Simple Lie algebras of absolute toral rank 2 in characteristic 2.

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1 Introduction

After the classification of the finite dimensional simple Lie algebras over a field of characteristic $p > 3$ (see [PS]) the main problem in the theory of finite dimensional Lie algebras is the classification of simple Lie algebras over a field of characteristic 2 and 3. The first step in the direction of this classification was made by S. Skryabin ([Sk]). In his paper S. Skryabin proved that all finite dimensional Lie algebras of absolute toral rank 1 over a field of characteristic 2 are solvable.

In the present paper we make the following step: describe the simple finite dimensional Lie algebras over a field of characteristic 2 of absolute toral rank 2. The main resalt of the paper is:

Theorem 1.1. *All simple finite dimensional Lie algebras over a field of charactyeristic 2 of absolute toral rank 2 are classical.*

Recall the definition of the classical Lie algebras. Let k be an algebraically closed field of characteristic $p > 0$. Let B be a Chevalley $\mathbb{Z}$-form of a finite dimensional complex simple Lie algebra. The Lie algebra $A = (B \otimes_{\mathbb{Z}} k)/Z$, where Z is the centre of $B \otimes_{\mathbb{Z}} k$, is called a classical Lie algebra over k. This

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is a universal definition of classical Lie algebras over k. Obviously, this definition is external with respect to the field k. If $p > 3$, then there exists an internal characterization of classical Lie algebras given by the following theorem.

Theorem 1.2. *[Pr] A Lie algebra L over a field k of characteristic $p > 3$ is classical if and only if L has no elements a such that $(\mathrm{ad}(a))^2 = 0$.*

2 Estimation of dimension of root spaces.

Let fix an algebraically closed field k of characteristic 2.

Let L be a Lie algebra over k and L_p is the 2-envelope of L in $Der(L)$. It means that L_p is the minimal 2-subalgebra of $Der(L)$ that contains L. The toral rang of L is equal 2 if L_p contains a toral subalgebra T of dimention 2 and does not contain a toral subalgebra of dimension 3.

Let L be of toral rank 2 and fix a toral subalgebra T of dimension 2 in L_p. Note that it is possible that $T \cap L = 0$. Denote $H = C_L(T) = \{x \in L | [x, T] = 0\}$, H_p is the 2-envelope of H in L_p and $H^p = C_{L_p}(T)$. Then H^p is a regular Cartan subalgebra in L_p. Let $L = H \oplus L_\alpha \oplus L_\beta \oplus L_\gamma$ be a Cartan decomposition of L, where $\alpha, \beta, \gamma = \alpha + \beta \in T^\star$ and $T = kt \oplus kh$, $\alpha(h) = \beta(t) = 0$, $\alpha(t) = \beta(h) = 1$. The main result of this section is the following Proposition.

Proposition 2.1. *In the notation above we have:*

$dimL_\alpha = dimL_\beta = dimL_\gamma = 1, 2, 4.$

3 Lie Algebras of type D_4.

As the corallary of Proposition 1 we have three cases when $dimL_\alpha = 1, 2$ or 4. In this section we conside the principal case when $dimL_\alpha = 4$. The Lie algebras with this property we will call algebras of type D_4. This definition is motivated by the structure of simple classical Lie algebra of type D_4.(see ([GM])). Let X be a variety in k^{11} which is given by the following equations:

$$aa_{23} = ac_{14} = ac_{23} = 0,$$

$$ab_{24} + a_{24}c_{14} + a_{13}c_{23} = 0.$$

We will denote the coordinate functions on k^{11} by

$a, a_{13}, a_{14}, a_{23}, a_{24}, b_{13}, b_{14}, b_{23}, b_{24}, c_{14}, c_{23}$.

For any $v \in X$ by L_v we denote a Lie algebra with the following generations:

$$e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4;\ t, h;$$

$$V = \{\sigma | \sigma \subseteq I = (1234)\}$$

and relations:

$$[t, h] = 0, [x, h] = 0, [x, t] = x, x \in \{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4\},$$

$$[y, h] = y,\ [y, t] = |y| y, y \in V.$$

$$
\begin{array}{lll}
[\emptyset, f_1] = 1, & [\emptyset, f_2] = 2, & [\emptyset, f_3] = 3, \\
[\emptyset, f_4] = 4, & [1, f_1] = 0, & [1, f_2] = 12, \\
[1, f_3] = 13, & [1, f_4] = 14, & [2, f_1] = 12, \\
[2, f_2] = 0, & [2, f_3] = 23, & [2, f_4] = 24, \\
[3, f_1] = 13, & [3, f_2] = 23, & [3, f_3] = 0, \\
[3, f_4] = 34, & [4, f_1] = 14, & [4, f_2] = 24, \\
[4, f_3] = 34, & [4, f_4] = 0, & [12, f_1] = a_{13}3 + a_{14}4, \\
[12, f_2] = a_{23}3 + a_{24}4, & [12, f_3] = 123, & [12, f_4] = 124, \\
[13, f_1] = a4, & [13, f_2] = 123, & [13, f_3] = 0, \\
[13, f_4] = 134, & [14, f_1] = 0, & [14, f_2] = 124, \\
[14, f_3] = 134, & [14, f_4] = 0, & [23, f_1] = 123, \\
[23, f_2] = 0, & [23, f_3] = 0, & [23, f_4] = 234, \\
[24, f_1] = 124, & [24, f_2] = 0, & [24, f_3] = 234, \\
[24, f_4] = 0, & [34, f_1] = 134, & [34, f_2] = 234, \\
[34, f_3] = 0, & [34, f_4] = 0, & [124, f_1] = a_{13}34 + b_{14}\emptyset, \\
[124, f_2] = a_{23}34 + b_{24}\emptyset, & [124, f_3] = I, & [124, f_4] = 0, \\
[134, f_1] = c_{14}\emptyset, & [134, f_2] = I, & [134, f_3] = 0, \\
[134, f_4] = 0, & [234, f_1] = I, & [234, f_2] = 0, \\
[234, f_3] = 0, & [234, f_4] = 0. &
\end{array}
\tag{1}
$$

