Simple classical Lie algebras in characteristic 2 and their graduations, II.

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June 20, 2002

1 Introduction

This paper is a continuation of [GG]. Here we prove Conjecture 5.1 [GG]. Recall some notations and definition of [GG].

Definition 1.1. *Let* $I_n = \{1, \ldots, n\}$. *We call* $\mathfrak{a} \subset \mathcal{P}(I_n) = \{\sigma \mid \sigma \subseteq I_n\}$ *an* **even** *set if for all* $\sigma, \tau \in \mathfrak{a}$, *we have* $|\sigma| \equiv |\tau| \equiv 0$ *and* $|\sigma \cap \tau| \equiv 0 \mod 2$.

We note that $\mathcal{P}(I_n)$ is an elementary abelian group with the operation $\sigma \triangle \tau = (\sigma \setminus \tau) \cup (\tau \setminus \sigma)$. For $\mathfrak{a} \subseteq \mathcal{P}(I_n)$, $< \mathfrak{a} >$ denotes the group generated by $\mathfrak{a}$.

Definition 1.2. *A subset* H *of* $\mathcal{P}(I_n)$ *is* **connected** *if, for every partition* $I_n = I \cup J$, *there is* $\sigma \in H$ *such that* $\sigma \cap I \neq \emptyset$ *and* $\sigma \cap J \neq \emptyset$.

Definition 1.3. *A subset* $\sigma \subseteq I_n$ *is called* $\mathfrak{a}$**-even** *if* $|\mu \cap \tau| \equiv 0 \mod 2$ *for all* $\tau \in \mathfrak{a}$. *A subset* $B \subseteq \mathcal{P}(I_n)$ *is called an* $\mathfrak{a}$**-even set** *if all its elements are* $\mathfrak{a}$**-even**.

*Supported by FAPEMIG as visitor to the Depto. de Matemática - UFV in February 2001
†Supported by FAPESP as visitor to IME-USP in February 2000

For an even set $\mathfrak{a} \subset \mathcal{P}(I_n)$, we construct an algebra $\tilde{S} = \tilde{S}(\mathfrak{a})$ with basis $\{ e_i,\ h_i,\ f_i,\ h^\sigma \mid i \in I_n,\ \sigma \in <\mathfrak{a}> \}$ and multiplication given by

$$e_i f_i = h_i, \qquad e_i h^\sigma = e_i, \quad f_i h^\sigma = f_i, \qquad \text{for } i \in \sigma\,, \tag{1}$$

and zero for any other cases. Denote $h_\mu = \sum_{i \in \mu} h_i\,,\ h_\emptyset = 0\,.$

It is easy to show that the algebra $\tilde{S}(\mathfrak{a})$ contains a central ideal I generated by $\{h^\sigma + h^\tau + h^{\sigma\triangle\tau} + h_{\sigma\cap\tau} \mid \sigma, \tau \in <\mathfrak{a}>\}$. We denote $S(\mathfrak{a}) = S = \tilde{S}(\mathfrak{a})/I$.

For every $\mathfrak{a}$-even set σ, define an S-module Λ_σ whose basis is $\{(\sigma, \mu) \mid \mu \subseteq \sigma\}$ and the S-action is given by

$$\begin{aligned}
(\sigma,\ \mu)\, e_i &= (\sigma,\ \mu \cup i), \quad i \in \sigma \setminus \mu;\\
(\sigma,\ \mu)\, f_i &= (\sigma,\ \mu \setminus i), \quad i \in \mu;\\
(\sigma,\ \mu)\, h_i &= (\sigma,\ \mu), \quad i \in \sigma;\\
(\sigma,\ \mu)\, h^\varphi &= \left(\frac{|\sigma \cap \varphi|}{2} + |\varphi \cap \mu|\right) (\sigma,\ \mu), \ \text{ for } \varphi \in \mathfrak{a},
\end{aligned} \tag{2}$$

and for any other cases the action is zero.

Now let $\Delta = \{0\} \cup \mathfrak{a}$.

Definition 1.4. *An algebra A is called a* **Δ-algebra** *if $A = \sum_{\alpha \in \Delta} \oplus A_\alpha$ and, for every $\alpha \neq \beta \in \mathfrak{a}$, we have $A_\alpha A_\beta \subseteq A_{\alpha\triangle\beta}$, $A_0^2 \subseteq A_0$, $A_0 A_\alpha \subseteq A_\alpha$, $A_\alpha A_\alpha \subseteq A_0 + A_\emptyset$ and $A_0 A_\emptyset = 0$.*

Define a commutative Δ-graded algebra Λ as follows. As a k-space, Λ is

$$\Lambda = \Lambda_0 \bigoplus \sum_{\sigma \in \mathfrak{a}} \oplus \Lambda_\sigma\,, \quad \text{where } \Lambda_0 = S(\mathfrak{a})\,. \tag{3}$$

Moreover, $S = S(\mathfrak{a})$ is a subalgebra of Λ and, by (2), each Λ_σ is an S-module. For $\sigma \neq \tau \in \mathfrak{a}$, the multiplication is given by

$$(\sigma,\ \mu)\,(\tau,\ \varphi) = (\sigma\triangle\tau,\ (\mu \setminus \tau) \cup (\varphi \setminus \sigma)), \ \text{ if } \mu \cap \varphi = \emptyset,\ \mu \cup \varphi \supset \sigma \cap \tau\,. \tag{4}$$

$$(\sigma,\,\mu)\,(\sigma,\,\varphi)\;=\;\begin{cases} e_i, & \mu\cap\varphi=i,\;\mu\cup\varphi=\sigma,\\ f_i, & \mu\cap\varphi=\emptyset,\;\mu\cup\varphi=\sigma\setminus i,\\ h^\sigma\,+\,\sum_{i\in\varphi}h_i\,+\,(\emptyset,\emptyset), & \mu\cap\varphi=\emptyset,\;\mu\cup\varphi=\sigma\,,\end{cases} \tag{5}$$

and all other products are zero.

