Three aspects of the exponential map

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1 Introduction.

It is well known that the classical exponential map: $exp(x) = \sum x^i/i!$ has many different applications in the various branches of mathematics. For instance, if k is a locally compact field of characteristic 0 then $exp : k \longrightarrow k^{\star}$ is a homomorphism from the additive group of the field k in the multiplicative group. If A is a locally nilpotent linear operator on a vector space V then $\{exp(tA) \| t \in k\}$ is 1-parametric group. Here and above $k^{\star} = k \setminus \{0\}$. In this paper we will give some characterizations and generalizations of the exponential map and their applications for the theory of groups, algebras and loops.

2 Analytic aspect of the exponential map.

Consider the followings categories $\mathbf{G} \subset \mathbf{M} \subset \mathbf{D}$, where $\mathbf{G}$ is the category of local Lie Groups, $\mathbf{M}$ the catagory of local analytic Moufang loops and $\mathbf{D}$ the catagory of local analytic diassociative loops. Recall that a loop is diassociative (Moufang) if every two elements generate a subgroup (this loop satisfies the identity $(xy \cdot z)x = x(y \cdot zx)$). In his work ([21]) Malcev proved that the tangent space for a loop from $\mathbf{D}$ has the structure of a Binary-Lie algebra (BL-algebra) and if this loop is Moufang then the corresponding BL-algebra is a Malcev algebra. Recall that an algebra is a BL-algebra (Malcev algebra) if it is anticommutative and every two elements genarate a Lie subalgebra (it satisfies the identity $(xy \cdot z)x + (yz \cdot x)x + (zx \cdot x)y = xz \cdot yx$). Hence for the categories $\mathbf{M}$ and $\mathbf{D}$ we have an analogy of the classical correspondence: Lie Groups$\longrightarrow$ Lie Algebras. In the same article ([21]) Malcev noted that for every BL-algebra B over a real field $\mathbf{R}$ we can construct the corresponding local analytic diassociative loop D, because the classical Campbell-Hausdorf series $H(x, y) = x + y + [x, y]/2 + ...$ depends only on two variables x, y and those two elemens generate Lie subalgebra. In this case we have the classical (local) exponential map $exp : B \longrightarrow D, exp(x) = x$. If the algebra B is a Malcev algebra (note that every Malcev algebra is a BL-algebra but the converse is not true) then the corresponding loop D is a local analytic Moufang loop. This is an important Theorem of Kuzmin ([20]). Therefore the theory of local analytic Moufang (diassociative) loops is equivalent to the theory of finite dimensional Malcev (BL-) algebras. The theory of finite dimensional Malcev and BL-algebras was developed in ([22],[18],[19],[4],[5], [6],[7],[8]).

Now we construct a global analytic Moufang (diassociative) loop for a given local analitic loop D. For Lie groups it is the famous Cartan Theorem. For Moufang loops this problem was solved by Kerdman in ([17]). But for diassociative loops the situation is more difficult. In ([9]) the author constructed a local analytic diassociative loop such the corresponding global analytic loop does not exist. Let B be a finite dimensional BL-algebra over $\mathbf{R}$. Denote by $B(a,b)$ a Lie subalgebra of B with two generators $a,b \in B$ and by $G(a,b)$ the corresponding simply connected Lie group. Then we have the exponential map: $exp : B(a,b) \longrightarrow G(a,b)$. We define a binary relation $\sim$ on B such that $a \sim b, a,b \in B$ iff $exp(a) = exp(b)$. It is clear that if for a local analytic diassociative loop D corresponding to the BL-algebra B a global analytic loop exists then the relation $\sim$ is an equivalence.

Conjecture 1 *For a given finite dimensional real BL-algebra the corresponding global analytic diassociative loop exists iff the relation $\sim$ on this BL-algebra is an equivalence.*

The first step in the direction of the proof of this conjecture was done in ([14]):

Theorem 1 *Let B be a finite dimensional real BL-algebra such that the relation $\sim$ on B is equality. Then the corresponding global analytic diassociative loop exists.*

