The automorphisms Group of the Multiplicative Cartan Decomposition of Lie algebra E_8

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Alexandr N. Grishkov

Omsk State University(Russia) and University of Sao Paulo(Brazil)

e-mail: grishkov@ime.usp.br

1 Introduction.

Suppose that a group G or a Lie algebra L acts by automorphisms or derivations respectively on some algebra A. Moreover, we assume that as a G-module (respectively L-module), A is a direct sum of irreducible modules $\Lambda_1, ..., \Lambda_m$. We will call such algebra A by Λ-algebras, where $\Lambda = \Lambda_1 \oplus ... \oplus \Lambda_m$. In [2], [3] we proposed a method of studying a category of Λ-algebras from a fixed variety $\mathcal{M}$. In the present paper we describe this method and apply it for the classification of the simple algebras from a certain category which contains the exceptional simple Lie algebra E_8. This new construction of the Lie algebra E_8 defines a basis with a simple multiplication. Finally, we apply this basis to obtain the multiplicative Cartan decomposition (MCD) of the Lie algebra E_8 and to compute the generators of the automorphism group of MCD. Observe that in the original work [6] J. Thompson called this decomposition by Dempwolff's decomposition. Let $E_8(k)$ be the exceptional algebraic group of type E_8 over a field k. In his thesis [4], Peter Smith constructed a certain subgroup D of $E_8(\mathbf{C})$ called the Dempwolff group. Here D is a non-split extension of Z_2^5 by $L_5(2)$, which was used by Thompson to construct his sporadic simple group Th. Smith produced 248×248 matrices which generate D and preserve the Multiplicative Cartan Decomposition. For his construction he used a computer. These matrices have rational entries with denominators being powers of 2.

In this paper we give a more simple description of the above 248×248 matrices of P. Smith. The main result (Theorem 2) is based on the construction of the exceptional Lie algebra of type E_8 (Theorem 1).

Let k be a field of characteristic $p > 2$, such that the equation $x^2 + 1 = 0$ has a solution in k. We shall use the following standard notation: $\mathbf{Q}$ is the field of rational numbers, Z_2 is the group of two elements, $k\{X\} = kX$ is the $k-$space with a basis X.

2 Construction of the Lie algebras of type E_7 and E_8.

In this chapter we recall some results from the theory of graded varieties [3] and apply them to construction of simple Lie algebras.

We fix a set ∇ and shall call by ∇-space (∇-algebra, ∇-module) a space V with a fixed ∇-grading: $V = \sum_{\alpha \in \nabla} \oplus V_\alpha$. We can consider the ∇-space V as an algebra with unary operations $\{\alpha | \alpha \in \nabla\}$ such that for $a \in A$: $(a)_\alpha = a_\alpha$ if $a = \sum_{\beta \in \nabla} a_\beta$. Let $A = \sum_{\alpha \in \nabla} \oplus A_\alpha$ be a ∇-algebra. Then a ∇-identity on A is a (non-associative) polynomial $f(x, y, ...)$ in signature $(+, \cdot, \alpha \in \nabla)$ such that $f(a, b, ...) = 0$ for all elements $a, b, ... \in A$. For example, $f(x, y) = (x_\alpha y_\beta)_\gamma$. Let $V = \sum_{\alpha \in \triangle} \oplus V_\alpha$, $W = \sum_{\beta \in \triangle} \oplus W_\beta$ be two given ∇-spaces, we define the *contraction* of the ∇-spaces as following

$$V \square W = \sum_{\alpha \in \nabla} \oplus V_\alpha \otimes W_\alpha.$$

Thus $V \square W$ is a ∇-space too.

If A and B are two ∇-algebras then we define the *contraction* of ∇-algebras A and B as a ∇-space $A \square B$ with the multiplication rule

$$a_\alpha \otimes b_\alpha \cdot a_\beta \otimes b_\beta = \sum_{\gamma \in \nabla} c_\gamma \otimes d_\gamma,$$

where $a_\alpha a_\beta = \sum_{\gamma \in \nabla} c_\gamma, b_\alpha b_\beta = \sum_{\gamma \in \nabla} d_\gamma$.

Definition 1 *A set $\mathcal{N}$ of algebras over k is called ∇-variety if $\mathcal{N}$ is the set of all ∇-algebras over k which satisfy a given set of ∇-identities.*

For a given set X of ∇-algebras or ∇-identities we denote by $\{X\}$ the minimal ∇-variety which contains all ∇-algebras from X or satisfies all identities from X.

If $\mathcal{N}$ and $\mathcal{M}$ are two ∇-varieties then we define a contraction operation by

$$\mathcal{N}\Box\mathcal{M} = \{A\Box B | A \in \mathcal{N}, B \in \mathcal{M}\},$$

and a division operation by

$$\mathcal{N}/\mathcal{M} = \begin{array}{l}\{A| \ \ \forall B \in \mathcal{M}, A\Box B \in \mathcal{N}, \ \ \textit{and} \ \ A \ \ \textit{satisfies} \\ \textit{all the identities of} \ \ \mathcal{M} \ \ \textit{of the type} \ \ (a_\alpha b_\beta)_\gamma = 0\}.\end{array}$$

It is obvious that $(\mathcal{N}/\mathcal{M})\Box\mathcal{M} \subset \mathcal{N}$.

Between these two operations (contraction and division) there is some difference. If we have a set X of ∇-polynomials such that $\mathcal{N}$=$\{X\}$ and an ∇-algebra A such that $\mathcal{M}$ =$\{A\}$ then finding a set Z such that $\mathcal{N}\Box\mathcal{M}$ =$\{Z\}$ may require non-trivial efforts. On the other hand there is a simple algorithm for constructing the set of ∇-identities Z such that $\mathcal{N}/\mathcal{M}$ =$\{Z\}$. First we have to take the absolutely free ∇-algebra $F = F(x_1, ...)$, where $\{x_i\}$ are the homogeneous free generators of F. Let $B = \{a_i | i = 1, ...\}$ be a homogeneous basis of the ∇-algebra A. For any ∇-identity $f(x_{\alpha_1}, ..., x_{\alpha_n})$ of X and any subset $T = \{a_{i_1}, ..., a_{i_n}\}$ of B such that $a_i \in A_{\alpha_i}$ we can construct the following set of ∇- identities: $G(f,T) = \{g_1(x_\alpha, ...), ..., g_m(x_\alpha, ...)\}$, where $f(x_{\alpha_1} \otimes a_{i_1}, ..., x_{\alpha_n} \otimes a_{i_n}) = \sum_{j=1}^m g_j \otimes a_{k_j}$ and $k_j \neq k_i$, if $j \neq i$.

Proposition 1 *If $\mathcal{N}$ and $\mathcal{M}$ are ∇-varieties such that $\mathcal{N}$=$\{X\}$, $\mathcal{M}$ =$\{A\}$ and $B = \{a_i | i = 1, ...\}$ is a basis of the ∇-algebra A then*

$$\mathcal{N}/\mathcal{M} = \{M_2, G(f,T) | f \in X, T \subset B\},$$

where M_2 is the set of identities of the variety $\mathcal{M}$ of the type $(x_\alpha x_\beta)_\gamma = 0$.

This proposition is usefull for the classification of simple ∇-algebras from a given variety $\mathcal{N}$ which has the form $A\Box\Lambda$ for a given ∇-algebra Λ.

Let $\mathcal{N}$ be a variety (not nesessarily graded). Then a Z_2-graded algebra A is called a $\mathcal{N}$-superalgebra if $A\Box G \in \mathcal{N}$, where G is the Grassman algebra. It means that Z_2-variety $\mathcal{N}_2$ of $\mathcal{N}$-superalgebras is the variety $\mathcal{N}/Gr$, where $Gr = \{G\}$. And it is well known that there is a simple algorithm to constract the graded identities of $\mathcal{N}$-superalgebras if we know the identities of the variety $\mathcal{N}$.

In this section we describe some aplication of this notion for a constraction of the simple exceptional finite dimentional Lie algebras of the types E_7 and E_8.