Recall the definition of the product of two Δ-algebras. Let $A = \sum_{\alpha\in\Delta} \oplus A_\alpha$ and $B = \sum_{\alpha\in\Delta} \oplus B_\alpha$ be two Δ-algebras. Then $A\Box B = \sum_{\alpha\in\Delta} \oplus\, A_\alpha \otimes B_\alpha$ is a Δ-algebra with multiplication $[\,\cdot\,,\,\cdot\,]$ given by

$$[a_\alpha \otimes b_\alpha,\, a_\beta \otimes b_\beta] = \sum_{\gamma\in\Delta} c_\gamma \otimes d_\gamma\,, \qquad \text{if } a_\alpha a_\beta = \sum_{\gamma\in\Delta} c_\gamma\,,\; b_\alpha b_\beta = \sum_{\gamma\in\Delta} d_\gamma.$$

Proposition 1.1. *Let $\mathfrak{a}$ be an even set, $<\mathfrak{a}>$ be the group generated by $\mathfrak{a}$, $\Lambda = \Lambda(<\mathfrak{a}>)$ and $\Delta = \{0\}\cup\mathfrak{a}$. Let $M = M_0 \oplus \sum_{\sigma\in\mathfrak{a}} \oplus M_\sigma$ be a commutative Δ-algebra. Then the algebra $L = \Lambda\Box M$ is a Lie algebra if and only if M satisfies a list of Δ-identities given in Proposition 3.1 [GG].*

We recall some of the Δ-identities which will be used in this paper. $\sigma\triangle\tau \neq$

$$\mu a_\sigma b_\tau \cdot c_\lambda + b_\tau c_\lambda \cdot a_\sigma + c_\lambda a_\sigma \cdot b_\tau = 0, |\sigma \cap \tau \cap \lambda| = 0, \sigma \neq \tau \neq \lambda \neq \sigma \neq \tau \triangle \lambda, \quad (6)$$

$$(a_\sigma b_\sigma)_\emptyset c_\tau = 0, \qquad (a_\sigma b_\sigma)_\emptyset c_\tau = 0, \ (a_\sigma b_\sigma)_0 c_\tau = a_\sigma c_\tau \cdot b$$

(7)

$$a_\sigma c_\tau \cdot b_\sigma = a_\sigma \cdot c_\tau b_\sigma + (a_\sigma b_\sigma)_\emptyset c_\tau , \qquad |\sigma \cap \tau| =$$

(8)

$$(a_\sigma b_\tau \cdot c_\lambda)_0 = (a_\sigma \cdot b_\tau c_\lambda)_0, \qquad \lambda = \sigma \triangle$$

(9)

$$(ab)_0 c = (cb)_0 a, \ \ (ca)_\emptyset b = 0, \qquad a, b, c \in M_\tau, \ |\tau| =$$

(10)

$$(ab)_0 c + (bc)_0 a = (ac)_\emptyset b, \qquad a, b, c \in M_\tau, \ |\tau| =$$

(11)

$$(a_\sigma b_\sigma)_0 c_\tau = 0, \qquad |\sigma| >$$

(12)

$$\text{or } (a_\emptyset b_\sigma \cdot c_\sigma)_\emptyset + (b_\sigma c_\sigma)_\emptyset \cdot a_\emptyset + (c_\sigma a_\emptyset \cdot b_\sigma)_\emptyset = 0 \qquad \forall$$

(13)

$$(a_\sigma b_\sigma)_0 \cdot c_0 = (a_\sigma c_0 \cdot b_\sigma)_0 , \qquad \sigma \neq$$

(14)

(15)

We observe that $L = \Lambda \Box M$ is not necessarily a simple algebra, even though M is simple, but $L/Z(L)$ is simple, where $Z(L)$ is the center of L.

2 Novo artigo

Let $\mathfrak{a}$ be an even connected set, $< \mathfrak{a} >$ be the group generated by $\mathfrak{a}$, and $\Delta = \{0\} \cup \mathfrak{a}$. Let $\mathcal{M}$ be the variety of Δ-algebras satisfying the list of identities of Proposition 3.1 [GG]. Let $M = M_0 \oplus \sum_{\sigma \in \mathfrak{a}} \oplus M_\sigma \oplus M_\emptyset$ be a commutative

Δ-algebra in $\mathcal{M}$. In [GG] (see Theorem 3.1 [GG]) we classified the simple Δ-algebras of the variety $\mathcal{M}$, for which $M_\emptyset = 0$. Now we consider the case when $0 \neq M_\emptyset$ is abelian.

Recall the S-module $\Lambda = \Lambda_0 \bigoplus \sum_{\sigma\in\mathfrak{a}} \oplus\Lambda_\sigma$, where $\Lambda_0 = S(\mathfrak{a})$ and the corresponding Lie algebra $L = M\Box\Lambda$.

In the final section of [GG], we remarked that Theorem 3.1 [GG] is not true if we omit the condition $\emptyset \notin \mathfrak{a}$ and we formulated the following conjecture.