The idea of the proof of this theorem is the following. Let B be a BL-algebra over $\mathbf{R}$ with a basis $\{b_1, ..., b_n\}$ and $K = \mathbf{R}[x_1, x_2, ..., y_1, y_2, ...]$ is the ring of polynomiais. Then $B \otimes_{\mathbf{R}} K$ is a BL-algebra over K. Denote by B_0 the Lie subalgebra of $B \otimes_{\mathbf{R}} K$ with two generators $X = \sum_{i=1}^{n} x_i b_i, Y = \sum_{i=1}^{n} y_i b_i$. Suppose that we have a faithful matrix representation π of B_0 over K: $\pi : B_0 \longrightarrow \mathcal{T}_m(K)$ where $\mathcal{T}_m(K)$ is the set of triangular matricies over K. Consider the matrix

$$Z = exp^{-1}(exp(\pi(X))exp(\pi(Y))).$$

It is clear that $Z \in M_m(\overline{K})$, where $\overline{K} = \mathbf{R}[[x_1, ...]]$ is the ring of power series. Suppose that

$$Z = \sum_{i=1}^{l} f_i \pi(Z_i),$$

where $Z_i = p_{ij} b_j \in B \otimes_{\mathbf{R}} K, p_{ij} \in K$ and $f_i \in \overline{K}$. If the series $f_1, ..., f_l$ have an analytic extension then we can define a multiplication on $\mathbf{R}^n$ by:

$$(x_1, ..., x_n) \cdot (y_1, ..., y_n) = (z_1, ..., z_n),$$

where $z_i = \sum_{j=1}^{l} f_j(x_1, ..., y_1, ...)p_{ji}(x_1, ..., y_1, ...)$.

As all finite dimensional semisimple BL-algebras over $\mathbf{R}$ are Malcev algebras we can suppose (at first) that the BL-algebra B is solvable. In ([13],[14]) we proved

Theorem 2 *Let L be a completly solvable Lie algebra over K such that L is free as K-module. Then L has a faithful triangulable representation over K.*

For the proof of the Conjecture 1 the following is usefull:

Conjecture 2 *If $X, Y \in M_m(K)$ are triangular matricies then there exists a matrix $Z \in M_m(\overline{K})$ such that $exp(Z) = exp(X)exp(Y), Z = \sum_{i=1}^{l} f_i Z_i, Z_i \in Lie\{X, Y\}$, the Lie algebra over $\mathbf{R}$ with generators X, Y, and $f_1, ..., f_l$ have analytic extension.*

In order to establish Theorem 1 we proved in ([14]) a weak version of the Conjecture 2.

Now we illustrate the above construction:

Example 1 *Let B be Bl-algebra with a basis $\{t, a, b, c\}$ and the multiplication*

$$at = bt = ac = bc = 0, ab = c, ct = c.$$

Denote as above $X = x_1 t + x_2 a + x + x_3 b + x_4 c, Y = y_1 t + y_2 a + y_3 b + y_4 c$, then $[X, Y] = Z = f \cdot c$, where $f = (x_2 y_3 - x_3 y_2 + x_4 y_1 - x_1 y_4)$, $[Z, X] = x_1 Z$. We have

$$B_0 = KX \oplus KY \oplus KZ.$$

Define a representation π of B_0 by the following formulas:

$$\pi(X) = \begin{pmatrix} -x_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\pi(Y) = \begin{pmatrix} 0 & 1 \\ 0 & y_1 \end{pmatrix},$$

$$\pi(Z) = \begin{pmatrix} 0 & -x_1 \\ 0 & 0 \end{pmatrix}.$$

It is clear that

$$exp(\pi(X))exp(\pi(Y)) = exp(C) = \begin{pmatrix} e^{-x_1} & e^{-x_1}(e^{y_1}-1)/y_1 \\ 0 & e^{y_1} \end{pmatrix}.$$

Hence

$$C = \pi(X) + \pi(Y) + \frac{1-\tau}{x_1}\pi(Z) = \begin{pmatrix} -x_1 & \tau \\ 0 & y_1 \end{pmatrix},$$

where $\tau = \frac{(x_1+y_1)(e^{y_1}-1)}{y_1(e^{y_1}-e^{-x_1})}$. *It is not difficult to prove that the space* $B = \mathbf{R}^4$ *with multiplication*

$$X \cdot Y = (x_1 + y_1)t + (x_2 + y_2)a + (x_3 + y_3) + (x_4 + y_4 + \frac{1-\tau}{x_1} f)c$$

is a diassociative analytic loop which corresponds to the BL-algebra B.

