QUANTUM IMAGINARY VERMA MODULES FOR AFFINE LIE ALGEBRAS

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Abstract. Let $\mathfrak{g}$ be an untwisted affine Kac-Moody algebra and $M(\lambda)$ an imaginary Verma module for $\mathfrak{g}$ with S-highest weight $\lambda \in P$. We construct quantum imaginary Verma modules $M^q(\lambda)$ over the quantum group $U_q(\mathfrak{g})$, investigate their properties and show that $M^q(\lambda)$ is a true quantum deformation of $M(\lambda)$ in the sense that the weight structure is preserved under the deformation.

1. Let $\dot{\mathfrak{g}}$ be a finite-dimensional simple complex Lie algebra with root system $\dot{\Delta}$. Denote by $\dot{\Delta}_+$ and $\dot{\Delta}_-$, the positive and negative roots of $\mathfrak{g}$. Let $\mathfrak{g}$ be the untwisted affine Kac-Moody algebra associated to $\dot{\mathfrak{g}}$, with Cartan subalgebra $\mathfrak{h}$. The root system Δ of $\mathfrak{g}$ is given by

$$\Delta = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}, n \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}, k \neq 0\},$$

where δ is the indivisible imaginary root. Let I be the indexing set for the simple roots.

Consider the partition $\Delta = S \cup -S$ of the root system of $\mathfrak{g}$ where $S = \{\alpha + n\delta \mid \alpha \in \Delta_+, n \in \mathbb{Z}\} \cup \{k\delta \mid k > 0\}$. This is a non-standard partition of the root system Δ in the sense that S is not Weyl equivalent to the set Δ_+ of positive roots.

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The algebra $\mathfrak{g}$ has a triangular decomposition $\mathfrak{g} = \mathfrak{g}_{-S} \oplus \mathfrak{h} \oplus \mathfrak{g}_S$, where $\mathfrak{g}_S = \oplus_{\alpha \in S} \mathfrak{g}_\alpha$. Let $U(\mathfrak{g}_S)$ (resp. $U(\mathfrak{g}_{-S})$) denote the universal enveloping algebra of $\mathfrak{g}_S$ (resp. $\mathfrak{g}_{-S}$).

Let $\lambda \in P$, the weight lattice of $\mathfrak{g}$. A weight (with respect to $\mathfrak{h}$) $U(\mathfrak{g})$-module V is called an S-highest weight module with highest weight λ if there is some nonzero vector $v \in V$ such that

(i) $u^+ \cdot v = 0$ for all $u^+ \in U(\mathfrak{g}_S)$;

(ii) $V = U(\mathfrak{g}) \cdot v$.

Let $\lambda \in P$. We make $\mathbb{C}$ into a 1-dimensional $U(\mathfrak{g}_S \oplus \mathfrak{h})$-module by picking a generating vector v and setting $(x + h) \cdot v = \lambda(h)v$, for all $x \in \mathfrak{g}_S, h \in \mathfrak{h}$. The induced module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_S \oplus \mathfrak{h})} \mathbb{C}v = U(\mathfrak{g}_{-S}) \otimes \mathbb{C}v$$

is called the *imaginary Verma module* with S-highest weight λ. Imaginary Verma modules are in many ways similar to ordinary Verma modules except they contain both finite and infinite-dimensional weight spaces. They were studied in [Fu], from which we summarize (cf. [Fu, Proposition 1, Theorem 1]).

Proposition 1. *Let $\lambda \in P$, and let $M(\lambda)$ be the imaginary Verma module of S-highest weight λ. Then $M(\lambda)$ has the following properties.*

(i) *The module $M(\lambda)$ is a free $U(\mathfrak{g}_{-S})$-module of rank 1 generated by the S-highest weight vector $1 \otimes 1$ of weight λ.*

(ii) *$M(\lambda)$ has a unique maximal submodule.*

(iii) *Let V be a $U(\mathfrak{g})$-module generated by some S-highest weight vector v of weight λ. Then there exists a unique surjective homomorphism $\phi : M(\lambda) \mapsto V$ such that $\phi(1 \otimes 1) = v$.*

(iv) $\dim M(\lambda)_\lambda = 1$. *For any $\mu = \lambda - k\delta$, k a positive integer,* $0 < \dim M(\lambda)_\mu < \infty$. *If $\mu \neq \lambda - k\delta$ for any integer $k \geq 0$ and* $\dim M(\lambda)_\mu \neq 0$, *then* $\dim M(\lambda)_\mu = \infty$.