Conjecture 2.1. *Let M be an arbitrary simple finite dimensional $\triangle$-algebra which satisfies all the list of identities of Proposition 3.1 [GG] and $M_\emptyset^2 = 0$. Then the corresponding Lie algebra $L = M\Box\Lambda$ is a simple Lie algebra of type $B_{2\ell}$, C_ℓ, $D_{2\ell+1}$, E_7 or E_8.*

For each $\emptyset \neq \sigma \in \mathfrak{a}$, define $M_\sigma^0 = \{x \in M_\sigma \,|\, xM_\sigma \subseteq M_\emptyset\} = \{x \in M_\sigma \,|\, (xM_\sigma)_0 = 0\}$.

Lemma 2.1. $I = \displaystyle\sum_{\sigma\in\mathfrak{a}\setminus\emptyset} \oplus M_\sigma^0 \bigoplus \sum_{\sigma\in\mathfrak{a}} (M_\sigma M_\sigma^0)$ *is an ideal in M.*

Proof. (a) First we prove that $M_\tau M_\sigma^0 \subseteq M_{\sigma\triangle\tau}^0$, for all $\sigma \neq \tau \in \mathfrak{a}$. Indeed, by (9), for $a_\tau \in M_\tau$, $b_\sigma \in M_\sigma^0$, $c_\lambda \in M_{\sigma\triangle\tau}^0$, we have $(a_\tau\, b_\sigma \cdot c_\lambda)_0 = (b_\sigma \cdot a_\tau\, c_\lambda)_0 = 0$.

(b) Now we prove that $(M_\sigma M_\sigma^0)M_\tau \subseteq M_\tau^0$, for all $\tau \in \mathfrak{a}$. We need to prove that $(((b_\sigma\, c_\sigma)_\emptyset\, a_\tau)d_\tau)_0 = 0$ for all $a_\tau \in M_\tau$, $b_\sigma \in M_\sigma^0$, $c_\sigma \in M_\sigma$, $d_\tau \in M_\tau$. We have two cases:

(b.1) $\sigma \neq \tau$. If $|\,\sigma \cap \tau\,| = 2$, we have by (7) that $(((b_\sigma\, c_\sigma)_\emptyset\, a_\tau)d_\tau)_0 = 0$. If $|\,\sigma \cap \tau\,| = 0$ then, by (8) and (9), $(((b_\sigma\, c_\sigma)_\emptyset\, a_\tau)d_\tau)_0 = ((b_\sigma\, a_\tau \cdot c_\sigma)d_\tau)_0 + ((c_\sigma\, a_\tau \cdot b_\sigma)d_\tau)_0 = (b_\sigma\, a_\tau \cdot c_\sigma\, d_\tau)_0 + (b_\sigma \cdot (c_\sigma\, a_\tau \cdot d_\tau))_0 = (b_\sigma \cdot a_\tau(c_\sigma\, d_\tau))_0 = 0$.

(b.2) $\sigma = \tau$. If $|\,\sigma\,| = 2$, then by (11) we have $(((b_\sigma\, c_\sigma)_\emptyset\, a_\sigma)d_\sigma)_0 = ((b_\sigma\, a_\sigma)_0\, c_\sigma)d_\sigma)_0 + ((c_\sigma\, a_\sigma)_0\, b_\sigma)d_\sigma)_0 \subseteq k\,(b_\sigma\, d_\sigma)_0 = 0$, as $b_\sigma \in M_\sigma^0$. If $|\,\sigma\,| = 4$, then by (10) $(((b_\sigma\, c_\sigma)_\emptyset\, a_\sigma)d_\sigma)_0 = 0$.

This proves the lemma. $\square$

By Lemma 2.1, if M is simple then $I = 0$ and, for each $\sigma \neq \emptyset$, $M_\sigma^0 = 0$.

Lemma 2.2. *For M simple as defined above and $\sigma \in \mathfrak{a}$ we have*

1. *If $\mid \sigma \mid = 4$ then $M_\sigma = k\, a_\sigma$ where $a_\sigma^2 = s$.*

2. *If $\mid \sigma \mid = 2$ then*

2.1) $M_\sigma = k\, a_\sigma$, where $(a_\sigma^2)_0 = s$, or

2.2) $M_\sigma = k\, a_\sigma \oplus k\, b_\sigma$, where $(a_\sigma^2)_0 = (b_\sigma^2)_0 = s$ and $(a_\sigma\, b_\sigma)_0 = 0$ or

2.3) $M_\sigma = k\, a_\sigma \oplus k\, b_\sigma$, where $(a_\sigma\, b_\sigma)_0 = s$ and $(a_\sigma^2)_0 = (b_\sigma^2)_0 = 0$.

Proof. Let $\mid \sigma \mid = 4$ and $a_\sigma \in M_\sigma$. By Lemma 2.1, there exists $b_\sigma \in M_\sigma$ such that $(a_\sigma\, b_\sigma)_0 = s$, then we have on the one hand $(a_\sigma\, b_\sigma)_0\, a_\sigma = a_\sigma$ and on the other hand, by (10), $(a_\sigma\, b_\sigma)_0\, a_\sigma = (a_\sigma\, a_\sigma)_0\, b_\sigma = \alpha\, b_\sigma$, if $(a_\sigma\, a_\sigma)_0 = \alpha\, s$. Hence $a_\sigma = \alpha\, b_\sigma$. If $c \in M_\sigma$ and $(b\, c)_0 = \gamma\, s$, then by (10), $c = (a\, b)_0\, c = (b\, c)_0\, a = \gamma\, a$. Hence $\dim M_\sigma = 1$ and we have proved part 1.