Let $\mathcal{A}_0$ be a set of some subsets of $I_n = \{1, \dots, n\}$ and $\mathcal{A} = \mathcal{A}_0 \cup I_n$. Then we can define an $\mathcal{A}-$algebra $\Lambda(\mathcal{A}) = \Lambda$ with the basis

$$B = B(\mathcal{A}) = \{e_i, f_i, h_i, i = 1, ..., n; (\sigma, \varphi) \mid \varphi \subseteq \sigma \in \mathcal{A}\}$$

and an $\mathcal{A}-$graduation

$$\Lambda_i = ke_i \oplus kh_i \oplus f_i, i \in I_n, \Lambda_\sigma = \sum_{\mu \subseteq \sigma} k(\sigma, \mu).$$

We also define the multiplication by the rule

$$e_i f_i = -f_i e_i = h_i, e_i h_i = -h_i e_i = 2e_i, h_i f_i = -f_i h_i = 2f_i,$$

$$e_i(\sigma, \varphi) = -(\sigma, \varphi)e_i = (\sigma, \varphi \cup i), i \in \sigma \setminus \varphi;$$

$$(\sigma, \varphi)f_i = -f_i(\sigma, \varphi) = (\sigma, \varphi \setminus i), i \in \varphi;$$

$$(\sigma, \varphi)h_i = -h_i(\sigma, \varphi) = (\sigma, \varphi), i \in \varphi;$$

$$(\sigma, \varphi)h_i = -h_i(\sigma, \varphi) = -(\sigma, \varphi), i \in \sigma \setminus \varphi;$$

$$(\sigma, \varphi)(\sigma, \psi) = \begin{cases} (-1)^{|\psi|+1} e_i, & \varphi \cap \psi = i, \varphi \cup \psi = \sigma; \\ (-1)^{|\psi|} f_i, & \varphi \cap \psi = \emptyset, \varphi \cup \psi = \sigma \setminus i; \\ \frac{(-1)^{|\psi|}}{2} (\sum_{i \in \varphi} h_i - \sum_{j \in \psi} h_j), & \varphi \cap \psi = \emptyset, \varphi \cup \psi = \sigma; \end{cases}$$

$$(\sigma, \varphi)(\tau, \psi) = \begin{cases} (-1)^{|\sigma \cap \psi|} (\sigma \triangle \tau, (\varphi \setminus \tau) \cup (\psi \setminus \sigma)), \\ \sigma \neq \tau, \varphi \cap \psi = \emptyset, \sigma \cap \tau \subseteq \varphi \cup \psi, \sigma \triangle \tau \in \mathcal{A}, \end{cases} \quad (1)$$

All other products are equal to zero. Here and below we use the standard notation $\sigma \triangle \tau = \sigma \setminus \tau \cup \tau \setminus \sigma$ for symmetric difference.

We observe that every variety $\mathcal{N}$ defines the ∇-variety which we can denote by the same letter $\mathcal{N}$ and this ∇-variety consists of all ∇-algebras from $\mathcal{N}$.

Proposition 2 *Let $\mathcal{A}_0$ be a set $\{\sigma \mid \sigma \subseteq I_n, |\sigma| = 4\}$, $\mathcal{A} = \mathcal{A}_0 \cup I_n$ and $\mathcal{L}$ be a $\mathcal{A}$-variety of Lie algebras. Then the $\mathcal{A}-$variety $\mathcal{L}/\{\Lambda(\mathcal{A})\}$ is defined by the following $\mathcal{A}-$identities*

$$(a_i \cdot b_i)_j = (a_i \cdot b_i)_\sigma = (a_\sigma \cdot b_\tau)_\lambda = 0, i \neq j, \sigma \neq \tau, \lambda \neq \sigma \triangle \tau. \quad (2)$$

$$(a_\sigma \cdot b_\sigma)_\lambda = (a_\sigma \cdot b_\sigma)_i = 0, \quad i \notin \sigma. \tag{3}$$

$$a_\sigma \cdot b_\tau = (-1)^{|\sigma\cap\tau|+1} b_\tau \cdot a_\sigma, \quad \sigma \neq \tau. \tag{4}$$

$$a_\sigma \cdot b_\sigma = b_\sigma \cdot a_\sigma. \tag{5}$$

$$\begin{aligned} a_i \cdot b_j = \delta_i^j b_j \cdot a_i, \quad & a_i \cdot b_\sigma = b_\sigma \cdot a_i, \quad i \in \sigma, \\ & a_i \cdot b_\sigma = 0, \qquad\quad i \notin \sigma. \end{aligned} \tag{6}$$

$$(a_\sigma \cdot b_i) \cdot c_j = a_\sigma \cdot (b_i \cdot c_j), (a_i \cdot b_i) \cdot c_i = a_i \cdot (b_i \cdot c_i). \tag{7}$$

$$\begin{aligned} &(a_\sigma \cdot b_\tau) \cdot c_\lambda = 0, \quad \sigma \neq \tau \neq \lambda \neq \sigma \neq \sigma\triangle\tau, |\sigma\cap\tau\cap\lambda| > 1. \\ &\qquad\qquad or \quad |\sigma\cap\tau\cap\lambda| = 1, |\sigma\cap\tau| + |\sigma\cap\lambda| \equiv 1 (mod 2). \end{aligned} \tag{8}$$

$$\begin{aligned} &(a_\sigma \cdot b_\tau) \cdot c_\lambda + (-1)^{|\tau\cap\lambda|}(a_\sigma \cdot c_\lambda) \cdot b_\tau = 0, \\ &\sigma \neq \tau \neq \lambda \neq \sigma \neq \sigma\triangle\tau, \qquad\qquad |\sigma\cap\tau\cap\lambda| = 1. \end{aligned} \tag{9}$$

$$\begin{aligned} &(-1)^{|\sigma\cap\lambda|}(a_\sigma \cdot b_\tau) \cdot c_\lambda + (-1)^{\tau\cap\sigma}(b_\tau \cdot c_\lambda) \cdot a_\sigma + \\ &(-1)^{|\tau\cap\lambda|}(c_\lambda \cdot a_\sigma) \cdot b_\tau = 0, |\sigma\cap\tau\cap\lambda| = 0. \end{aligned} \tag{10}$$

$$\begin{aligned} &(a_\sigma \cdot b_\tau) \cdot c_\lambda = (-1)^{|\sigma|}(b_\tau \cdot c_\lambda) \cdot a_\sigma = -(c_\lambda \cdot a_\sigma) \cdot b_\tau, \\ &|\sigma\cap\tau| = 2, \sigma \neq \tau. \end{aligned} \tag{11}$$

$$((a_\sigma \cdot b_\tau) \cdot c_\lambda)_i = \begin{cases} ((b_\tau \cdot c_\lambda) \cdot a_\sigma)_i, & i \in \sigma \setminus \tau, \\ ((c_\lambda \cdot a_\sigma) \cdot b_\tau)_i, & i \in \tau \setminus \sigma, \end{cases} \tag{12}$$

$$where, \lambda = \sigma\triangle\tau \in \mathcal{A}_0.$$

$$((a_\sigma \cdot b_\sigma)_i \cdot c_\sigma) = ((b_\sigma \cdot c_\sigma)_j \cdot a_\sigma), i, j \in \sigma. \tag{13}$$

$$((a_\sigma \cdot b_\sigma)_i \cdot c_\tau) + ((c_\tau \cdot a_\sigma) \cdot b_\sigma) = 0, i \in \sigma \cap \tau; |\sigma \cap \tau| = 2. \qquad (14)$$

$$((a_\sigma \cdot b_\sigma)_i \cdot c_\tau) = 0, i \in \sigma \cap \tau; |\sigma \cap \tau| = 1 \quad or \quad 3. \qquad (15)$$

Proof. All $\mathcal{A}-$identities (2-15) follow from the identities of Lie algebras by straightforward calculations. □

Proposition 3 *Let P be a simple finite dimensional $\mathcal{A}-$algebra from $\mathcal{L}/\{\Lambda\}$, $P = \sum_{\alpha \in \mathcal{A}} P_\alpha$ and $D = \{\alpha \in \mathcal{A}_0 \mid P_\alpha \neq 0\}$. If k is algebraically closed then $dim_k P_\sigma = 1, \sigma \in D$ and one of the following equalities holds*

$(i) D = \{(2i-1, 2i, 2j-1, 2j) \mid 1 \leq i < j \leq n\}$,

$(ii) D = \mathcal{E}_7 = \{(1234), (1256), (1357), (3456), (2367), (2457), (1467)\}$,

$(iii) D = \mathcal{E}_8 = \mathcal{E}_7 \cup \{\sigma \mid \bar{\sigma} = I_8 \setminus \sigma \in \mathcal{E}_7\}$.