(v) *Let $\lambda, \mu \in \mathfrak{h}^*$. Any non-zero element of $Hom_{U(\mathfrak{g})}(M(\lambda), M(\mu))$ is injective.*

(vi) *The module $M(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.*

2. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$. Then $U_q(\mathfrak{g})$ is an associative algebra with 1 over $\mathbb{C}(q)$ with generators $E_i, F_i, K_i^{\pm 1}$ $(i \in I)$ and $D^{\pm 1}$. Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by E_i (resp. F_i), $i \in I$, and let $U_q^0(\mathfrak{g})$ denote the subalgebra generated by $K_i^{\pm}$ $(i \in I)$ and $D^{\pm}$.

Beck [Be1, Be2] has given a total ordering of the root system Δ and a PBW like basis for $U_q(\mathfrak{g})$. Here we follow the construction in [BK] and let E_β denote the root vectors for each $\beta \in \Delta$ counting with multiplicity for the imaginary roots.

Let $U_q(\pm S)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_\beta \mid \beta \in \pm S\}$, and let B_q denote the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_\beta \mid \beta \in S\} \cup U_q^0(\mathfrak{g})$.

Let $\lambda \in P$. A $U_q(\mathfrak{g})$ weight module V^q is called an S-highest weight module with highest weight λ if there is a non-zero vector $v \in V^q$ such that:

(i) $u^+ \cdot v = 0$ for all $u^+ \in U_q(S) \setminus \mathbb{C}(q)^*$;

(ii) $V^q = U_q(\mathfrak{g}) \cdot v$.

Note that, in the absence of a general quantum PBW theorem for non-standard partitions, we cannot immediately claim that an S-highest weight module V^q is generated by $U_q(-S)$. This is in contrast to the classical case.

Let $\mathbb{C} \cdot v$ be a 1-dimensional vector space. Let $\lambda \in P$, and set $E_\beta \cdot v = 0$ for all $\beta \in S$, $K_i^{\pm 1} \cdot v = q^{\pm\lambda(h_i)} v$ $(i \in I)$ and $D^{\pm 1} \cdot v = q^{\pm\lambda(d)} v$. Define $M^q(\lambda) = U_q(\mathfrak{g}) \otimes_{B_q} \mathbb{C}v$. Then $M^q(\lambda)$ is an S-highest weight U_q-module called the *quantum imaginary Verma module* with highest weight λ.

Although we cannot say in general that an S-highest weight module is generated by $U_q(-S)$, in the case of imaginary Verma modules, we can make use of Beck's explicit ordered basis and the grading given in [BK, Proposition 1.8]to obtain the following result.

Theorem 2. *As a vector space, $M^q(\lambda)$ is isomorphic to the space spanned by the ordered monomials $E_{-\alpha-n\delta} \ldots E_{-k\delta} \ldots E_{-\alpha+k\delta}$, $\alpha \in \dot{\Delta}_+$, $n \geq 0$, $k > 0$.*

Let G be the subalgebra of $U_q(\mathfrak{g})$ generated by the imaginary root vectors. Let $M^q(\lambda)$ be the quantum imaginary Verma module over $U_q(\mathfrak{g})$ with S-highest weight $\lambda \in P$ and generating vector v. Consider the G-submodule of $M^q(\lambda)$ generated by v, $H^q(\lambda) = G \cdot v$. The G-module $H^q(\lambda)$ is irreducible iff $\lambda(c) \neq 0$.

Denote by $H_0^q(\lambda)$ the unique maximal submodule of $H^q(\lambda)$. Denote $M_0^q(\lambda) = U_q(\mathfrak{g})H_0^q(\lambda)$. and set $\widetilde{M^q(\lambda)} = M^q(\lambda)/M_0^q(\lambda)$.