Now let $\mid \sigma \mid = 2$.

(a) There exists $a \in M_\sigma$ such that $(a^2)_0 = s$. If $\dim M_\sigma = 1$, then we have case 2.1. Suppose that there exists $b \in M_\sigma \setminus k\, a$. If $(a\, b)_0 = \alpha s \neq 0$ then we can replace b by $b + \alpha a = \tilde{b}$ and we get $(\tilde{b}\, a)_0 = 0$. Hence we can suppose that b satisfies $(a\, b)_0 = 0$.

(a.1) Suppose that for all $a \in M_\sigma$ such that $(a\, b)_0 = 0$ we have $(b^2)_0 = 0$. By Lemma 2.1, there exists $c \in M_\sigma$ such that $(c\, b)_0 = s$. We can suppose that $(c\, a)_0 = 0$ (by replacing c by $c + \alpha a = \tilde{c}$ as before). Now, using identity (11), we get $(ab)_\emptyset\, c = (cb)_0\, a + (ac)_0\, b = a$, $(bc)_\emptyset\, a = (ba)_0\, c + (ca)_0\, b = 0$ and $(bc)_\emptyset\, c = (bc)_0\, c + (cc)_0\, b = c$. Hence, $[(ab)_\emptyset, (bc)_\emptyset]\, c = a \neq 0$, contradicting the fact that $M_\emptyset$ is abelian.

(a.2) There exists $b \in M_\sigma$ such that $(b^2)_0 = s$ and $(a\, b)_0 = 0$. If $\dim M_\sigma = 2$ then we have part 2.2.

Suppose that $\dim M_\sigma > 2$. By Lemma 2.1, there exists $c \in M_\sigma$ such that $(a\,c)_0 = (b\,c)_0 = 0$.

(a.3) If $(c^2)_0 = s$ then by (11), $(ac)_\emptyset\, c = (cc)_0\, a = a$, $(ab)_\emptyset\, c = 0$ and $(ab)_\emptyset\, a = b$. Hence $[(ac)_\emptyset, (ab)_\emptyset]\, c = b \neq 0$, contradicting the fact that $M_\emptyset$ is abelian.

(a.4) Suppose that for all $c \in M_\sigma$ such that $(a\,c)_0 = (b\,c)_0 = 0$ we have $(c^2)_0 = 0$. By Lemma 2.1, there exists $d \in M_\sigma$ such that $(d^2)_0 = 0$ and $(c\,d)_0 = s$. Then, by identity (11), $(ab)_\emptyset\, a = b$, $(ab)_\emptyset\, d = 0$ and $(ac)_\emptyset\, d = a$. Hence $[(ab)_\emptyset, (ac)_\emptyset]\, d \neq 0$ and again we contradict the fact that $M_\emptyset$ is abelian.

(b) For all $a \in M_\sigma$, $(a^2)_0 = 0$. By Lemma 2.1, there exist $a, b \in M_\sigma$ such that $(a^2)_0 = (b^2)_0 = 0$ and $(a\,b)_0 = s$. If $\dim M_\sigma = 2$, then we have case 2.3.

If $\dim M_\sigma > 2$, then by Lemma 2.1 there exist $c, d \in M_\sigma$ such that $(a\,c)_0 = (a\,d)_0 = (b\,c)_0 = (b\,d)_0 = (c^2)_0 = (d^2)_0 = 0$ and $(c\,d)_0 = s$. In this case, by (11), $(ab)_\emptyset\, a = a$ and $(ac)_\emptyset\, d = a$. Hence $[(ab)_\emptyset, (ac)_\emptyset]\, d = a \neq 0$, contradicting the fact that $M_\emptyset$ is abelian. This proves the lemma. $\square$

Lemma 2.3. *Let $\mathfrak{a} \subset \mathcal{P}(I_n)$ be an even set and $\Delta = \{0\} \cup \mathfrak{a}$. Let $\mathcal{M}$ be the variety of Δ-algebras satisfying the list of identities of Proposition 3.1 [GG]. If $M \in \mathcal{M}$ is a simple Δ-algebra (containing no graded ideals), then $\mathfrak{M} = \{\sigma \in \mathfrak{a} \mid M_\sigma \neq 0\}$ is connected and is one of the following sets:*

(i) $\{(2i-1,\, 2i,\, 2j-1,\, 2j) \mid 1 \le i < j \le \ell\} = \mathcal{D}_{2\ell}$,

(ii) $\{(2i-1,\, 2i,\, 2j-1,\, 2j),\ (2i-1,\, 2i) \mid 1 \le i < j \le \ell\} = \mathcal{B}_{2\ell}$,

(iii) $\{\,(1234), (1256), (1357), (3456), (2457), (2367), (1467)\,\} = \mathcal{E}_7$,

(iv) $\mathcal{E}_7 \cup \{\,\overline{\sigma} \mid \sigma \in \mathcal{E}_7,\ \overline{\sigma} = I_8 \setminus \sigma\,\} = \mathcal{E}_8$.

Proof. The proof of this lemma in [GG] is based on the following facts:

(1) for all $\sigma \in \mathfrak{M}$, we have $|\sigma| = 2$ or 4.

(2) If $\sigma \neq \tau \in \mathfrak{M}$ and $\sigma \cap \tau \neq \emptyset$ then $\sigma \triangle \tau \in \mathfrak{M}$.

