Q)Q$ and Lemma 5 to show that λ is a homomorphism. However, we choose a different approach. We show that the epimorphism $\mu : \mathcal{D}(Q) \to Mlt(Q)$ defined in (24) can be factored through $\tau : \mathcal{D}(Q) \to \mathcal{E}(Q)$. To this end, we have to show that $Z_S(\mathcal{D}(Q))$ is contained in $Ker\,\mu$. Let $A \in Z_S(\mathcal{D}(Q))$. Then $A \in H = C_{\mathcal{D}(Q)}(\sigma)$. By Lemma 8, A is expressed as a word in $T_{(x)}$ and $R_{(x,y)}$. Observe that

$\tau(T_{(x)}) = [(T_x, x^{-3}), 1] = [(\mu(T_{(x)}), x^{-3}), 1]$ and

$\tau(R_{(x,y)}) = [(R_{x,y}, [\![x,y]\!]), 1] = [(\mu(R_{(x,y)}), [\![x,y]\!]), 1]$.

Consequently, $\tau(A) = [(\mu(A), a), 1]$, where a is a suitable companion of $\mu(A)$; and $\tau(H) = C_{\mathcal{E}(Q)}(\sigma) = PsInn(Q)$ (we have identified $PsInn(Q)$ with its image in $\mathcal{E}(Q)$). On the other hand, $\tau(A) = [(1,1),1]$, since $A \in \ker\tau$. Hence, $\mu(A) = 1$ as is required.
Consequently, there exists a homomorphism $\lambda : \mathcal{E}(Q) \to Mlt(Q)$ such that the following diagram commutes:



Take an arbitrary $[(A,a),x] \in \mathcal{E}(Q)$. We have $[(A,a),x] = [(A,a),1][(1,1),x]$. Since $[(A,a),1] \in PsInn(Q)$, by the above discussion there exists $W \in H$ such that $\tau(W) = [(A,a),1]$ and $A = \mu(W)$. Hence,

$\lambda([(A,a),x]) = \lambda(\tau(W)\tau(R_{(x)})) = \mu(W)\mu(R_{(x)}) = AR_x$.

It is now clear that λ is an S-homomorphism iff μ is an S-homomorphism, which holds iff $Mlt(Q)$ admits Glauberman's triality (23). □

Observe that $K = Ker\,\lambda$ consists of all elements of $\mathcal{E}(Q)$ of the form $[(1,a),1]$ and, for λ to be an S-homomorphism, K must be S-invariant. However, $[(1,a),1]^\rho = [(1,a),a] \notin K$ unless $a = 1$. Therefore, K must be trivial. Hence, we have

Corollary 2 *The multiplication group $Mlt(Q)$ of a Moufang loop Q admits Glauberman's triality (23) if and only if the natural epimorphism $\lambda : PsInn(Q) \to \mathcal{I}(Q)$, which acts by $\lambda : (A,a) \mapsto A$, is an isomorphism.*

We put forward the following conjecture:

Conjecture 1 *Let $(Q,\,.\,)$ be a Moufang loop. The kernel of the natural epimorphism $\lambda : PsInn(Q) \to \mathcal{I}(Q)$ is generated by the elements $(1,c^3)$ for all $c \in C(Q)$ and $(1,[\![m,n]\!])$ for all pairs $m,n \in Q$ such that $(x.m).n = x.(m.n)$ for all $x \in Q$.*

3 Examples

In this section we give some examples of groups with triality.

Example 1 *Let P be a group, let $P_i \cong P$, $i = 1,\dots,4$. Put $Q = P_1 \times P_2 \times P_3 \times P_4$. The symmetric group S_4 acts on Q naturally. It is well known that $S_4 = SK$, where $S = \langle \sigma = (12), \rho = (123)\rangle$ and $K = \langle a = (12)(34), b = (14)(23)\rangle$. Denote $G = KQ$.*

Proposition 1 *The group G defined above is a group with triality with respect to the action of S by conjugation and the corresponding Moufang loop $M = \mathcal{M}(G)$ is Chein's duplication $M(P,2)$ of P.*

Proof. It is easy to see that

$$M = \xi(G) = \{\ (x^{-1},x,1,1),\ a(1,1,x^{-1},x) \mid x \in P\ \}$$

and $mm^\rho m^{\rho^2} = 1$ for all $m \in M$. Denote $t_x = (x^{-1},x,1,1)$ and identify a with $a(1,1,1,1)$. Then we have $a(1,1,x^{-1},x) = a.t_x$ and

$t_x.t_y = t_{xy}$, $t_x.(a.t_y) = a.t_{x^{-1}y}$,

$(a.t_x).t_y = a.t_{yx}$, $(a.t_x).(a.t_y) = t_{yx^{-1}}$.