$$
\begin{array}{ll}
[123, f_1] = a24 + a_{14}34 + b_{13}\emptyset, & [123, f_2] = a_{24}34 + b_{23}\emptyset, \\
[123, f_3] = c_{23}\emptyset, & [123, f_4] = I, \\
[I, f_1] = c_{14}2 + b_{14}3 + b_{13}4, & [I, f_2] = b_{24}3 + b_{23}4, \\
[I, f_3] = c_{23}4, & [I, f_4] = 0.
\end{array}
\tag{2}
$$

$$
[\sigma, e_i] = \sigma \setminus i, \ i \in \sigma; \ [\sigma, e_i] = 0, \ i \notin \sigma.
$$

For define the next relations we introduce the elements $\{m_{ij}, m_j^i | 0 < i, j < 5\}$ wich acts on the generators of L_v as follows:

$$[f_i, m_i^j] = f_j,\ [f_i, m_{ij}] = e_j,\ [e_j, m_i^j] = e_i,$$

$$[\sigma, m_i^j] = (\sigma \cup j) \setminus i,\ i \in \sigma,\ j \notin \sigma,$$

$$[\sigma, m_{ij}] = \sigma \setminus (ij),\ (ij) \subseteq \sigma$$

and as zero in the other cases. Now we can define the products $[f_i, f_j], [f_i, e_j]$ and $[e_i, e_j]$.

$$\begin{aligned} [f_1, f_2] = &\ a_{13}m_1^3 + a_{23}m_2^3 + a_{14}m_1^4 + a_{24}m_2^4 + \\ &\ b_{13}m_1^3 + b_{23}m_2^3 + b_{14}m_1^4 + b_{24}m_2^4, \\ [f_1, f_3] = &\ am_1^4 + b_{13}m_{12} + c_{14}m_{14} + c_{23}m_{23}, \\ [f_2, f_3] = &\ b_{23}m_{12} + c_{23}m_{13}, \\ [f_1, f_4] = &\ b_{14}m_{12} + c_{14}m_{13}, \end{aligned} \tag{3}$$

$$\begin{aligned} &[f_2, f_4] = b_{24}m_{12}, && [f_3, f_4] = 0, \\ &[f_1, e_4] = a_{14}m_{12} + am_{13}, && [f_2, e_4] = a_{24}m_{12}, \\ &[f_1, e_3] = a_{13}m_{12}, && [f_2, e_3] = a_{23}m_{12}, \end{aligned} \tag{4}$$

and zero in the other cases.

For $\psi, \phi \subseteq I = (1234)$ define $\psi \cdot \phi$ by the following:

$$\phi \cdot \psi = \begin{cases} e_i, & \phi \cap \psi = i, \phi \cup \psi = I; \\ f_i, & \phi \cap \psi = \emptyset, \phi \cup \psi = I \setminus i; \\ h + |\psi| t, & \phi \cap \psi = \emptyset, \phi \cup \psi = I_4. \end{cases}$$

Here $|\sigma|$ is the number of elements of $\sigma \subseteq \{1, 2, 3, 4\}$.

Note that this definition coinsider with the multiplication in the classical Lie algebra of type D_4 obtaned in ([GM].)

We define the product $[V,V]$ by the following:

$$[\sigma,\mu]=\sigma\cdot\mu+\sigma,\mu,$$

where

$$\begin{array}{lll}
\{1,124\}=a_{13}m_{12}, & \{1,123\}=am_{13}+a_{14}m_{12}, & \{2,124\}=a_{23}m_{12},\\
\{2,123\}=a_{24}m_{12}, & \{12,13\}=am_{13}+a_{14}m_{12}, & \{12,14\}=a_{13}m_{12},\\
\{12,23\}=a_{24}m_{12}, & \{12,24\}=a_{23}m_{12}, & \{12,124\}=a_{13}e_1+a_{23}e_2,\\
12,123=a_{14}e_1+a_{24}e_2, & 12,I=[f_1,f_2], & 13,123=ae_1,\\
13,I=[f_1,f_3], & 14,I=[f_1,f_4], & 23,I=[f_2,f_3],\\
24,I=[f_2,f_4], & 123,124=[f_1,f_2], & 123,134=[f_1,f_3],\\
123,234=[f_2,f_3], & 123,I=c_{23}e_3+b_{23}e_2+b_{13}e_1, & 124,134=[f_1,f_4],\\
[124,234]=[f_2,f_4], & 124,I=b_{24}e_2+b_{14}e_1, & 134,I=c_{14}e_1
\end{array} \tag{5}$$

and zero in the other cases.

Proposition 3.1. *Let $v\in X$ and L_v be the corresponding Lie algebra defined above. Then L_v has the toral rank 2 if and only if $v=0$ and and in this case L_0 is a classical Lie algebra of type D_4.*

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