Proof. Let $i \in I_n$. It follows from (6) and (7) that P_i is a commutative and associative subalgebra of P. If P_i contains zero divisors $a, b \in P_i, a \cdot b = 0$, then $aP \cdot bP = 0$, a contradiction. Hence $P_i = ks_i, s_i^2 = s_i$. Note that $P_\sigma^2 \neq 0$. Indeed, if $P_\sigma^2 = 0$ then we see from (14) that $(P_\sigma \cdot P_\tau)^2 = 0$ for every $\tau \in D$, which is impossible. If $\sigma \in D$ and $a, b \in P_\sigma$ such that $a \cdot b = s_i, i \in \sigma$, then from (13) we obtain that $(a \cdot b)_i \cdot b = (b \cdot b)_i \cdot a = b$, hence $dim_k P_\sigma = 1$. If $\sigma, \tau \in D$ and $|\sigma \cap \tau| = 1$ or 3 then from (15) we have $P_\sigma = 0$, a contradiction. Hence we obtain $|\sigma \cap \tau| = 2$ or 0 for any $\sigma, \tau \in D$. At last we note that from (14) it follows that $\sigma \triangle \tau \in D$ for any $\sigma, \tau \in D$ such that $|\sigma \cap \tau| = 2$. Now it is easy to prove that if $D = \{\sigma \mid \sigma \subseteq I_n, |\sigma| = 4, |\sigma \cap \tau| = 2 \text{ or } 0; |\sigma \cap \tau| = 2 \Rightarrow \sigma \triangle \tau \in D, \ \forall \sigma, \tau \in D\}$ then D satisfies one of the conditions (i)-(iii). □

Our purpose is to construct a simple algebra $P = P(D)$ from $\mathcal{L}/\Lambda$ for $D = \mathcal{E}_7$ or $\mathcal{E}_8$. Let $\mathbf{O}$ be the Caley-Dikson algebra with the standard generators i, j, k, then $B_0 = \{1, i, j, k, ij, ik, jk, ij \cdot k\}$ is a basis of $\mathbf{O}$. We note that $B = \pm B_0$ is a Moufang loop and $B_1 = B/\{\pm 1\}$ is an elementary 2-group. Let us remind that a loop is Moufang if it satisfies the identity $xy \cdot zx = (x \cdot yz)x$. Let us fix an isomorphism $\bar{t} : B_1 \to D_7$, where $D_7 = \mathcal{E}_7 \cup \{\emptyset\}$ with a product $\sigma \triangle \tau$ and $\bar{t}(i) = (1234), \bar{t}(j) = (1256), \bar{t}(k) = (1357)$. We can consider a set $M_7 = \pm D_7$ as a Moufang loop such that $t : B \to M_7$, $t(ax) = at(x), a \in \{\pm\}, x \in B_0$, is an isomorphism. Set $M_8 = M_7 \times Z_2$, where

$Z_2 = \{e, b | b^2 = e\}$. Identify M_8 with the set $\pm\mathcal{E}_8 \cup \pm\{\emptyset, I_8\}$ via $a\sigma \cdot e = a\sigma$, $a\sigma \cdot b = a\bar{\sigma}, \sigma \in \mathcal{E}_7$ and $a1 \cdot e = a\emptyset, a1 \cdot b = aI_8$, where $a \in \{\pm\}$.

Let P_8 be an $\mathcal{A}-$algebra with a basis $\{s_1, ..., s_8, \sigma | \sigma \in \mathcal{E}_8\}$ and the multiplication rule

$$\sigma \star \sigma = \sum_{i \in \sigma} s_i, \sigma \star \tau = a\sigma \triangle \tau,$$

if $\sigma \cdot \tau = a\sigma \triangle \tau$ in the loop M_8. It is obvious that the space P_7 with the basis $\{s_1, ..., s_7, \sigma | \sigma \in \mathcal{E}_7\}$ is a subalgebra of P_8.

Proposition 4 *Let $P = P(D)$ be a simple finite dimensional $\mathcal{A}-$algebra from $\mathcal{L}/\Lambda$ and $D = \mathcal{E}_7$ or $D = \mathcal{E}_8$. Then P isomorphic to P_7 or P_8.*

Proof. Let $P = P(D)$ be a simple $\mathcal{A}-$algebra and $D = \mathcal{E}_7$. From Proposition 3 we have that P has a basis $\{s_1, ..., s_7, \sigma | \sigma \in \mathcal{E}_7\}$ and we can normalize this basis such that $\sigma \cdot \sigma = \sum_{i \in \sigma} s_i$. Then from (12) one gets that $((\sigma \cdot \tau) \cdot \lambda)_i = ((\tau \cdot \lambda) \cdot \sigma)_i$ for $\sigma, \tau \in \mathcal{E}_7, \sigma \neq \tau$. Hence we can assume that $\sigma \cdot \tau = a\lambda$, where $\lambda = \sigma \triangle \tau$ and a^2 does not depend on σ, τ. But it follows from (14) that $a^2 = 1$. Now we note that $\sigma \cdot \tau = -\tau \cdot \sigma$ follows from (4) and (9),(4) lead to

$$(\sigma \cdot \tau) \cdot \lambda = -(\sigma \cdot \lambda) \cdot \tau = \tau \cdot (\sigma \cdot \lambda) = -(\tau \cdot \sigma) \cdot \lambda,$$

if $\sigma \neq \lambda$. This means that we are done already in the case of $\mathcal{E}_7$, since in the loop M_7 we have the same equalities $xy = -yx, (xy)z = -x(yz)$, if x, y, z are linear independent over Z_2.

In the case $D = \mathcal{E}_8$ the proof is analogous. $\square$

Now we can prove the main result of this chapter.

Theorem 1 *A simple Lie algebra of type E_8 over an algebraically closed field of characteristic $p > 2$ has a basis*

$$B_8 = \{e_i, h_i, f_i, i = 1, ..., 8; (\sigma, \mu) | \mu \subseteq \sigma \in \mathcal{E}_8\}$$

and the following multiplication rules in this basis:

$$e_i f_i = h_i, e_i h_i = 2e_i, h_i f_i = 2f_i,$$

$$e_i(\sigma,\varphi) = (\sigma, \varphi \cup i), i \in \sigma \setminus \varphi;$$

$$(\sigma,\varphi) f_i = (\sigma, \varphi \setminus i), i \in \varphi;$$

$$(\sigma,\varphi) h_i = (\sigma,\varphi), i \in \varphi;$$

$$(\sigma,\varphi) h_i = -(\sigma,\varphi), i \in \sigma \setminus \varphi;$$

$$(\sigma,\varphi)(\sigma,\psi) = \begin{cases} (-1)^{|\psi|+1} e_i, & \varphi \cap \psi = i, \varphi \cup \psi = \sigma; \\ (-1)^{|\psi|} f_i, & \varphi \cap \psi = \emptyset, \varphi \cup \psi = \sigma \setminus i; \\ \frac{(-1)^{|\psi|}}{2}(\sum_{i \in \psi} h_i - \sum_{j \in \varphi} h_j)/2, & \varphi \cap \psi = \emptyset, \varphi \cup \psi = \sigma; \end{cases}$$

$$(\sigma,\varphi)(\tau,\psi) = \begin{cases} (-1)^{|\sigma \cap \psi|}(\sigma \star \tau, (\varphi \setminus \tau) \cup (\psi \setminus \sigma)), \\ \sigma \neq \tau, \varphi \cap \psi = \emptyset, \sigma \cap \tau \subseteq \varphi \cup \psi, \sigma \triangle \tau \in \mathcal{A}, \end{cases} \tag{16}$$

where $\sigma \star \lambda$ is the product in the Moufang loop M_8.

Proof. Let L be the algebra from the hypotheses of the Theorem. From Proposition 4 we have that L is a Lie algebra. It is obvious that $H = k\{h_1, ..., h_8\}$ is a Cartan subalgebra of L and L has the following Cartan decomposition

$$L = H \oplus \sum_{i=1}^{8} (ke_i \oplus kf_i) \oplus \sum_{\mu \subseteq \sigma \in \mathcal{E}_8} k(\sigma,\mu).$$

Hence we can identify the set of roots ∇_8 of L with the following subset of the space $\mathbf{Q}^8$

$$\nabla_8 = \{\pm\alpha_i = (\underbrace{0,\ldots,0}_{i-1}, \pm 2, 0, \ldots, 0), i = 1, \ldots, 8, \quad \alpha(\sigma,\mu) =$$

$$(\pm\varepsilon_1, ..., \pm\varepsilon_8), \mu \subseteq \sigma \in \mathcal{E}_8, \varepsilon_i = 1, i \in \mu; \varepsilon_i = -1, i \in \sigma \setminus \mu; \varepsilon_i = 0, i \in I_8 \setminus \sigma\}.$$

It is easy to prove that the following symmetric bilinear form on L is invariant and non-degenerate:

$$(h_i, h_j) = 2\delta_i^j, (e_i, f_j) = -\delta_i^j,$$
$$((\sigma, \mu), (\sigma, \bar{\mu})) = (-1)^{|\mu|+1}, \tag{17}$$

and the others products are zero. The corresponding form on the roots is

$$(\alpha_i, \alpha_j) = 2\delta_i^j,$$
$$(\alpha(\sigma, \mu), \alpha(\tau, \psi)) = (|\mu \cap \psi| + |\bar{\mu} \cap \bar{\psi}| - |\bar{\mu} \cap \psi| - |\mu \cap \bar{\psi}|)/2, \tag{18}$$

where $\bar{\mu} = \sigma \setminus \mu, \bar{\psi} = \tau \setminus \psi$. We define an order on $\mathbf{Q}^8$: $v > w$, if $v - w = (0, ..., 0, a, ..), a \in \mathbf{Q}, a > 0$. Then the following set is a set of the simple positive roots:

$$(1, -1, -1, -1, 0, 0, 0, 0), \alpha_4, (0, 0, 1, -1, -1, -1, 0, 0), \alpha_6,$$
$$(0, 0, 0, 0, 1, -1, -1, -1), \alpha_8, (0, 1, -1, 0, -1, 0, 0, -1), \alpha_7.$$

Now it is easy to construct the Dynkin diagram of the root system ∇_8, which is of type E_8. □

3 The multiplicative Cartan decomposition of the exceptional Lie Algebra E_8.

In this chapter we construct an elementary abelian subgroup Z_2^5 in the group $Aut_k(E_8)$ such that the corresponding grading of the Lie algebra E_8 is the famous MCD (multiplicative Cartan decomposition) [6].