Hence $(M,.)$ is Chein's duplication of P (see [2]). □

Example 2 *Suppose that Q is a group. Then $\mathcal{W}(Q)$ is isomorphic to the semidirect product $Aut(Q)(Q \times Q)$, where $Aut(Q)$ acts on $Q \times Q$ componentwise, and the triality automorphisms act as follows:*

$$(A, x, y) \stackrel{\rho}{\longmapsto} (AT_y, y^{-1}, y^{-1}x), \qquad (A, x, y) \stackrel{\sigma}{\longmapsto} (AT_y, y^{-1}x, y^{-1})$$

for all $A \in Aut(Q)$, $x, y \in Q$, where $Inn(Q) \ni T_y : x \mapsto x^y$. The subgroup of this group generated by the elements $(T_x, x^{-3}, 1)$ and $(1, x, x^{-1})$ for all $x \in Q$ is isomorphic to $\mathcal{E}(Q)$.

Proof. The required isomorphism is the map

$$\mathcal{W}(Q) \ni [(A, a), x] \;\longmapsto\; (A, ax, x) \in Aut(Q)(Q \times Q).$$

All the needed properties are easily verified. □

Example 3 *Let $M = \mathbb{Z}_3 = \langle v \mid v^3 = 1\rangle$ be a cyclic group of order 3. Then $\mathcal{D}(M) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 = \langle a, b \mid a^3 = b^3 = [a, b] = 1\rangle$, where $a = L_{(v)}$, $b = R_{(v)}$ and $a^\sigma = b^{-1}$, $b^\sigma = a^{-1}$, $a^\rho = b$, $b^\rho = a^{-1}b^{-1}$;*

Also, $\mathcal{W}(M) \cong \mathbb{Z}_2(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and $\mathcal{E}(M) \cong \mathbb{Z}_3$ as follows from the previous example. There are two other basic groups with triality associated with M; namely $S_3 = S$ with the S-action by conjugation, and $S_3 = S$ with σ-action by conjugation and trivial ρ-action. These two groups are S-embedded into $\mathcal{W}(M)$ by Theorem 2.

4 Open problems and conjectures

In this concluding section we give a number of most important and interesting, in our opinion, problems in the theory of groups with triality and Moufang loops.

Conjecture 2 *A simple (infinite) group G admits non-trivial triality S if and only if $G \cong D_4(k)$, where k is a field, and S is the group of graph automorphisms of G.*

This conjecture is both important and difficult. The only encouraging fact is that the corresponding problem for simple Lie algebras with triality was solved in the affirmative (see [11]).

Conjecture 3 (G.P. Nagy, P. Vojtěchovský, [10]) *Let G be a simple group with triality $S = \langle\rho, \sigma\rangle$ such that $G = [G, S]$ and $Z_S(G) = 1$. Then $C_G(\sigma) = \{x \in G \mid x^\sigma = x\}$ and $C_G(\rho) = \{x \in G \mid x^\rho = x\}$ are simple groups.*

Denote by $M(q)$ the finite simple Paige loop of order $\frac{1}{d}q^3(q^4-1)$, where q is a prime power and $d=(2,q-1)$. Note that this order is the product of two coprime numbers $q^3(q^2-1)$ and $\frac{1}{d}(q^2+1)$. Let M be a finite Moufang loop. A prime p is called *"bad"* for M if there exists a composition factor of M isomorphic to $M(q)$ for some q such that p divides $\frac{1}{d}(q^2+1)$. Otherwise the prime p is *"good"* for M.

Conjecture 4 (Sylow's Theorem) *A finite Moufang loop M contains a Sylow p-subloop if and only if p is a "good" prime for M. Moreover, two Sylow p-subloops of M are conjugate by an inner automorphism of M.*

Define the *Frattini subloop* $\Phi(M)$ of a Moufang loop M as the intersection of all maximal subloops of M, provided M has maximal subloops, and $\Phi(M)=M$, otherwise. As in the case of groups, $\Phi(M)$ is the normal subloop of M that consists of all non-generating elements of M.

Conjecture 5 *The Frattini subloop of a finite Moufang loop is nilpotent.*

Define the following group:

$$\widetilde{F}_{n,m}=\langle x_1,\ldots,x_n,\, y_1\ldots,y_n;\, a_1,\ldots,a_m,\, b_1,\ldots,b_m,\, c_1,\ldots,c_m \mid [x_i,y_i]=1, i=1,\ldots,n\rangle$$

and the action of $S=\langle\rho,\sigma\rangle$ on $\widetilde{F}_{n,m}$ as follows

$x_i^\sigma=x_i^{-1},\quad y_i^\sigma=x_iy_i,\quad x_i^\rho=y_i,\quad y_i^\rho=x_i^{-1}y_i^{-1},$

$a_j^\sigma=a_j,\quad b_j^\sigma=c_j,\quad a_j^\rho=b_j,\quad b_j^\rho=c_j.$

Let $N_{n,m}$ be the minimal normal subgroup of $\widetilde{F}_{n,m}$ such that $F_{n,m}=\widetilde{F}_{n,m}/N_{n,m}$ is a group with triality S. Let $F_n=F_{n,0}$. It is not difficult to show that the following fact holds:

Proposition 2 *The loop $M_n=\mathcal{M}(F_n)$ is a free n-generated Moufang loop.*

Problem 1 *For which primes p does the group F_n (the loop M_n) have a p-torsion?*