3 Algebraic aspect of the exponential map.

In this section we will introduce an algebraic analogy of the exponential map from a Lie algebra into a Lie group.

Definition 1 *Let* k *be a field of characteristic 0,* G *be an algebraic* k*-group and* $L = L(G)$ *be the corresponding Lie algebra.*

An algebraic map (or rational map) $E : L \longrightarrow G$ *is called* **exponential algebraic map** *or* **EA-map** *if*

1.$E(kx)$ *is an additive 1-parametric subgroup of* G *for every* $x \in N(L)$ *where* $N(L)$ *is the nilradical of* G.

2.$E(k^\star t)$ *is a multiplicative 1-parametric subgroup of* G *if* $x \in L$ *is semisimple.*

The main problem about EA-maps is the existence of an EA-map for a given algebraic group G.

Theorem 3 *[11] Let* $T_n(k)$ *be the group of triangular nondegenerate matricies over* k*. Then for every closed algebraic subgroup of the rang one of* $T_n(k)$ *there exists an* EA*-map.*

Example 2 *[11] Let* $G = G(n,m), n, m \in \mathbf{Z}^\star$ *be the following algebraic group:*

$$G = \left\{ \left(\begin{array}{ccc} 1 & a & b \\ 0 & x^n & c \\ 0 & 0 & x^m \end{array} \right) \middle| \begin{array}{c} x \in k^\star, \\ a, b, c \in k. \end{array} \right\}$$

Then the corresponding Lie algebra has the form:

$$L = \left\{ \left(\begin{array}{ccc} 0 & p & r \\ 0 & nz & q \\ 0 & 0 & mz \end{array} \right) \middle| \; p, q, r, z \in k. \right\}$$

In this case the EA*-map may be defined by the formula:*

$$E \left(\begin{array}{ccc} 0 & p & r \\ 0 & nz & q \\ 0 & 0 & mz \end{array} \right) = \left(\begin{array}{ccc} 1 & p[\frac{(1+z)^n-1}{nz}] & \tau \\ 0 & (1+z)^n & \eta \\ 0 & 0 & (1+z)^m \end{array} \right),$$

where

$$\eta = q[\frac{(1+z)^n - (1+z)^m}{(n-m)z}],$$

$$\tau = r[\frac{(1+z)^m - 1}{mz}] + pq[\frac{(m-n) - m(1+z)^n + n(1+z)^m}{mn(m-n)z^2}].$$

As in the analytic case, we can apply the EA-map for construction of diassociative (in this case algebraic) loops.

Example 3 *(Grishkov ([11])) Let $B = B(n,m)$ be a BL-algebra over k with a basis $\{t, a, b, c\}$ and multiplication:*

$$at = na, bt = nb, ct = mc, ab = c, ac = bc = 0, n, m \in k.$$

If $m, n \neq 0, m, n \in \mathbf{Z}, n \neq m$ then the following local algebraic loop is diassociative and corresponds to B:

$$G(n,m) = \{(x_1, x_2, x_3, x_4) | x_1 \neq -1, x_i \in k, i = 1, ..., 4\}.$$

$$(x_1, ..., x_4) \cdot (y_1, ..., y_4) = (\alpha, \beta, \gamma, \tau),$$

where

$$\alpha = x_1 + y_1 + x_1 y_1,$$

$$\beta = \frac{\alpha y_2}{y_1} + \frac{\alpha[(1+x_1)^n - 1](x_2 y_1 - x_1 y_2)(1+y_1)^n}{x_1 y_1 [(1+x_1)^n (1+y_1)^n - 1},$$

$$\gamma = \frac{\alpha y_3}{y_1} + \frac{\alpha[(1+x_1)^n - 1](x_3 y_1 - x_1 y_3)(1+y_1)^n}{x_1 y_1 [(1+x_1)^n (1+y_1)^n - 1},$$

$$\tau = \frac{\alpha y_4}{y_1} + \frac{\alpha[(1+x_1)^n - 1](x_2 y_3 - x_3 y_2)(1+y_1)^n}{x_1 y_1 [(1+x_1)^n (1+y_1)^n - 1](n-m)} +$$

$$\frac{\alpha[(1+x_1)^m - 1](1+y_1)^m [(n-m)(x_4 y_1 - x_1 y_4) + x_2 y_3 - x_3 y_2]}{x_1 y_1 (n-m)[(1+x_1)^m (1+y_1)^m - 1]}$$

Recall some important theorems from the theory of algebraic group.