In previous chapter we constructed the Lie Algebra E_8 with Z_2^4 grading

$$L = \sum_{\sigma \in \mathcal{E}_8 \cup \{\emptyset\}} \oplus L_\sigma,$$

where $L_\sigma = \sum_{\mu \subseteq \sigma} k(\sigma, \mu)$, $Z_2^4 = G_8 = \mathcal{E}_8 \cup I_8 \cup \{\emptyset\}$ is a group with product $\sigma \triangle \tau$ and $L_\emptyset = \bar{S}$. Denoting $L_\sigma \oplus L_{\bar{\sigma}}$ by V_σ we get

$$L = V_\emptyset \oplus \sum_{\sigma \in \mathcal{E}_7} V_\sigma \tag{19}$$

It is easy to see that (19) is Z_2^3-grading, where $Z_2^3 = G_7 = \mathcal{E}_7 \cup \{\o\} \subseteq G_8$.

From the Theorem 1 we have that the following involution κ is the Cartan involution

$$h_i^\kappa = -h_i, e_i^\kappa = -f_i, i \in I_8, (\sigma, \mu)^k = (\sigma, \sigma \setminus \mu), \tag{20}$$

which preserves G_7-grading (19).

We define another involution r of L

$$h_i^r = h_i, e_i^r = -e_i, f_i^r = -f_i, i \in I_8, (\sigma, \mu)^r = (-1)^{|\mu|}(\sigma, \mu). \tag{21}$$

Let G_0 denote the elementary abelian 2-group with generators $\{\sigma, \kappa, r \mid \sigma \in G_7\}$. Then $G_0 = Z_2^5$ and :

$$L = \sum_{\sigma \in \mathcal{E}_7} H_\sigma^\pm \oplus \sum_{\sigma \in \mathcal{E}_7} A_\sigma^\pm, \tag{22}$$

where

$$A_\o^- = \{h_1, ..., h_8\}, H_\o^- = \{e_1 + f_1, ..., e_8 + f_8\}, A_\o^+ = 0,$$
$$H_\o^+ = \{e_1 - f_1, ..., e_8 - f_8\},$$
$$A_\sigma^\pm = \{(\sigma, \mu) \pm (\sigma, \bar{\mu}), (\bar{\sigma}, \lambda) \pm (\bar{\sigma}, \bar{\lambda}) \mid \sigma \in \mathcal{E}_7, \mu \subseteq \sigma, \lambda \subseteq \bar{\sigma}; \mid \mu \mid, \mid \lambda \mid \in 2\mathbf{Z}\},$$
$$H_\sigma^\pm = \{(\sigma, \mu) \pm (\sigma, \bar{\mu}), (\bar{\sigma}, \lambda) \pm (\bar{\sigma}, \bar{\lambda}) \mid \sigma \in \mathcal{E}_7, \mu \subseteq \sigma, \lambda \subseteq \bar{\sigma}; \mid \mu \mid, \mid \lambda \mid \in 2\mathbf{Z}+1\}.$$

Is is easy to check that

$$[H_\sigma^p, H_\tau^q] \subseteq A_{\sigma \triangle \tau}^{pq}, [A_\sigma^p, A_\tau^q] \subseteq A_{\sigma \triangle \tau}^{pq},$$

$$[A_\sigma^p, H_\tau^q] \subseteq H_{\sigma \triangle \tau}^{pq}, \sigma, \tau \in \mathcal{E}_7, p, q \in \{\pm\},$$

but all of the subspaces of the grading (22) are the Cartan subalgebras. Hence the decomposition (22) is the MCD.

4 The automorphism Group of MCD

In this chapter we calculate the automorphism group G of MCD. Moreover, we obtain the generators of this group and their action on the basis B_8 of E_8. Fix the sets $\sigma_1 = (1234), \sigma_2 = (1256), \sigma_3 = (1357)$.

Lemma 1 *Let G_1 be the automorphism group of the Moufang loop M_8. Then $G_1 = (GL_3(2) \rtimes V_3) \times V_4$, where V_i is an $i-$dimensional Z_2-space. Moreover, there exists an embedding of G_1 into $G = Aut(MCD)$.*

Proof. Note that $Z(M_8) = \{a \in M_8 \mid [a,x] = (x,y,a) = (x,a,y) = (a,x,y) = 1, \forall x,y \in M_8\} = \{\pm 1, \pm I_8\}$ and $[M_8, M_8] = \{\pm 1\}$, here $(x,y,z) = ((xy)z)(x(yz))^{-1}$ and $[x,y] = xyx^{-1}y^{-1}$. Hence for $\phi \in G_1$ we have: $(\pm 1)^\phi \subseteq \{\pm 1\}$, $(\pm I_8)^\phi \subseteq \{\pm I_8\}$.

The factor loop $M_8/\{\pm 1\}$ is a 4-dimensional Z_2-space with a base $\{\sigma_1, \sigma_2, \sigma_3, I_8\}$. We consider $GL_4(2)$ as linear automorphisms of this Z_2-space. Then $Stab(I_8) = \{\phi \in GL_4(2) \mid I_8^\phi = I_8\} = (GL_3(2) \times V_3)$. For every $A \in Stab(I_8)$ we can construct an automorphism ϕ_A such that $\sigma_i^{\phi_A} = \sigma_i A, i = 1,2,3;\quad I_8^{\phi_A} = I_8,\quad (-1)^{\phi_A} = -1$. But $Z_{Aut(M_8)}\{\phi_A \mid A \in Stab(I_8)\} = \{\psi \in Aut(M_8) \mid [\psi, \phi_A] = 1, \forall A \in Stab(I_8)\} = \{\psi \in Aut(M_8) \mid \sigma_i^\psi = \pm\sigma_i, i = 1,2,3; I_8^\psi = \pm I_8\} = V_4$ and the first part of Lemma is proved.

To prove the second part we define a homomorphism $\Phi : G_1 \to S_8$ by the rule: $\Phi(\alpha) = \hat{\alpha}$, where $\alpha \in G_1, \sigma_i^\alpha = \pm\mu_i, \hat{\alpha} \in S_8$ and $\hat{\alpha}(i) = p_i$, where

$$p_1 = \mu_1 \cap \mu_2 \cap \mu_3, p_2 = (\mu_1 \cap \mu_2) \setminus p_1, p_3 = \mu_1 \cap \mu_3 \setminus p_1,$$

$$p_4 = \mu_1 \setminus \{p_1, p_2, p_3\}, p_5 = \mu_2 \cap \mu_3 \setminus p_1, p_6 = \mu_2 \setminus \{p_1, p_2, p_5\},$$

$$p_7 = \mu_3 \setminus \{p_1, p_3, p_5\}, p_8 = I_8 \setminus \{\mu_1 \cup \mu_2 \cup \mu_3\}.$$

Note that $ker\Phi = V_4$.

Now we can define an embedding $\Psi : G_1 \to Aut(MCD)$. If $\alpha \in V_4 \subseteq G_1$ then $\Psi(\alpha) = \psi \in Aut(MCD)$, where $x^\psi = x, x \in S, (\sigma,\mu)^\psi = \varepsilon(\sigma,\mu)$, $\varepsilon \in \{\pm 1\}$ and $\sigma^\alpha = \varepsilon\sigma$ for $\sigma \in M_8$. If $\alpha \in GL_3(2) \times V_3$ then by definition $\Psi(\alpha) = \psi$ where $x_i^\psi = x_{\hat{\alpha(i)}}, x \in \{e,h,f\}$, $(\sigma,\mu)^\psi = (\hat{\alpha}(\sigma), \hat{\alpha}(\mu))$ here $\hat{\alpha}(\sigma)$ is the action of S_8 on the set of all subsets of I_8. Lemma is proved. □

Now we construct the generators of the group $G = Aut(MCD)$. It is obvious that $G = \{\varphi \in Aut_k(L) \mid G_0^\varphi = \varphi^{-1}G_0\varphi \subseteq G_0\}$. We fix a basis of G_0 as a Z_2-space: $\{\sigma_1, \sigma_2, \sigma_3, \kappa, r\}$. Then every $\varphi \in G$ has a unique form $\varphi = \varphi_1\varphi_2$, where $\varphi_1 \in GL_5(2), \varphi_2 \in Z_{Aut_k(L)}(G_0)$.