A more difficult question is to describe the structure of the Moufang loop corresponding to the group $F_{n,m}$, $m>0$. It is unknown (and, seemingly, non-trivial) even in the case of the group $F_{0,1}$. It is probable that the following is true:

Conjecture 6 *The Moufang loop $\mathcal{M}(F_{0,1})$ is infinitely generated.*

Let G be a group with triality and let $M = \mathcal{M}(G)$. Take $m, n, k \in M$. By (iv) of Lemma 4, the elements m, n, k associate (i.e., $(m, n, k) = 1$) if and only if

$$f(m, n, k) = [[m^\rho, n^{\rho^2}], k] = 1 \tag{25}$$

By Moufang's theorem [3, p.93] the relation $(m, n, k) = 1$ implies that the subloop of M generated by m, n, k is a group. Therefore, we have

Proposition 3 *Let $M = \mathcal{M}(G)$, where G is a group with triality. If elements $m, n, k \in M$ satisfy (25) then $f(x, y, z) = 1$ for all $x, y, z \in \mathcal{M}(G_1)$, where G_1 is the S-subgroup of G generated by m, n, k.*

This proposition is equivalent to Moufang's theorem. However, we have not found a short group-theoretic proof of this proposition that would not use Moufang's theorem.

Another consequence of Moufang's theorem is the fact that the loop $M_2 = \mathcal{M}(F_2)$ is a free group. In this case, the group with triality F_2 is isomorphic to the S-subgroup in $G = M_2 \times M_2 \times M_2$ generated by the elements of the form $(x^{-1}, x, 1)$ for $x \in M_2$. It is easy to see that

$F_2 = \{(x, y, z) \in G \mid xyz \in [M_2, M_2]\}, \quad \mathcal{M}(F_2) = \{(x^{-1}, x, 1) \mid x \in M_2\} \cong M_2.$

This remark gives a simple criterion to verify relations in two variables $m, n \in M$ in an arbitrary group with triality. For example, we showed in Lemma 2 that $[m^\rho, n^{-\rho^2}] = [m^{-\rho^2}, n^\rho]$. Verify this using the above criteria. Let $m = (x^{-1}, x, 1)$ and $n = (y^{-1}, y, 1)$. Then

$m^\rho = (1, x^{-1}, x), \quad n^{-\rho^2} = (y^{-1}, 1, y), \quad m^{-\rho^2} = (x^{-1}, 1, x), \quad n^\rho = (1, y^{-1}, y)$

and we have $[m^\rho, n^{-\rho^2}] = (1, 1, [x, y]) = [m^{-\rho^2}, n^\rho]$.

Let G be a group with triality and $M = \mathcal{M}(G)$. Let $N = Nuc(M)$ and define the *Moufang nucleus* $Nuc(G)$ of G to be the S-subgroup of G generated by N. Then $Nuc(G)$ is a normal subgroup of G and $\mathcal{M}(Nuc(G)) = N$.

Problem 2 *Describe perfect (i.e., equal to the commutator subgroup) algebraic (finite) groups with triality, with trivial Moufang nucleus.*

There is hope that the perfect algebraic groups with triality with trivial Moufang nucleus over an arbitrary field have structure similar to the characteristic zero case, which is described in [11] for Lie algebras with triality and is easily extended to the algebraic groups over an algebraically closed field of characteristic zero.

References

[1] R. H. Bruck, A survey of binary systems (Springer-Verlag, 1958).

[2] O. Chein, Moufang loops of small order, *Mem. of the Amer. Math. Soc.*, **13**, N 197,issue 1 (1978).

[3] H. O. Pflugfelder, Quasigroups and loops: introduction, *Sigma Series in Pure Mathematics*, 7. Berlin: Heldermann Verlag. (1990).

[4] M. W. Liebeck, The classification of finite simple Moufang loops, *Math. Proc. Camb. Phil. Soc.*, **102**, (1987), 33-47.

[5] G. Glauberman, On loops of odd order II, *J. Algebra*, **8**, (1968), 393–414.

[6] S. Doro, Simple Moufang loops, *Math. Proc. Camb. Phil. Soc.*, **83**, (1978), 377-392.

[7] P. O. Mikheev, Enveloping groups of Moufang loops, *Russ. Math. Surv.*, **48**, N 2 (1993), 195–196.

[8] J. D. Phillips, Moufang loop multiplication groups with triality, *Rocky Mt. J. Math.*, **29**, N 4 (1999), 1483–1490.

[9] R. H. Bruck., Some theorems on Moufang loops, *Math. Z.*, **73**, (1960), 59–78.

[10] G. P. Nagy, P. Vojtěchovský, Octonions and simple Moufang loops, *Quasigroups and related systems*, **10** (2003), 65–94.

[11] A. Grishkov, Lie algebras with triality, *J. Algebra*, **266**, N 2 (2003), 698–722.

[12] A. N. Grishkov, A. V. Zavarnitsine, Lagrange's theorem for Moufang loops, to appear in *Math. Proc. Camb. Phil. Soc.*

[13] A. N. Grishkov, A. V. Zavarnitsine, Maximal subloops of simple Moufang loops, submitted to *J. Algebra.*