Theorem 4 *[3]Let L be a finite dimensional solvable Lie algebra over an algebraicly closed field k of characteristic 0. Then L is algebraic (it means that there is an algebraic group G such that $Lie(G) = L$) iff we have:*
1.$L = T \oplus N$, where T is a torus of L and N is the nilradical.
2.$N = \sum_{\alpha \in T^*} \oplus N_\alpha$, where $N_\alpha = \{x \in N | xt = \alpha(t)x, \forall t \in T\}$ and

$$dim_{\mathbf{Q}}\{\alpha | N_\alpha \neq 0\} = dim_k\{\alpha | N_\alpha \neq 0\}.$$

Theorem 5 *[23]If G is a local algebraic group then there exists a global algebraic group G_1 such that $G \simeq G_1$ as local algebraic groups.*

The following theorems show that the Theorems of C.Chevalley and A.Weyl are not valid for algebraic diassociative loops.

Theorem 6 *(Grishkov,([11]))Let $B = B(n, m)$ the Bl-algebra from Example 3. Then B is* **algebraic** *iff $m \neq 0$ and $n/m \in \mathbf{Q}$ or $n = m = 0$.*

Theorem 7 *[11]Let $B = B(n, m)$ be the BL-algebra as above. Then for B there exists the corresponding* **global algebraic** *diassociative loop $G_1 = G_1(n, m)$ iff $n/m \in \mathbf{Z}$.*

If $n = ms, 2 \leq s, n, m \in \mathbf{N}$ then

$$G_1 = \{(x_1, ..., x_4)|x_1 \neq 0, x_i \in k, i = 1, ..., 4\},$$

$$(x_1, ..., x_4) \cdot (y_1, ..., y_4) = (x_1y_1, x_2y_1^n + y_2, x_3y_1^n + y_3, \alpha),$$

where

$$\alpha = x_4y_1^m + y_4 + y_1^n(x_2y_3 - x_3y_2)(\sum_{j=0}^{s-2}\sum_{i=0}^{j} x_1^{in}y_1^{jn})/n(s-1).$$

Theorems 6 and 7 confirm the following Conjecture:

Conjecture 3 *Let B be a finite dimensional solvable BL-algebra over an algebraically closed field k of characterstic 0. Then B is algebraic (local) iff we have:*

(i)$B = T \oplus N, T$ is a torus and N is the nil radical of B.
(ii)$N = \sum_{\alpha \in T^\star} \oplus N_\alpha, dim_{\mathbf{Q}}\{\alpha|N_\alpha \neq 0\} = dim_k\{\alpha|N_\alpha \neq 0\}$.
(iii)$N_0^2 \subset N_0$.
Moreover, B is an algebraic global iff we have (i)-(iii) and
(iv) $N_\alpha N_\alpha \subset \sum_{p \in \mathbf{Z} \setminus 0} \oplus N_{p\alpha}, \forall \alpha \in T^\star$.

Note that for Moufang loops this conjecture was proved in [16].

4 Arithmetic aspect of the exponential map.

In this section we will discuss the analogues of the classical exponential map for the case of the fields of characteristic $p > 0$.

Chevalley showed that the correspondence **Lie algebras** − **Algebraic groups** breaks down completely in characteristic $p > 0$. Thus it is important to search for a good substitute for the Lie algebra of an algebraic group.

In this section we will consider a solution to this problem in the following particular case: the correspondence between $\mathcal{LN}_n(F)$ and $GN_n(F)$ where $\mathcal{LN}_n(F)$ is the set of all Lie p-subalgebras of the Lie algebra

$$N_n(F) = \{a \in M_n(F)|a_{ij} = 0, i \geq j\},$$

F is a field of characteristic $p > 2$, $GN_n(F)$ is a set of all closed algebraic subgroups of

$$U_n(F) = \{a \in M_n(F) | a_{ii} = 1, a_{ij} = 0, i > j\}.$$

It is clear that for $G \in GN_n(F)$ the corresponding Lie algebra $L(G) \in \mathcal{LN}_n(F)$. If the field F has characteristic 0 then the classical exponential map gives a good correspondence between $GN_n(F)$ and $\mathcal{LN}_n(F)$. In this case every algebra $L \in \mathcal{LN}_n(F)$ can be considered as a group from $GN_n(F)$, since Campbell-Hausdorf series converges in every nilpotent Lie algebra. But this series has no sense for a field F of characteristic p and $n \geq p$. There is no hope to find a "good" correspondence between $GN_n(F)$ and $\mathcal{LN}_n(F)$ because there are examples of non-isomorphic groups $G_1, G_2 \in GN_n(F)$ such that $L(G_1) \simeq L(G_2)$. But we can reformulate the question:

Problem 1 *Find a* **canonical** *(in some sense) function $\mathcal{F} : \mathcal{LN}_n(F) \longrightarrow GN_n(F)$ such that $L(\mathcal{F}(L)) \simeq L$.*

This problem is still open. We say that this problem has a solution in the classical sence if there exists an **exponential** map $\mathcal{E} : L(G) \longrightarrow G, L = L(G) \in \mathcal{LN}_n(F), G \in GN_n(F)$ such that L as a group with the multiplication: $a \cdot b = \mathcal{E}^{-1}(\mathcal{E}(a)\mathcal{E}(b))$ is isomorphic to G. I think that this problem has no solution in the classical sense but there is a hope that for a certain subclass of p-subalgebras of $N_n(F)$ the map $\mathcal{E}$ exists.

Let F be a field of characteristic 3 and

$$\mathcal{JN}_n(F) = \{L \in \mathcal{LN}_n(F) | \forall a, b, c \in L : \{a, b, c\} = abc + cba \in L\}.$$

Note that in this definition the products abc and cba are the usual products of matrices.

Theorem 8 *[15] In the notations above there exists a series:*

$$\mathcal{E}(x) = 1 + \sum_{i=1}^{\infty} a_i x^i,$$

such that for every $L \in \mathcal{JN}_n(F)$ we can consider L as an algebraic group with the multiplication: $a \cdot b = \mathcal{E}^{-1}(\mathcal{E}(a)\mathcal{E}(b)) \in L$ and the Lie algebra corresponding to this group is isomorphic to L.

We will call the series $\mathcal{E}(x)$ from Theorem 8 the **3-exponential map**. Note that the series $\mathcal{E}$ from Theorem 8 is not unique and we can describe all such series (3-exponential maps):

Theorem 9 *[15] Let $\mathbf{Z}_3$ be the ring of integral 3-adic numbers and $E(x) = 1 + \sum_{i=0}^{\infty} A_i x^i \in \mathbf{Z}_3[[x]]$ be a series such that*

$$E'(x) = \Lambda(x)E(x),$$

where $E'(x)$ is the derivation of $E(x)$ and $\Lambda(x) = 1+\sum_{i=1}^{\infty} \lambda_i x^{2i}$. Then the serie $\overline{E}(x) = 1 + \sum_{i=1}^{\infty} \overline{A}_i x^i$ is a 3-exponential map. Here $\overline{A}, A \in \mathbf{Z}_3$ is the element from the field $\mathbf{Z}_3/3\mathbf{Z}_3$ that corresponds to A.

I hope that the converse of this theorem is valid too.

Example 4 *Let*

$$E(x) = exp(\sum_{i=0}^{\infty} x^{3^i}/3^i) \in \mathbf{Z}_3[[x]]$$

be the famouse Artin-Hasse exponent for $p = 3$. Then we have

$$E'(x) = (\sum_{i=0}^{\infty} x^{3^i-1})E(x), E(0) = 1.$$

Hence $\overline{E}(x)$ is a 3-exponential map.

Example 5 *[15] Let*

$$E(x) = x + \sqrt{1+x^2},$$

then

$$E'(x) = (\sqrt{1+x^2})^{-1}E(x), E(0) = 1.$$

Hence $\overline{E}(x)$ is a 3-exponential map.