Corollary 1 *Let π be a homomorphism G to $GL_5(2), \pi(\varphi) = \varphi_1$. Then*

$$\pi(\Psi(G_1) = GL_3(2) = \{\|a_{ij}\| \mid a_{ij} \in Z_2, a_{ij} = 0, if\ \ 4 \le i \ne j \le 5\}.$$

$K = \Psi(Aut(M_8)) \cap ker\pi$ is a 2-group of type $2^{4+3}.Z(K)$ has a basis $\{\sigma_1, \sigma_2, \sigma_3\}$ and K is generated by $\{\phi_1, \phi_2, \phi_3, \phi_4\}$, where $\phi_i = \Psi(\bar{\phi}_i)$. Here $\bar{\phi}_i$ is the unique automorphism of M_8 such that $\bar{\phi}_i(\sigma_j) = \sigma_j, \bar{\phi}_i(\sigma_i) = \bar{\sigma}_i, \bar{\phi}_i(I_8) = I_8, i \ne j = 1,2,3, \phi_4(\sigma_i) = \sigma_i, i = 1,2,3, \phi_4(I_8) = -I_8$. Moreover, $[\phi_i, \phi_4] = \sigma_i, i = 1,2,3$.

Corollary 2 *Let $Stab_{G_1}W$ be the stabilizer of the elementary 2-group W with the basis $\{\sigma_1, \sigma_2, \sigma_3\}$ in G_1. Then $Stab_{G_1}W = G_2 \times \langle I_8 \rangle, G_2 \simeq GL_3(2) \cdot W$.*

Lemma 2 *Let φ_1 and φ_2 be the reflections in the roots (σ, i) and $(\bar{\sigma}, p)$, where $\sigma \in \mathcal{E}_8, i \in \sigma, p \in \bar{\sigma}$. Then $\varphi = \varphi(\sigma, i, p) = \varphi_1\varphi_2 \in G$.*

Proof. We denote $\sigma = (ijln), \bar{\sigma} = (pqst), \mu = (ip), G_\sigma = \{\emptyset, \sigma, \bar{\sigma}, I_8\} \subseteq G_8$. Then every set $\tau \triangle G_\sigma$ contains an unique element $\tau \in \mathcal{E}_8$, such that $\mu \subseteq \tau$. Suppose that $\tau = (ijpq)$. Then $\bar{\tau} = (lnst), \lambda = \tau \triangle \sigma = (lnpq), \bar{\lambda} = \tau \triangle \bar{\sigma} = (ijst)$.

Our aim is to describe the action of the involution φ on the basis B_8 of L. If $x, y \in B_8$ then we wright $\langle x, y \rangle$ if $x^\varphi = y$ and $\langle \pm x \rangle$ if $x^\varphi = \pm x$. We denote by θ one of the sets σ or $\bar{\sigma}$. If we have in M_8 that $\sigma \star \tau = \varepsilon\lambda, \varepsilon \in \{\pm\}$, then from Theorem 1 we can obtain by straightforward calculations that

$$\langle e_m, (\theta, \theta) \rangle, \langle f_m, (\theta, \emptyset) \rangle, m \in \mu \cap \theta;$$

$$\langle e_m, -(\theta, (\theta \cap \mu) \cup m) \rangle, \langle f_m, -(\theta, \theta \cap (\mu \cup m) \rangle, m \in \theta \setminus \mu;$$

$$\langle (\theta, \rho) \rangle, |\rho \triangle \mu| = 3; \langle (\theta, \rho), (\theta, \theta \setminus \rho) \rangle, \rho \in \{(i), (p)\};$$

$$\langle (\tau, \rho) \rangle, \rho \in \{\emptyset, (ij), (pq), \tau\}; \langle (\bar{\tau}, \rho) \rangle, \rho \in \{(ls), (lt), (ns), (nt)\};$$

$$\langle -(\lambda, \rho) \rangle, \rho \in \{(lpq), (npq), (l), (n)\}; \langle -(\bar{\lambda}, \rho) \rangle, \rho \in \{(ijs), (ijt), (s), (t)\};$$

$$\langle (\tau, ijp), \varepsilon(\bar{\lambda}, \bar{\lambda}) \rangle; \langle (\tau, ijq), -\varepsilon(\bar{\lambda}, ij) \rangle; \langle (\tau, ip), -(\bar{\tau}, \bar{\tau}) \rangle;$$

$$\langle (\tau, iqp), \varepsilon(\lambda, \lambda) \rangle; \langle (\tau, jpq), -\varepsilon(\lambda, pq) \rangle; \langle (\tau, iq), -(\bar{\tau}, ln) \rangle;$$

$$\langle (\tau, jp), (\bar{\tau}, st) \rangle, \langle (\tau, i), \varepsilon(\lambda, ln) \rangle, \langle (\tau, q), -e(\bar{\lambda}, \emptyset) \rangle,$$

$$\langle (\tau, jq), -(\bar{\tau}, \emptyset) \rangle, \langle (\tau, j), -\varepsilon(\lambda, \emptyset) \rangle, \langle (\tau, p), e(\bar{\lambda}, st) \rangle,$$

$$\langle (\bar{\tau}, lns), \varepsilon(\bar{\lambda}, is) \rangle, \langle (\bar{\tau}, lnt), \varepsilon(\bar{\lambda}, it) \rangle, \langle (\bar{\tau}, lst), \varepsilon(\lambda, pl) \rangle,$$

$$\langle (\bar{\tau}, tns), \varepsilon(\lambda, pn) \rangle, \langle (\bar{\tau}, l), -\varepsilon(\lambda, lq) \rangle, \langle (\bar{\tau}, n), -\varepsilon(\lambda, nq) \rangle,$$

$$\langle (\bar{\tau}, t), -\varepsilon(\bar{\lambda}, jt) \rangle, \langle (\bar{\tau}, s), -\varepsilon(\bar{\lambda}, js) \rangle, \langle (\lambda, nlp), (\bar{\lambda}, ist) \rangle,$$

$$\langle (\lambda, lnq), -(\bar{\lambda}, i) \rangle, \langle (\lambda, p), (\bar{\lambda}, jst) \rangle, \langle (\lambda, q), (\bar{\lambda}, j) \rangle.$$

We point out that in order to verify that φ is an automorphism of L it is sufficient to check it on an invariant subalgebra D_φ of type D_8 with a basis $\{e_i, f_i, h_i, i = 1, ..., 8; (\alpha, \beta) \mid \beta \subseteq \varphi \in \{\sigma, \bar{\sigma}, \tau, \bar{\tau}, \lambda, \bar{\lambda}\}\}$. $\square$

Lemma 3 *In above notation we have:*

$$A = \pi\{\varphi(\sigma, i, p) \mid \sigma \in \{\sigma_1, \sigma_2, \sigma_3\}, i \in \sigma, p \in \bar{\sigma}\} =$$

$$\{e + e_{i4} + e_{4i} + e_{ii} + e_{44} \mid e = e_{11} + ... + e_{55}, i = 1, 2, 3;\}$$

and A *with* $\pi(Aut(M_8))$ *generate a subgroup* $Q \simeq GL_4(2)$ *of* $GL_5(2)$ *where* $Q = \{||a_{ij}|| \in GL_5(2) \mid a_{5i} = a_{i5} = 0, i = 1, ..., 4\}$.

Proof. We have by a straightforward calculation from Lemma 2 that $\varphi\sigma\varphi^{-1} = \varphi\sigma\varphi = r$, if $\varphi = \varphi(\sigma, i, p)$. For instance

$$e_i \xrightarrow{\varphi} (\sigma, \sigma) \xrightarrow{\sigma} -(\sigma, \sigma) \xrightarrow{\varphi} -e_i,$$

$$(\sigma, i) \,{}^{\varphi}\!\!\rightarrow (\sigma, jln) \xrightarrow{\sigma} -(\sigma, jln) \xrightarrow{\varphi} -(\sigma, i),$$

$$(\sigma, \sigma) \xrightarrow{\varphi} e_i \xrightarrow{\sigma} e_i \xrightarrow{\varphi} (\sigma, \sigma).$$

Analogously, $\varphi r\varphi = \sigma$, for instance

$$(\sigma, \sigma) \xrightarrow{\varphi} e_i \xrightarrow{r} -e_i \xrightarrow{\varphi} -(\sigma, \sigma),$$

$$(\sigma, i) \,{}^{\varphi}\!\!\rightarrow (\sigma, jln) \xrightarrow{r} -(\sigma, jln) \xrightarrow{\varphi} -(\sigma, i),$$

$$e_i \xrightarrow{\varphi} (\sigma, \sigma) \xrightarrow{r} (\sigma, \sigma) \xrightarrow{\varphi} e_i.$$

Moreover, $\varphi\tau\varphi = \tau$, if $\tau \in \{\sigma_1, \sigma_2, \sigma_3\} \setminus \{\sigma\}$ and $\varphi\kappa\varphi = \kappa$. □

Lemma 4 *Let* ω *be a linear map from* L *onto* L *such that*
$h^\omega = h, \forall h \in H, e_j^\omega = ie_j, f_J^\omega = -if_j, j \in I_8; (\sigma, \mu)^\omega = i^{2+2\varepsilon(\sigma)+|\mu|}(\sigma, \mu)$, *where* $i = \sqrt{-1}$ *and* $\varepsilon(\sigma) \in \{0, 1\}$ *and the map* $\sigma \to (-1)^{\varepsilon(\sigma)}$ *is a homomorphism from* G_8 *to* $Z_2 = \{0, 1\}$ *such that* $\varepsilon(\sigma_j) = 1, j = 1, 2, 3; \quad \varepsilon(I_8) = 0$. *Then* $\omega \in G, \omega^4 = 1$ *and* $\pi(\omega) = e + e_{54}$.