The item (2) may be proved as in [GG]. Let us prove item (1). Suppose that $\sigma \in \mathfrak{M}$ and $|\sigma| > 4$. Thus, by (12), $(M_\sigma M_\sigma)_0 M_\sigma = 0$, hence $(M_\sigma M_\sigma)_0 = 0$ and $M_\sigma = M_\sigma^0$. But by Lemma 2.1, $M_\sigma^0 = 0$. □

Theorem 2.1. *Let $M \in \mathcal{M}$ be a simple Δ-algebra such that $M_\emptyset \neq 0$ and $M_\emptyset^2 = 0$. Then $\mathfrak{M} = \mathcal{B}_{2\ell}$ and M has a basis*

$$\{ s, d_{ij}, a_i, b_i, \lambda \mid 1 \le i < j \le \ell \}$$

with one of the following set of multiplication rules:

$$\begin{aligned} d_{ij} d_{jk} &= d_{ik}, & d_{ij} a_j &= b_i, \\ d_{ij} b_j &= a_i, & a_i b_j &= d_{ij}, \\ (a_i b_i)_\emptyset &= \lambda, & \lambda a_i &= b_i, \\ \lambda b_i &= a_i, & (d_{ij}^2)_0 &= s, \\ (a_i^2)_0 &= (b_i^2)_0 = s \end{aligned} \tag{16}$$

or

$$\begin{aligned} d_{ij} d_{jk} &= d_{ik}, & d_{ij} a_i &= a_j, \\ d_{ij} b_j &= b_i, & a_i b_j &= d_{ij}, \\ a_i b_i &= s + \lambda, & (d_{ij}^2)_0 &= s, \\ \lambda a_i &= a_i, & \lambda b_i &= b_i, \end{aligned} \tag{17}$$

where $M_{(2i-1,2i,2j-1,2j)} = k\, d_{ij}$, $M_{(2i-1,2i)} = k\, a_i \oplus k\, b_i$ and $M_\emptyset = k\,\lambda$.

We will denote by $\mathcal{D}_{2\ell+1}$ the Δ-algebra M with multiplication rules given by (16) and by $\mathcal{C}_{2\ell+1}$ the one with multiplication rules given by (17).

Proof. If $\mathfrak{M} = \mathcal{B}_{2\ell} = \{(2i-1, 2i, 2j-1, 2j), (2i-1, 2i) \mid 1 \le i < j \le \ell\}$, then by Lemma 2.2 for $\sigma = (2i-1, 2i) \in \mathfrak{M}$ we have three cases

(a) $M_\sigma = k\, a_\sigma$, where $(a_\sigma^2)_0 = s$, or

(b) $M_\sigma = k\,a_\sigma \oplus k\,b_\sigma$, where $(a_\sigma^2)_0 = (b_\sigma^2)_0 = s$ and $(a_\sigma\, b_\sigma)_0 = 0$ or

(c) $M_\sigma = k\,a_\sigma \oplus k\,b_\sigma$, where $(a_\sigma\, b_\sigma)_0 = s$ and $(a_\sigma^2)_0 = (b_\sigma^2)_0 = 0$.

Let us consider each case.

(a) For $\mid \sigma \mid = 2$, by identity (11), we have $(a_\sigma\, a_\sigma)_\emptyset\, a_\sigma = 2\,(a_\sigma^2)_0 a_\sigma = 0$ and by (7), for $\mid \sigma \mid = \mid \tau \mid = 2$ with $\sigma \cap \tau = \emptyset$ $(a_\sigma\, a_\sigma)_\emptyset\, a_\tau = 2\,a_\sigma\, a_\tau \cdot a_\sigma = 0$. If $\mid \sigma \mid = 2$ and $\mid \sigma \cap \tau \mid = 2$ with $\sigma \neq \tau$, then by (7) $(a_\sigma\, a_\sigma)_\emptyset\, c_\tau = 0$. Therefore, $(a_\sigma\, a_\sigma)_\emptyset \in Z(M) = 0$. Let $\mu \in \mathfrak{M}$ such that $M_\mu = k\,d$ and $\sigma \subseteq \mu$, $\tau = \mu \setminus \sigma$. Denote $b_\tau = d\,a_\sigma$. If $c \in M_\tau$ then, by (7), $d\,c \cdot d = c(dd)_0 = c$. But $d\,c \in M_\sigma = k\,a_\sigma$. Thus $c = d\,c \cdot d = \alpha\, a_\sigma\, d = \alpha\, b_\tau$ and $M_\tau = k\,b_\tau$. In this case, M is the algebra obtained in [GG].

(b) Let $d = d_{12} \in M_{(1234)}$ and denote

$$b_2 = d\,b_1, \qquad a_2 = d\,a_1, \qquad a_1\, a_2 = \alpha\, d, \qquad b_2\, b_1 = \beta\, d$$

By (7), $d\,b_2 = d \cdot d\,b_1 = (d\,d)_0 b_1 = b_1$ and $d\,a_2 = d \cdot d\,a_1 = (d\,d)_0 a_1 = a_1$.

Now, by (9), we have $(b_2\, b_2)_0 = (d\,b_1 \cdot b_2)_0 = (d \cdot b_1\, b_2)_0 = (d\,b_2 \cdot b_1)_0 = s$. Hence, $\beta = 1$. Moreover, $(a_2\, a_2)_0 = (d\,a_1 \cdot a_2)_0 = (d\,a_2 \cdot a_1)_0 = (d \cdot a_1\, a_2)_0 = s$. Hence, $\alpha = 1$.