Note that the proof of the Theorem 8 is based on the fact that the 3-exponential map $E(x)$ from the last example is algebraic:

$$E(x)^2 - 2xE(x) - 1 = 0,$$

moreover the inverse series (3-logarithmic map) is rational:

$$\overline{E}^{-1}(x) = (x - x^{-1})/2.$$

We can consider the Lie algebras from $\mathcal{JN}_n(F)$ as algebras with two operations: binary $[,]$ and ternary $\{,,\}$. It is easy to prove that these algebras satisfy the following identities:

$$x^2 = [[x,y],z] + [[y,z],x] + [[z,x],y] = 0, \tag{1}$$

$$\{x,y,z\} = \{z,y,x\}, \tag{2}$$

$$\{x,y,z\} - \{y,x,z\} = [[x,y],z], \tag{3}$$

$$[\{x,y,z\},t] = \{[x,t],y,z\} + \{x,[y,t],z\} + \{x,y,[z,t]\}, \tag{4}$$

$$\{\{x,y,z\},t,u\} = \{\{x,t,u\},y,z\} - \{x,\{y,t,u\},u\} + \{x,y,\{z,t,u\}\}. \tag{5}$$

We will call an algebra with two operations $[,]$ and $\{,,\}$ a **Lie-Jordan algebra** (LJ-algebra) if it satisfies the identities (1-5).

Theorem 10 *[15] Let L be a LJ-algebra from $\mathcal{JN}_n(F)$ and $\mathcal{E}$ be a 3-exponential map. Then there exists a series $H(a,b)$ in the signature of the operations $[,]$ and $\{,,\}$ such that*

$$a \star b = \mathcal{E}^{-1}(\mathcal{E}(a)\mathcal{E}(b)) = H(a,b).$$

Note that in view of Theorems 8 and 10 the LJ-algebra $L \in \mathcal{JN}_n(F)$ with operation $a \star b = H(a,b)$ is an algebraic group. The series

$$H(a,b) = a + b - [a,b] + \{a,b,a\} + \{b,a,b\} + ...$$

is the analogy of the classical Campbell-Hausdorf serie in characteristic 3. Now suppose that we have any nilpotent LJ-algebra L over F, then we can consider L as a loop with the multiplication $a \star b = H(a,b)$. From the following Theorem hence that this loop is a group.

Theorem 11 *[10] Let L be a nilpotent LJ-algebra over field F of characteristic 3. Then L is a group with the miltiplication $a \star b = H(a,b)$.*

Now we can apply the series $H(a,b)$ for the construction of nilpotent diassociative loops. We call an algebra B with two operations $[,]$ and $\{,,\}$ a binary LJ-algebra if every two elements of B generate a LJ-subalgebra of B. It is clear that if L is a binary LJ-algebra then L with the operation $a \star b = H(a,b)$ is an diassociative loop.

Definition 2 *Let B be an algebra with two operations $[,]$ and $\{,,\}$ over a field of characteristic 3. Then B is a MJ-algebra (Malcev-Jordan algebra) if and only if it satisfies the identities (1),(2),(4),(5) and*

$$\{x,y,x\} - \{y,x,x\} = [[x,y],x].$$

It is clear that if B is a nilpotent MJ-algebra over a field of characteristic 3 then we can consider B as a diassociative (algebraic) loop. It is possible that all loops of this type are Moufang.

Conjecture 4 *Let B be nilpotent a MJ-algebra over a field of characteristic 3. Then the loop $(B,\star)$ is Moufang where $a \star b = H(a,b)$.*

This conjecture is a corollary of the following Conjecture 5.

Conjecture 5 *Let B be a MJ-algebra over a field of characteristic 3. Then there exists an alternative algebra A and a homomorphism π of MJ-algebras $\pi : B \longrightarrow A^{(\pm)}$ such that $ker(\pi) = 0$ and $A^{(\pm)}$ is a MJ-algebra with operations: $[a,b] = ab - ba$ and $\{a,b,c\} = (ab)c + (cb)a$.*

This Conjecture is intresting even if the binary operation $[,]$ is trivial and an alternative algebra A is commutative.

Now we will try to generalize the above theory to the case of characteristic $p \geq 3$. Let F be a field of characteristic $p \geq 3$. For a definition of the class $\mathcal{JN}_n(F)$ we have to define the analogy of the associative polynomial $abc + cba$. Let A be free associative ring over F with two free generators a, b. It is well known that the element $(a+b)^p - a^p - b^p$ belongs to the Lie subalgebra $Lie(a,b) \otimes_{\mathbf{Z}} F$ of $A_F = A \otimes_{\mathbf{Z}} F$ with generators a, b. It means that there exists a Lie polynomial $p(a,b) \in Lie(a,b) \subset A$ such that $(a+b)^p - a^p - b^p - p(a,b) = pf(a,b)$, $f(a,b) \in A$. Note that $f(a,b)$ is unique only modulo $Lie(a,b)$.