Theorem 3. *For any $\lambda \in P$,*

(i) *$M^q(\lambda)$ is irreducible iff $\lambda(c) \neq 0$;*

(ii) *$\widetilde{M^q(\lambda)}$ is irreducible iff $\lambda(h_i) \neq 0$, $i \in I$;*

(iii) *Let $\lambda(c) = 0$ and $\lambda(h_i) \neq 0$ for all $i \in I$. Then $M^q(\lambda)$ has an infinite filtration with irreducible quotients $\widetilde{M^q(\lambda + k\delta)}$, $k \geq 0$;*

(iv) *Let $\lambda(c) = 0$, $\lambda(h_i) \neq 0$, $i \in I$ and N be a submodule of $M^q(\lambda)$. Then N is generated by $N \cap H^q(\lambda)$.*

3. We have constructed quantum imaginary Verma modules and determined some of their properties. Now we show that these quantum imaginary Verma modules are quantum deformations of imaginary Verma modules defined over the affine algebra. That is, the weight-space structure of a given module $M^q(\lambda)$ is the same as that of its classical counterpart $M(\lambda)$ for any $\lambda \in P$. To do this, we construct an intermediate module, called an $\mathbb{A}$-form.

Following [Lu], for each $i \in I$, $s \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we define the *Lusztig numbers* $\begin{bmatrix} K_i\ ;\ s \\ n \end{bmatrix}$ and $\begin{bmatrix} D\ ;\ s \\ n \end{bmatrix}$ in $U_q(\mathfrak{g})$. Let $\mathbb{A} = \mathbb{C}[q, q^{-1}, \frac{1}{[n]_{q_i}}, i \in I, n > 0]$. Define the $\mathbb{A}$-form, $U_{\mathbb{A}}(\mathfrak{g})$, of $U_q(\mathfrak{g})$ to be the $\mathbb{A}$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements $E_i, F_i, K_i^{\pm 1}, \begin{bmatrix} K_i\ ;\ s \\ n \end{bmatrix}$, $i \in I$, $D^{\pm 1}$, $\begin{bmatrix} D\ ;\ s \\ n \end{bmatrix}$. Let $U_{\mathbb{A}}^+$ (resp. $U_{\mathbb{A}}^-$) denote the subalgebra of $U_{\mathbb{A}}$ generated by the E_i, (resp. F_i), $i \in I$, and let $U_{\mathbb{A}}^0$ denote the subalgebra of $U_{\mathbb{A}}$ generated by the elements $K_i^{\pm}, \begin{bmatrix} K_i\ ;\ s \\ n \end{bmatrix}, D^{\pm 1}, \begin{bmatrix} D\ ;\ s \\ n \end{bmatrix}$.

The algebra $U_{\mathbb{A}}$ inherits the standard triangular decomposition of $U_q(\mathfrak{g})$. In particular, any element u of $U_{\mathbb{A}}$ can be written as a sum of monomials of the form $u^- u^0 u^+$ where $u^{\pm} \in U_{\mathbb{A}}^{\pm}$ and $U^0 \in U_{\mathbb{A}}^0$. In fact, we can say rather more. ¿¿From the construction of Beck's basis, it follows that all the root vectors E_β, $\beta \in \Delta$ are in $U_{\mathbb{A}}$, too.

Let $\lambda \in P$, and let $M^q(\lambda)$ be the imaginary Verma module over $U_q(\mathfrak{g})$ with S-highest weight λ and highest weight vector v_λ. The $\mathbb{A}$-form of $M^q(\lambda)$, $M^{\mathbb{A}}(\lambda)$, is defined to be the $U_{\mathbb{A}}$ submodule of $M^q(\lambda)$ generated by v_λ. That is, we set $M^{\mathbb{A}}(\lambda) = U_{\mathbb{A}} \cdot v_\lambda$.

Proposition 4. *As a vector space, $M^{\mathbb{A}}(\lambda)$ is isomorphic to the space spanned by the ordered monomials $E_{-\alpha-n\delta}\dots E_{-k\delta}\dots E_{-\alpha+k\delta}$, $\alpha\in\dot{\Delta}_+$, $n\geq 0$, $k>0$.*

Proof. This essentially follows from Theorem 2 and the fact that all the root vectors are in $U_{\mathbb{A}}$. One must also check the action of the Lusztig numbers. □

Define a weight structure on $M^{\mathbb{A}}(\lambda)$ by setting $M^{\mathbb{A}}(\lambda)_\mu = M^{\mathbb{A}}(\lambda)\cap M^q(\lambda)_\mu$ for each $\mu\in P$. Then $M^{\mathbb{A}}(\lambda)$ is a weight module with the weight decomposition $M^{\mathbb{A}}(\lambda)=\oplus_{\mu\in P}M^{\mathbb{A}}(\lambda)_\mu$, and, for each $\mu\in P$, $M^{\mathbb{A}}(\lambda)_\mu$ is a free $\mathbb{A}$-module such that $\mathrm{rank}_{\mathbb{A}}M^{\mathbb{A}}(\lambda)_\mu=\dim_{\mathbb{C}(q)} M^q(\lambda)_\mu$.