Again by (9), $(b_2\, a_2)_0 = (d\,b_1 \cdot a_2)_0 = (d \cdot b_1\, a_2)_0 = (d\,a_2 \cdot b_1)_0 = 0$. Hence, $b_1\, a_2 = 0$. Analogously, $a_1\, b_2 = 0$.

Now denote $\tau = (a_1\, a_1)_\emptyset$, $\xi = (b_1\, b_1)_\emptyset$, $\lambda = (a_1\, b_1)_\emptyset$. For $c \in M_{(12)}$, by identity (11), we have $(a_1\, a_1)_\emptyset\, c = 2(a_1\, c)_0\, a_1 = 0$.

By (8), for $c \in M_{(2i-1,2i)}$, $i \neq 1$, we have $(a_1\, a_1)_\emptyset\, c = 2a_1\, c \cdot a_1 = 0$. Hence, $(a_1\, a_1)_\emptyset \in Z(M) = 0$ and analogously $(a_i\, a_i)_\emptyset = (b_i\, b_i)_\emptyset = 0$.

Moreover, by (11), $\lambda\, a_1 = (a_1\, b_1)_\emptyset\, a_1 = (a_1\, a_1)_0 b_1 = b_1$ and analogously $\lambda\, b_1 = a_1$. By (7), $\lambda\, a_2 = (a_1\, b_1)_\emptyset\, a_2 = b_1\, a_2 \cdot a_1 = d\,a_1 = b_2$ and in the same way $\lambda\, b_2 = a_2$.

Now we denote $b_i = d_{i1}\, b_1$ and $a_i = d_{i1}\, a_1$. As above, we can prove that $b_i\, b_j = a_i\, a_j = d_{ij}$, $a_i\, b_j = 0$ and $(a_i\, a_i)_\emptyset = (b_i\, b_i)_\emptyset = 0$ and $(a_i\, b_i)_\emptyset = \lambda$. In this case, we have the multiplication rules given by (16).

(c) Let $d = d_{12} \in M_{(1234)}$ be such that $d^2 = s$ and denote

$$a_2 = d\,a_1, \qquad b_2 = d\,b_1, \qquad a_1\,b_2 = \alpha\,d,$$
$$a_1\,a_2 = \gamma\,d, \qquad a_2\,b_1 = \beta\,d, \qquad b_1\,b_2 = \tau\,d,$$

As in case (b), by (7), we have $d\,a_2 = d \cdot d\,a_1 = (d\,d)_0 a_1 = a_1$ and $d\,b_2 = d \cdot d\,b_1 = (d\,d)_0 b_1 = b_1$.

Now, by (9), we have $(a_2\,a_2)_0 = (d\,a_1 \cdot a_2)_0 = (d\,a_2 \cdot a_1)_0 = (a_1\,a_1)_0 = 0$, $(b_2\,b_2)_0 = 0$. Again by (9), $(a_2\,b_2)_0 = (d\,a_1 \cdot b_2)_0 = (d\,b_2 \cdot a_1)_0 = (b_1\,a_1)_0 = s$.

Now for $c \in M_{(12)}$, by identity (11), we have $(a_1\,a_1)_\emptyset\, c = 2(a_1\,c)_0\,a_1 = 0$ and by (8), for $c \in M_{(2i-1,2i)}$, $i \neq 1$, we have $(a_1\,a_1)_\emptyset\, c = 2a_1\,c \cdot a_1 = 0$. Hence, $(a_i\,a_i)_\emptyset = (b_i\,b_i)_\emptyset \in Z(M) = 0$.

By (11) we have $(a_1\,b_1)_\emptyset\, a_1 = (a_1\,b_1)_0\, a_1 = a_1$ and, by (7), $(a_1\,b_1)_\emptyset\, a_2 = a_2\,b_1 \cdot a_1 = d\,a_1 = a_2$.

¿From this, analogously to the previous case, for $\lambda = (a_1\,b_1)_\emptyset$ we get $\lambda\,a_i = a_i$ and $\lambda\,b_i = b_i$.

Now for $d_{ij} \in M_{(2i-1,2i,2j-1,2j)}$, it is clear, by (7), (8) and the fact that $\mid \sigma \mid \leq 4$, that $\lambda\,d_{ij} = (a_1\,b_1)_\emptyset\, d_{ij} = 0$ □

Theorem 2.2. *Let M be a Δ-algebra as in Theorem 2.1 and V be an irreducible M-module. Then*

1. $M = \mathcal{D}_{2\ell+1}$ and

1.1. $V = < v_1, \ldots, v_\ell, \xi, \mu >$, where $v_i \in V_{(2i-1,2i)}$, $\xi, \mu \in V_\emptyset$ and

$$v_i\,d_{ij} = v_j, \qquad (v_i\,a_i)_\emptyset = \xi, \qquad (v_i\,b_i)_\emptyset = \mu,$$
$$\xi\,a_i = v_i, \qquad \mu\,b_i = v_i, \qquad \lambda\,\mu = \xi, \quad \lambda\,\xi = \mu \tag{18}$$

and all the other products are zero.