Proof. Let $(\sigma, \varphi), (\tau, \xi)$ are the basis elements of L. If $[(\sigma, \varphi), (\tau, \xi)] = 0$ then (1) leads to $\varphi \cap \xi \neq \emptyset$ or $\sigma \cup \tau \not\subseteq \varphi \cup \xi$. Hence, by definition, $[(\sigma, \varphi)^\omega, (\tau, \xi)^\omega] = 0$. If $[(\sigma, \varphi), (\tau, \xi)] = (-1)^{|\sigma\cap\xi|}(\alpha, \beta) \neq 0$ then $\alpha = \sigma \star \tau, \beta = (\varphi \setminus \tau) \cup (\xi \cap \sigma)$ and $\varphi \cap \xi = \emptyset, \sigma \cap \tau \subseteq \varphi \cup \xi$. Therefore

$$[(\sigma, \varphi)^\omega, (\tau, \xi)^\omega] = i^{2+2\varepsilon(\sigma)+|\varphi|+2+2\varepsilon(\tau)+|\xi|}(-1)^{|\sigma\cap\xi|}(\alpha, \beta) =$$

$$i^{2\varepsilon(\sigma\triangle\tau)+|\varphi\cup\xi|}(-1)^{|\sigma\cap\xi|}(\alpha, \beta) =$$

$$i^{2\varepsilon(\sigma\triangle\tau)+|(\varphi\setminus\tau)\cup(\sigma\cap\tau)\cup(\xi\setminus\sigma)|}(-1)^{|\sigma\cap\xi|}(\alpha, \beta) =$$

$$i^{2+2\varepsilon(\sigma\triangle\tau)+|(\varphi\setminus\tau)\cup(\xi\setminus\sigma)|}(-1)^{|\sigma\cap\xi|}(\alpha, \beta) =$$

$$(-1)^{|\sigma\cap\xi|}(\alpha, \beta)^\omega,$$

as $|\sigma \cap \tau| = 2$.

We point out that since $\sigma_1, \sigma_2, \sigma_3, r \in C(H) = \{\phi \in Aut_k(E_8) \mid h^\phi = h, \forall h \in H\}$ and $[C(H), C(H)] = 1$ then $\omega\sigma_j\omega^{-1} = \sigma_j, j = 1,2,3;\ \ \omega r \omega^{-1} = r$. Simultaneously $\kappa x \kappa = x^{-1}, \forall x \in C(H)$. Then $\kappa\omega\kappa = \omega^{-1} = \omega^3$ and $\omega\kappa\omega^{-1} = \kappa\omega^2$. But $\omega^2 = r$. Lemma is proved. □

We note that all automorphisms in G, which we had constructed so far belong to $N(H)$, the normalizer of H. Now we define an automorphism in G which does not belong to $N(H)$.

Lemma 5 *Let $\eta \in End_k(L)$ and*

$$e_i^\eta = (e_i + f_i - h_i)/2, f_i^\eta = (e_i + f_i + h_i)/2, h_i^\eta = (e_i - f_i), i \in I_8;$$

$$(\sigma, \tau)^\eta = \frac{(-1)^{|\tau|}}{4} \sum_{\mu \subseteq \sigma} (-1)^{|\mu| + |\mu \cap \bar{\tau}|} (\sigma, \mu), \bar{\tau} = \sigma \setminus \tau, \sigma \in \mathcal{E}_8.$$

Then $\eta \in G, \eta^2 = \kappa r$ and $\pi(\eta) = e + e_{45} + e_{54} + e_{44} + e_{55}$.

Proof. First we check that $\eta \in Aut_k(L)$. Let $(\sigma, \tau), (\varphi, \psi)$ belong to B_8 and $\alpha = \sigma \star \varphi$. Then

$[(\sigma, \tau)^\eta, (\varphi, \psi)^\eta] = R =$

$\frac{(-1)^{|\tau|+|\psi|}}{16} (\sum_{\mu \subseteq \sigma} (-1)^{|\mu| + |\mu \cap \bar{\tau}|} (\sigma, \mu)) (\sum_{\lambda \subseteq \varphi} (-1)^{|\lambda| + |\lambda \cap \bar{\psi}|} (\varphi, \lambda)) =$

$(-1)^{|\tau \cup \psi|}/16 \sum_{(\mu,\lambda) \in A} (-1)^{|\mu| + |\mu \cap \bar{\tau}| + |\lambda| + |\lambda \cap \bar{\psi}| + |\sigma \cap \lambda|} (\alpha, (\mu \setminus \varphi) \cup (\lambda \setminus \sigma)),$

where $A = \{(\mu, \lambda) \mid \mu \subseteq \sigma, \lambda \subseteq \varphi, \mu \cap \lambda = \emptyset, \sigma \cap \varphi \subseteq \mu \cup \lambda\}$. Denote $A_\xi = \{(\mu, \lambda) | \xi = (\lambda \setminus \sigma) \cup (\mu \setminus \varphi), \}$ and prove

$$\sum_{(\mu,\lambda) \in A_\xi} (-1)^{|\mu| + |\mu \cap \bar{\tau}| + |\lambda| + |\lambda \cap \bar{\psi}| + |\sigma \cap \lambda|} = \begin{cases} 4(-1)^{|\xi| + |\xi \cap \overline{(\tau \setminus \varphi) \cup (\psi \setminus \sigma)}| + |\sigma \cap \psi|}, \\ \tau \cap \psi = \emptyset, \sigma \cap \varphi \subseteq \tau \cup \psi; \\ 0, \tau \cap \psi \neq \emptyset \ \text{ or } \ \sigma \cap \varphi \nsubseteq \tau \cup \psi. \end{cases} \tag{23}$$

Let us fix $(\mu,\lambda)\in A_\xi$ and consider the following partition of $\sigma\cup\tau$:

$$Q_1=(\tau\cap\mu)\setminus\varphi|,\quad Q_2=|(\bar\tau\cap\mu)\setminus\varphi\quad Q_3=\tau\setminus(\mu)\cup\varphi),$$

$$Q_4=\bar\tau\setminus(\mu)\cup\varphi),\quad Q_5=(\psi\cap\lambda)\setminus\sigma,\quad Q_6=(\bar\psi\cap\lambda)\setminus\sigma,$$

$$Q_7=\psi\setminus(\lambda\cup\sigma),\quad Q_8=\bar\psi\setminus(\lambda\cup\sigma),\quad Q_9=\tau\cap\lambda\cap\psi,$$

$$Q_{10}=\tau\cap\lambda\cap\bar\psi,\quad Q_{11}=\tau\cap\mu\cap\psi,\quad Q_{12}=\tau\cap\mu\cap\bar\psi,$$

$$Q_{13}=\bar\tau\cap\lambda\cap\psi,\quad Q_{14}=\bar\tau\cap\lambda\cap\bar\psi,\quad Q_{15}=\bar\tau\cap\mu\cap\psi,$$

$$Q_{16}=\bar\tau\cap\mu\cap\bar\psi.$$

We denote $P_i=|Q_i|$, $i=1,...16$ then

$$\begin{aligned}&|\mu|+|\mu\cap\bar\tau|+|\lambda|+|\lambda\cap\bar\psi|+|\sigma\cap\lambda|=\\&P_1+P_4+P_{11}+P_{12}+P_{14}+\\&P_{16}+P_4+P_{14}+P_{16}+P_5+P_6+P_9+P_{10}+P_{13}+P_{14}+\\&P_6+P_{10}+P_{14}+P_9+P_{10}+P_{13}+P_{14},\end{aligned}\tag{24}$$

$$\begin{aligned}&|\xi|+|\xi\cap\overline{(\tau\setminus\varphi)\cup(\psi\setminus\sigma)})|+|\sigma\cap\psi|=\\&P_1+P_4+P_5+P_6+P_4+P_6+P_9+P_{11}+P_{13}+P_{15}.\end{aligned}\tag{25}$$

If $\tau\cap\psi=\emptyset,\sigma\cap\varphi\subseteq\tau\cup\psi$ then $P_9=P_{11}=P_{14}=P_{16}=0$ and from (24) and (25) we have:

$$|\mu|+|\mu\cap\bar\tau|+|\lambda|+|\lambda\cap\bar\psi|+|\sigma\cap\lambda|\equiv P_1+P_5+P_{10}+P_{12}(mod2),$$

$$|\xi|+|\xi\cap\overline{(\tau\setminus\varphi)\cup(\psi\setminus\sigma)})|+|\sigma\cap\psi|\equiv P_1+P_5+P_{13}+P_{15}.$$

But $P_{10}+P_{12}+P_{13}+P_{15}=|\sigma\cap\varphi|=2$ and hence

$$|\mu|+|\mu\cap\bar\tau|+|\lambda|+|\lambda\cap\bar\psi|+|\sigma\cap\lambda|\equiv|\xi|+|\xi\cap\overline{(\tau\setminus\varphi)\cup(\psi\setminus\sigma)})|+|\sigma\cap\psi|,$$

which proves the first part of (23).