4. Next, we take the classical limits of the $\mathbb{A}$-forms of the quantum imaginary Verma modules, and show that they are isomorphic to the imaginary Verma modules of $U(\mathfrak{g})$.

Let $\mathbb{J}$ be the ideal of $\mathbb{A}$ generated by $q-1$. Then there is an isomorphism of fields $\mathbb{A}/\mathbb{J}\cong\mathbb{C}$ given by $f+\mathbb{J}\mapsto f(1)$ for any $f\in\mathbb{A}$. Set $U'=(\mathbb{A}/\mathbb{J})\otimes_{\mathbb{A}}U_{\mathbb{A}}$. Then $U'\cong U_{\mathbb{A}}/\mathbb{J}U_{\mathbb{A}}$. Denote by u' the image in U' of an element $u\in U_{\mathbb{A}}$. Then $(D')^2=1$ and $(K_i')^2=1$ for all $i\in I$ [DK]. If K' denotes the ideal of U' generated by $D'-1$ and $\{K_i'-1\mid i\in I\}$, then $\overline{U}=U'/K'\cong U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.

For $\lambda\in P$, let $M'(\lambda)=\mathbb{A}/\mathbb{J}\otimes_{\mathbb{A}}M^{\mathbb{A}}(\lambda)$. Then $M'(\lambda)\cong M^{\mathbb{A}}(\lambda)/\mathbb{J}M^{\mathbb{A}}(\lambda)$ and $M'(\lambda)$ is a U'-module. For $\mu\in P$, let $M'(\lambda)_\mu=\mathbb{A}/\mathbb{J}\otimes_{\mathbb{A}}M^{\mathbb{A}}(\lambda)_\mu$. Since $M^{\mathbb{A}}(\lambda)=\oplus_{\mu\in P}M^{\mathbb{A}}(\lambda)_\mu$, we must have $M'(\lambda)=\oplus_{\mu\in P}M'(\lambda)_\mu$. For $\mu\in P$, $\dim_{\mathbb{A}/\mathbb{J}}M'(\lambda)_\mu=\mathrm{rank}_{\mathbb{A}}M^{\mathbb{A}}(\lambda)_\mu$.

Proposition 5. *The elements D' and K_i' $(i\in I)$ in U' act as the identity on the U' module $M'(\lambda)=\mathbb{A}/\mathbb{J}\otimes_{\mathbb{A}}M^{\mathbb{A}}(\lambda)$.*

Since $M'(\lambda)$ is a U'-module, $\overline{M}(\lambda)=M'(\lambda)/K'M'(\lambda)$ is a $\overline{U}=U'/K'$-module. But K' was the ideal generated by $D'-1$ and the $K_i'-1$, and D' and each K_i' acts as the identity on $M'(\lambda)$, so $\overline{M}(\lambda)=M'$. Since $\overline{U}\cong U(\mathfrak{g})$, this means $\overline{M}(\lambda)$ has a $U(\mathfrak{g})$-structure. The module $\overline{M}(\lambda)$ is called the classical limit of $M^{\mathbb{A}}(\lambda)$. For $v\in M^{\mathbb{A}}(\lambda)$, let $\overline{v}$ denote the image of v in $\overline{M}(\lambda)$.

Proposition 6. *Let v_λ be the generating vector for $M^{\mathbb{A}}(\lambda)$. Then as a $U(\mathfrak{g})$-module, $\overline{M}(\lambda)$ is an S-highest weight $U(\mathfrak{g}_{-S})$-module generated by $\overline{v_\lambda}$ and such that, for any $\mu\in P$, $\overline{M}(\lambda)_\mu$ is the μ-weight space of $\overline{M}(\lambda)$.*

Proof. The crucial part of the proof is observing that the images in $\overline{U}$ of the ordered monomials in the root vectors E_β, $\beta \in -S$, form a basis for $U(\mathfrak{g}_{-S})$. □

Assembling these ingredients, we obtain the following result.

Theorem 7. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra. Let $\lambda \in P$. Then the imaginary Verma module $M(\lambda)$ admits a quantum deformation to the quantum imaginary Verma module $M^q(\lambda)$ over $U_q(\mathfrak{g})$ in such a way that the weight space decomposition is preserved.*

Details of the proofs and additional results will be given in a later paper.

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