1.2. V is the adjoint module.

2. $M = \mathcal{C}_{2\ell+1}$

2.1. $V = < v_1, \ldots, v_\ell, \tau, \mu >$, *where* $v_i \in V_{(2i-1,2i)}$, $\tau, \mu \in V_\emptyset$ *and*

$$v_i\, d_{ij} = v_j, \qquad (v_i\, a_i)_\emptyset = \tau, \qquad (v_i\, b_i)_\emptyset = \mu,$$
$$\tau\, b_i = \mu\, a_i = v_i, \qquad \lambda\, \tau = \tau, \qquad \lambda\, \mu = \mu \tag{19}$$

and all the other products are zero.

2.2. V *is the adjoint module.*

Proof. 1. Let $V_0 \neq 0$ and $v_0 \in V_0$. Define

$$v_{ij} = v_0\, d_{ij}, \quad v_i = v_0\, a_i, \quad w_i = v_0\, b_i, \quad (v_i\, b_i)_\emptyset = \mu_i \quad (w_i\, a_i)_\emptyset = \xi_i\,. \tag{20}$$

By (14) we have $(v_i\, a_i)_0 = (v_0\, a_i \cdot a_i)_0 = v_0\,(a_i a_i)_0 = v_0$. Thus, by (8) and (11), we have that $\mu = \mu_1 = \cdots = \mu_\ell = \xi_1 = \cdots = \xi_\ell$ and $\mu_i\, a_i = (v_i\, b_i)_\emptyset\, a_i = (v_i\, a_i)_0\, b_i + (a_i\, b_i)_0\, v_i = v_0\, b_i = w_i$ and analogously $\mu\, b_i = v_i$.

Hence V has a basis $\{v_0, v_{ij}, v_i, w_i, \mu \,|\, i \leq i, j \leq \ell\}$ and V is the adjoint M-module.

Now suppose that $V_0 = 0$ and take V_μ, $\mu \neq 0$. As $\mid \mu \cap (2i-1, 2i) \mid = 0$ or 2, then $(2i-1, 2i) \subseteq \mu$ or $\mu \cap (2i-1, 2i) = \emptyset$ for all $1 \leq i \leq \ell$. Suppose $(1,2) \subseteq \mu$. If $(2i-1, 2i) \subseteq \mu$, $i > 2$, then $\sigma_{1i} = (1, 2, 2i-1, 2i) \subseteq \mu$ and, by (12), $V_\mu = s\, V_\mu = (d_{1i}\, d_{1i})_0\, V_\mu = 0$, a contradiction. Hence, $\mu = (12)$. Let $0 \neq v_1 \in V_{(12)}$ and denote

$$v_i = v_1\, d_{1i}, \qquad (v_i\, b_i)_\emptyset = \tau_i \qquad (v_i\, a_i)_\emptyset = \mu_i\,. \tag{21}$$

Now by (6), we have $v_i\, d_{ij} = v_1\, d_{1i} \cdot d_{ij} = v_1\, d_{ij} \cdot d_{1i} + v_1 \cdot d_{1i} d_{ij} = v_1\, d_{1j} = v_j$.

By (11), $\mu_i\, a_i = (v_i\, a_i)_\emptyset\, a_i = (v_i\, a_i)_0\, a_i + (a_i\, a_i)_0\, v_i = v_i$ and $\mu_i\, b_i = (v_i\, a_i)_\emptyset\, b_i = (v_i\, b_i)_0\, a_i + (a_i\, b_i)_0\, v_i = 0$. Analogously, $\tau_i\, a_i = 0$ and $\tau_i\, b_i = w_i$.

Moreover, by (8), $\mu_i\, a_j = (v_i\, a_i)_\emptyset\, a_j = v_i\, a_j \cdot a_i + v_i \cdot a_i\, a_j = v_i\, d_{ij} = v_j$.

Analogously, we prove that $\mu_i\, b_j = 0$, $\tau_i\, b_j = w_i$, $\tau_i\, a_j = 0$. Hence $\mu = \mu_1 = \cdots = \mu_\ell$ and $\tau = \tau_1 = \cdots = \tau_\ell$. Furthermore, by (13), $\lambda\mu = (a_1\, b_1)_\emptyset\, \mu =$

$(a_1\,\mu \cdot b_1)_\emptyset \;+\; (a_1 \cdot \mu\, b_1)_\emptyset \;=\; (v_1\, b_1)_\emptyset \;=\tau$ and, analogously, $\lambda\tau \;=\; \mu$. Hence $V \;= < v_1, \ldots, v_\ell, \tau, \mu >$, is the standard M-module.

2. Suppose $V_0 \neq 0$. As in case 1, we can prove that V is the adjoint M-module. Thus, let $V_0 = 0$. Again as in case 1 we can prove that there exists $\mu = (12)$ such that $V_\mu \neq 0$. Denote, as in the previous case,

$$v_i = v_1\, d_{1i}, \qquad (v_i\, a_i)_\emptyset = \mu_i \qquad (v_i\, b_i)_\emptyset = \tau_i \qquad v_i\, d_{ij} = v_j. \tag{22}$$

By (11), $\mu_i\, a_i \;=\; (v_i\, a_i)_\emptyset\, a_i \;=\; (v_i\, a_i)_0\, a_i \;+\; (a_i\, a_i)_0\, v_i \;=\; 0$ and $\mu_i\, b_i \;=\; v_i$, $\tau_i\, a_i \;=\; v_i$, $\tau_i\, b_i \;=\; 0$.