Let us suppose that $\tau\cap\psi\neq\emptyset$ or $\sigma\cap\varphi\not\subseteq\tau\cup\psi$. Then from (24) we obtain

$$|\mu|+|\mu\cap\bar\tau|+|\lambda|+|\lambda\cap\bar\psi|+|\sigma\cap\lambda|\equiv P_1+P_5+P_{10}+P_{11}+P_{12}+P_{14},$$

which imply

$$\sum_{(\mu,\lambda)\in A_\xi}(-1)^{|\mu|+|\mu\cap\bar{\tau}|+|\lambda|+|\lambda\cap\bar{\psi}|+|\sigma\cap\lambda|} =$$
$$(-1)^{P_1+P_5}\sum_{(\mu,\lambda)\in A_\xi}(-1)^{P_{11}+P_{10}+P_{12}+P_{14}}. \tag{26}$$

If $\sigma\cap\varphi=\{i,j\}$ then we have the following 4 possibilities:

1. $\tau\cap\psi=\emptyset, \bar{\tau}\cap\bar{\psi}=\{i\}, \tau\cap\varphi=\{j\}$; 2. $\tau\cap\psi=\emptyset, \bar{\tau}\cap\bar{\psi}=\{i,j\}$;

3. $\tau\cap\psi=\{j\}, \bar{\tau}\cap\bar{\psi}=\{i\}$; 4. $\tau\cap=\emptyset, \bar{\tau}\cap\bar{\psi}=\{i\}$.

It is easy to prove the second part of (23) for all of these cases.

From (23) we have $R=0$, if $\tau\cap\psi\neq\emptyset$ or $s\cap\varphi\not\subseteq\tau\cup\psi$, and

$$R=\frac{(-1)^{|\tau\setminus\varphi|+|\psi\setminus\sigma|+|\tau\cap\sigma|+|\sigma\cap\psi|}}{4}\sum_{\xi\subseteq\sigma\nabla\varphi}-1)^{|\xi|+|\xi\cap\overline{(\tau\setminus\varphi)\cup(\psi\setminus\sigma)}|}(\alpha,\xi),$$

if $\tau\cap\psi=\emptyset, \sigma\cap\varphi\subseteq\tau\cup\psi$.

On the other hand

$$(-1)^{|\sigma\cap\psi|}(\alpha,(\tau\setminus\varphi)\cup(\psi\setminus\sigma))^\eta =$$
$$\frac{(-1)^{|\tau\setminus\varphi|+|\psi\setminus\sigma|+|\sigma\cap\psi|}}{4}\sum_{\xi\subseteq\sigma\nabla\varphi}(-1)^{|\xi|+|\xi\cap\overline{(\tau\setminus\varphi)\cup(\psi\setminus\sigma)}|}(\alpha,\xi)=R.$$

It is easy to show that $e_i^{\eta^2}=f_i, f_i^{\eta^2}=e_i, h_i^{\eta^2}=-h_i, i=1,...,8$. Moreover,

$$(\sigma,\tau)^{\eta^2}=(-1)^{|\tau|}/4\sum_{\mu\subseteq\sigma}(-1)^{|\mu|+|\mu\cap\bar{\tau}|}(\sigma,\tau)^\eta =$$
$$(-1)^{|\tau|}/16\sum_{\lambda\subseteq\sigma}(-1)^\lambda\sum_{\mu\subseteq\sigma}(-1)^{2|\mu|+|\mu\cap\bar{\tau}|+|\lambda\cap\bar{\mu}|}(\sigma,\lambda) = \tag{27}$$
$$(-1)^{|\tau|}/16\sum_{\lambda\subseteq\sigma}(-1)^\lambda\sum_{\mu\subseteq\sigma}(-1)^{|\mu\cap\bar{\tau}|+|\lambda\cap\bar{\mu}|}(\sigma,\lambda).$$

If $\lambda=\bar{\tau}$ then

$$\sum_{\mu\subseteq\sigma}(-1)^{|\mu\cap\bar{\tau}|+|\bar{\tau}\cap\bar{\mu}|}=\sum_{\mu\subseteq\sigma}(-1)^{|\bar{\tau}|}=16(-1)^{|\bar{\tau}|}. \tag{28}$$

Let us suppose that $\lambda\not\subseteq\bar{\tau}$ and choose $i\in\lambda\setminus\bar{\tau}$. Then

$$
\begin{aligned}
&\sum_{\mu\subseteq\sigma}(-1)^{|\mu\cap\bar\tau|+|\lambda\cap\bar\mu|} =\\
&\sum_{i\in\mu\subseteq\sigma}(-1)^{|\mu\cap\bar\tau|+|\lambda\cap\bar\mu|}+\sum_{i\notin\mu\subseteq\sigma}(-1)^{|\mu\cap\bar\tau|+|\lambda\cap\bar\mu|} =\\
&\sum_{i\in\mu\subseteq\sigma}(-1)^{|\mu\cap\bar\tau|+|\lambda\cap\bar\mu|}+\sum_{i\notin\mu\subseteq\sigma}(-1)^{|(\mu\cup i)\cap\bar\tau|+|\lambda\cap\bar{\mu\cup i}|+1} =\\
&\sum_{i\in\mu\subseteq\sigma}[(-1)^{|\mu\cap\bar\tau|+|\lambda\cap\bar\mu|}+(-1)^{|\mu\cap\bar\tau|+|\lambda\cap\bar\mu|+1}] = 0.
\end{aligned} \tag{29}
$$

From (27),(28) and (29) we have

$$
\begin{aligned}
&(\sigma,\tau)^{\eta^2} = (-1)^{|\tau|+2|\bar\tau|}(\sigma,\bar\tau) = (-1)^{|\tau|}(\sigma,\bar\tau),\\
&\eta^2 = \kappa r, \quad \eta^3 = \eta^{-1}.
\end{aligned}
$$

It is obvious that $\sigma\eta = \eta\sigma, \sigma \in \mathcal{E}_7$. We prove that $\eta\kappa\eta^{-1} = r$ and $\eta r\eta^{-1} = \kappa$. Indeed $\eta\kappa\eta^{-1} = \eta\kappa\eta^3 = \eta\kappa\kappa r\eta = \eta r\eta$, since

$$
\begin{aligned}
&e_i \xrightarrow{\eta} (e_i + f_i - h_i)/2 \xrightarrow{r} (-e_i - f_i - h_i)/2 \xrightarrow{\eta} -e_i = e_i^r,\\
&h_i \xrightarrow{\eta} e_i - f_i \xrightarrow{r} (-e_i + f_i \xrightarrow{\eta} h_i = h_i^r,\\
&(\sigma,\tau) \xrightarrow{\eta} (-1)^{|\tau|}/4\sum_{\mu\subseteq\sigma}(-1)^{|\mu|+|\bar\tau\cap\mu|}(\sigma,\mu) \xrightarrow{r} (-1)^{|\tau|}/4\sum_{\mu\subseteq\sigma}(-1)^{|\bar\tau\cap\mu|}(\sigma,\mu)\\
&\xrightarrow{\eta} (-1)^{|\tau|}/16\sum_{\lambda\subseteq\sigma}(-1)^{\lambda}\sum_{\mu\subseteq\sigma}(-1)^{|\mu|+|\bar\tau\cap\mu|+|\bar\mu\cap\lambda|}(\sigma,\lambda).
\end{aligned}
$$

By analogy with (28) and (29) we can prove that

$$
\sum_{\mu\subseteq\sigma}(-1)^{|\mu|+|\bar\tau\cap\mu|+|\bar\mu\cap\tau|} = \sum_{\mu\subseteq\sigma}(-1)^{|\tau\cap\mu|+|\bar\mu\cap\tau|} = 16(-1)^{|\tau|}
$$

and if $\lambda \neq \tau$ then

$$
\sum_{\mu\subseteq\sigma}(-1)^{|\mu|+|\bar\tau\cap\mu|+|\bar\mu\cap\lambda|} = 0.
$$

Hence $(\sigma,\tau)^\eta = (-1)^{|\tau|}(\sigma,\tau)$ and $\eta r\eta = r$.