Denote

$$\mathcal{JN}_n(F) = \{L \in \mathcal{LJ}_n(F) | \forall a, b \in L : f(a,b) \in L\}.$$

Conjecture 6 *In the notations above there exists series over F:*

$$\mathcal{E}(x) = 1 + \sum_{i=1}^{\infty} a_i x^i$$

such that for every $L \in \mathcal{JN}_n(F)$ we can consider L as a group with multiplication: $a \cdot b = \mathcal{E}^{-1}(\mathcal{E}(a)\mathcal{E}(b)) \in L$ and the Lie algebra corresponding to this group is isomorphic to L.

As above we will call the series $\mathcal{E}(x)$ from this Conjecture (if it exists) by a **p-exponential map**.

Conjecture 7 *Let $\mathbf{Z}_p$ be the ring of integral p-adic numbers and $E(x) = 1 + \sum_{i=0}^{\infty} A_i x^i \in \mathbf{Z}_p[[x]]$ be a serie such that*

$$E'(x) = \Lambda(x)E(x),$$

where $E'(x)$ is the derivation of $E(x)$ and $\Lambda(x) = 1 + \sum_{i=1}^{\infty} \lambda_i x^{(p-1)i}$. Then the series $\overline{E}(x) = 1 + \sum_{i=1}^{\infty} \overline{A}_i x^i$ is a p-exponential map. Here $\overline{A}, A \in \mathbf{Z}_p$ is the element from the field $\mathbf{Z}_p/p\mathbf{Z}_p$ that corresponds to A.

It is not difficult to generalize the 3-exponential maps from Exampls 4 and 5 to the case of arbitrary $p \geq 3$:

$$\mathcal{E}(x) = exp(\sum_{i=0}^{\infty} x^{p^i}/p^i) \in \mathbf{Z}[[x]]$$

is the Artin-Hasse exponent and we have

$$E'(x) = (\sum_{i=0}^{\infty} x^{p^i - 1})E(x), E(0) = 1.$$

2. In the second case we have only the differential equation for $E(x) = 1 + \sum_{i=1}^{\infty} A_i x^i$:

$$E'(x) = (\sum_{i=0}^{\infty} \lambda_i x^{i(p-1)})E(x), A_{i(p-1)+1} = 0, i > 0. \qquad (6)$$

Conjecture 8 *The equation (6) has a unique solution $\mathcal{E}(x) \in \mathbf{Z}_p[[x]]$ and $\mathcal{E}(x)$ is an algebraic function.*

Suppose that $\mathcal{E}(x)$ is a solution of (6) Then we can wright: $\mathcal{E}(x) = e_0(x) + xe_1(x) + ... + x^{p-2}e_{p-2}(x)$ where $e_0, ..., e_{p-2} \in \mathbf{Z}_p[[x^{p-1}]], e_1(x) = 1$ and from (6) we have

$$\underbrace{(...(e_0'e_0)')e_0)')...)'e_0}_{p-1} = x. \quad (7)$$

It is clear that equations (6) and (7) have unique solutions $\mathcal{E}(x)$ and $e_0(x^{p-1})$ respectively in the ring $\mathbf{Q}[[x]]$ and the problem has the arithmetic nature. Is it true that the coefficients of those series are in $\mathbf{Q} \cap \mathbf{Z}_p$? As the first step for resolution of equation (7) over $\mathbf{Z}_p$ we have to solve this equation over the field $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. It is interesting that in this case the equation (7) over $\mathbf{F}_p$ is equivalent to the following:

$$(e_0^{p-1})^{(p-2)} = x. \quad (8)$$

This equivalence is the consequence of the following equality in the ring $\mathbf{F}_p[x_1, ..., x_{p-1}]$ [12]:

$$(x_1 + ... + x_{p-1})(x_1 + ... + x_{p-1} - 1)...(x_1 + ... + x_{p-1} - p + 2) =$$
$$\sum_{\sigma \in S_{p-1}} x_{\sigma 1}(x_{\sigma 1} + x_{\sigma 2} - 1)...(x_{\sigma 1} + ... + x_{\sigma(p-1)} - p + 2),$$

where S_{p-1} is the group of permutations.

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