Analogously we can prove that $\eta r\eta^{-1} = \eta\kappa\eta = \kappa$. Hence $\pi(\eta) = e + e_{45} + e_{54} + e_{44} + e_{55}$. Lemma is proved. □

From Lemmas 1-5 we have that $\pi(Aut_k MCD) = GL_5(2)$. Now we have to find $N = ker\pi$. Let $\sigma \in \mathcal{E}_8, p \in \sigma$ and t_σ^p, h_σ are the following linear maps $L \to L$

$$
h \xrightarrow{t_\sigma^p} h, \forall h \in H, e_i \xrightarrow{t_\sigma^p} (-1)^{|\sigma\cap i|}e_i, f \xrightarrow{t_\sigma^p} (-1)^{|\sigma\cap i|}f_i, i \in I_8,
$$

$$(\psi,\mu)\overset{t^p_\sigma}{\to}\begin{cases}(-1)^{|\sigma\cap\mu|}(\psi,\mu),\psi\in\{\sigma,\bar\sigma\}\quad or\quad \psi\notin\{\sigma,\bar\sigma\},p\in\psi,\\ \\ (-1)^{|\sigma\cap\mu|+1}(\psi,\mu),\psi\notin\{\sigma,\bar\sigma\},p\notin\psi.\end{cases}\tag{30}$$

$$h_i\overset{h_\sigma}{\to}(-1)^{|\sigma\cap i|}h_i,\forall i\in I_8,$$

$$e_i\overset{h_\sigma}{\to}e_i,i\notin\sigma,$$

$$e_i\overset{h_\sigma}{\to}-f_i\overset{h_\sigma}{\to}e_i,i\in\sigma,$$

$$(\psi,\mu)\overset{h_\sigma}{\to}\begin{cases}(\psi,(\mu\triangle\sigma)\cap\psi),\psi\in\{\sigma,\bar\sigma\}\\ \\ -(\psi,(\mu\triangle\sigma)\cap\psi),\psi\notin\{\sigma,\bar\sigma\}.\end{cases}\tag{31}$$

Lemma 6 *Let T_2 be the maximal elementary 2-group from the Cartan torus $T=\{\varphi\in Aut_kL|h^\varphi=h,\forall h\in H\}$. Then T_2 has the following basis $\{r,\sigma_1,\sigma_2,\sigma_3,t^1_{\sigma_1},t^1_{\sigma_2},t^1_{\sigma_3},t^5_{\bar\sigma_1}\}\subseteq G$.*

Proof. The only property we have to prove is that t^p_σ is an automorphism for $p\in\sigma\in\mathcal{E}_8$. Let $\mu\subseteq\psi\in\mathcal{E}_8,\varphi\subseteq\tau\in\mathcal{E}_8$ and $\mu\cap\varphi=\emptyset,\psi\cap\tau\subseteq\mu\cup\varphi$ then from (1) we obtain that

$$\{(\psi,\mu)(\tau,\varphi)\}^{t^p_\sigma}=(-1)^{|\psi\cap\varphi|}(\psi\star\tau,\mu\setminus\tau\cup\varphi\setminus\psi)^{t^p_\sigma}=$$

$$\varepsilon(-1)^{|\psi\cap\varphi|+|\sigma\cap(\mu\setminus\tau\cup\varphi\setminus\psi)|}(\psi\star\tau,\mu\setminus\tau\cup\varphi\setminus\psi),\tag{32}$$

$$(\psi,\mu)^{t^p_\sigma}(\tau,\varphi)^{t^p_\sigma}=\varepsilon_1\varepsilon_2(-1)^{|\psi\cap\varphi|+|\sigma\cap\mu|+|\sigma\cap\varphi|}(\psi\star\tau,\mu\setminus\tau\cup\varphi\setminus\psi),$$

where $\varepsilon,\varepsilon_1,\varepsilon_2\in\{\pm1\}$. But

$$\begin{array}{l}|\sigma\cap(\mu\setminus\tau\cup\varphi\setminus\psi)|\equiv|\sigma\cap(\mu\setminus\tau)|+|\sigma\cap(\varphi\setminus\psi)|\equiv\\ |\sigma\cap\mu|+|\sigma\cap\mu\cap\tau|+|\sigma\cap\psi|+|\sigma\cap\psi\cap\varphi|\equiv\\ |\sigma\cap\mu|+|\sigma\cap\psi|+|\sigma\cap\psi\cap\tau|.\end{array}\tag{33}$$

Suppose that $\{\psi,\tau,\psi\triangle\tau\}\cap\{\sigma,\bar\sigma\}=\emptyset$ and $p\in\psi\cap\tau$. Then $\varepsilon=-1,\varepsilon_1=\varepsilon_2=1$ and $|\sigma\cap\psi\cap\tau|=|p|=1$. Hence from (32) and (33) we have

$$\{(\psi,\mu)(\tau,\varphi)\}^{t^p_\sigma}=(\psi,\mu)^{t^p_\sigma}(\tau,\varphi)^{t^p_\sigma}.$$

The other cases are considered analogously.

□

Lemma 7 *$ker\pi$ is a 2-group of type 2^{10+5} with the generators*

$$\{h_{\sigma_1}, h_{\sigma_2}, h_{\sigma_3}, \phi_1, \phi_2, \phi_3, \phi_4, t^1_{\sigma_1}, t^1_{\sigma_2}, t^1_{\sigma_3}, \},$$

where $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ was defined in the Corollary 1 and $\{h_{\sigma_1}, h_{\sigma_2}, h_{\sigma_3}, t^1_{\sigma_1}, t^1_{\sigma_2}, t^1_{\sigma_3}, \}$ by (30),(31), $Z(ker\pi) = G_0$.

Proof. Let φ belong to $ker\pi$. If $h_i^{\varphi} \neq \pm h_i$ for some $i \in I_8$, then φ induces some nontrivial automorphism of the Moufang Loop M_8. In this case we can find $\phi \in \Psi(AutM_8) \cap ker\pi$ such that $h_i^{\varphi\phi} = \pm h_i, \forall i \in I_8$. If $h_i^{\varphi} = \pm h_i, \forall i \in I_8$ then $\sigma = \{i \mid h_i^{\phi} = -h_i\} \in \mathcal{E}_8$. Indeed, if $\sigma \notin \mathcal{E}_8$ then there exists $\xi \in \mathcal{E}_8$ such that $|\xi \cap \sigma| \equiv 1(mod 2)$. Moreover, it is obvious that $(\xi, \emptyset)^{\varphi} \in k(\xi, \xi \cap \sigma)$, hence $(A_{\xi}^{+})^{\varphi} \neq A_{\xi}^{+}$ and $\varphi \notin ker\pi$. Thus $\sigma \in \mathcal{E}_8$ and $\psi = \varphi h_{\sigma} \in T_2 \subseteq ker\pi$. Lemma is proved.
□

Now we formulate the main result of this paper:

Theorem 2 *The automorphisms constructed in Lemmas 1-6 generate the group $G = Aut_k MCD$.*

One can prove that the automorphisms $\pi(G_2)$ constructed in Corollary 2, $\varphi(\sigma_1, 1, 5)$ (from Lemma 2) and automorphisms constructed in Lemmas 3-5 generate the Dempwolff group, see [4],[1].

References

[1] U. Dempwolff, On extensions of an elementary abelian group of order 32 by GL(5,2), Rend. Sem.Mat. Univ. Padova, 48(1972), 359-364.

[2] A.N. Grishkov, On the conjugacy of Levi factors in finite dimensional Binary-Lie Algebras. Izvestiya Mathematics,v.28(1987),n.2,p.305-333.

[3] A.N. Grishkov, One category of Lie algebras with automorphisms and their applications to Malcev algebras and Moufang loops, submitted.

[4] P.E. Smith, On certain finite simple groups, Ph.D. Thesis, University of Cambridge, 1975.

[5] P.E. Smith, A simple subgroup of M? and E8(3). The Bull. of the London Math.Soc., v.8(1976),n.23,161-165.

[6] J.G. Thompson, A conjugacy Theorem for E_8. J.of Algebra,38,n.2,525-530 (1976).

5 ABSTRACT

We construct a new basis for exceptional simple Lie algebra L of type E_8 and describe the multiplication rule in this basis. It allows to find an action of generators of automorphism group of multiplicative Cartan decomposition of L on this basis.