By (8), $\mu_i\, a_j \;=\; (v_i\, a_i)_\emptyset\, a_j \;=\; v_i\, a_j \cdot a_i \;+\; v_i \cdot a_i\, a_j \;=\; 0$, $\mu_i\, b_j \;=\; (v_i\, a_i)_\emptyset\, b_j \;=\; v_i\, b_j \cdot a_i + v_i \cdot a_i\, b_j \;=\; v_i\, d_{ij} \;=\; v_j$. Analogously, $\tau_i\, a_j \;=\; v_i$, $\tau_i\, b_j \;=\; 0$. Hence $\mu = \mu_1 = \cdots = \mu_\ell$ and $\tau = \tau_1 = \cdots = \tau_\ell$. Furthermore, by (13), $\lambda\mu \;=\; (a_1\, b_1)_\emptyset\, \mu \;=\; (a_1\,\mu \cdot b_1)_\emptyset \;+\; (a_1 \cdot \mu\, b_1)_\emptyset \;=\; (a_1\, v_1)_\emptyset \;=\; \mu$ and, analogously, $\lambda\tau \;=\; \tau$. □

Recall some well known facts about quadratic forms over an algebraically closed field of characteristic 2 and its corresponding Lie algebras. Let V be a n-dimensional k-space and $f \;:\; V \times V \longrightarrow k$ be a non degenerated symmetric bilinear form. This means that $f(x,y) \;=\; f(y,x)$, for all $x,\, y \,\in\, V$ and $f(x,V) \;=\; 0$ implies $x \;=\; 0$. A non degenerated symmetric bilinear form f is called symplectic if $f(x,x) \;=\; 0$ and orthogonal otherwise. A vector space V has a unique orthogonal form f and in some basis $\{v_1, \ldots, v_n\}$ the form can be written as

$$f(v,w) \;=\; \sum_{i=1}^{n} x_i\, y_i$$

where $v = \sum_{i=1}^{n} x_i\, v_i$ and $w = \sum_{i=1}^{n} y_i\, v_i$.

A vector space V does not have a symplectic form if $\dim V$ is odd and has a unique symplectic form if $\dim V \;= 2\ell$. In this last case, the form can be written, in an appropriate basis $\{v_1, \ldots, v_\ell, w_1, \ldots, w_\ell\}$, as follows

$$f(v,w) \;=\; \sum_{i=1}^{\ell} (x_i t_i \;+\; y_i z_i)$$

where $v = \sum_{i=1}^{\ell} (x_i v_i + y_i w_i)$ and $w = \sum_{i=1}^{\ell} (z_i v_i + t_i w_i)$.

Let $End(V)$ be the associative algebra of all linear transformations of V. Consider the following sets

$$S(f) = \{a \in End(V) \mid f(va, w) = f(v, wa),\ \forall\, v, w \in V\},$$

$$O(f) = \{a \in End(V) \mid f(va, v) = 0,\ \forall\, v \in V\}.$$

It is clear that $O(f) \subseteq S(f)$. For f orthogonal, we denote $D_\ell = O(f)$ when $\dim V = 2\ell$ and $B_\ell = O(f)$ when $\dim V = 2\ell + 1$. For f symplectic, $C_\ell = O(f)$.

Theorem 2.3. *In the notation above we have*

1. $[S(f), S(f)] = O(f)$.

2. $Z(S(f)) = 0$ *if* $\dim V = 2\ell + 1$ *and* $Z(O(f)) = 1$ *if* $\dim V = 2\ell$.

3. $O(f)/Z(O(f))$ *is simple if* $\dim V > 2$ *and* $\dim V \neq 4$.

4. C_ℓ *is a* 2*-algebra.*

5. B_ℓ *and* D_ℓ *are not* 2*-algebras and* $S(f)$ *is the* 2*-envelope of* B_ℓ (D_ℓ) *in* $End(V)$.

6. $\dim C_\ell = 2\ell^2 - \ell$, $\dim B_\ell = 2\ell^2 + \ell$ *and* $\dim D_\ell = 2\ell^2 - \ell$.

Theorem 2.4. *Let M be a simple Δ-algebra in a Δ-variety $\mathcal{M}$ as described above and $L = M \square \Lambda$ be the corresponding Lie algebra. Then*

1. $L = C_{2\ell}$ *if* M *has a basis* $\{s, a_{ij} \mid 1 \leq i < j \leq \ell\}$, *where* $a_{ij} \in M_{(2i-1,2i,2j-1,2j)}$.

2. $L = B_{2\ell}$ *if* M *has a basis* $\{s, a_{ij}, a_i \mid 1 \leq i < j \leq \ell\}$, *where* $a_{ij} \in M_{(2i-1,2i,2j-1,2j)}$, $a_i \in M_{(2i-1,2i)}$.

3. $L = D_{2\ell+1}$ *or* $C_{2\ell+1}$ *if* M *has a basis* $\{s, a_{ij}, a_i, b_i, \lambda \mid 1 \leq i < j \leq \ell\}$, *where* $a_{ij} \in M_{(2i-1,2i,2j-1,2j)}$ *and the multiplication rules are given by* (16) *or* (17).

4. L *is a Lie algebra of type* E_7 *or* E_8, *if* $\mathfrak{M} = \sigma \mid M_\sigma \neq 0\} = \mathcal{E}_7$ *or* $\mathcal{E}_8$.

Proof. 1. By Theorem???? [GG], a Δ-algebra M has a module V with a basis $\{v_1, \ldots, v_\ell\}$, $v_i \in V_{(2i-1,2i)}$. This M-module admits an M-invariant bilinear form given by $(v_i, v_j) = \delta_{ij}$. Note that if a M-module $V = V_0 \oplus \sum \oplus V_\sigma$ admits an M-invariant symmetric bilinear form f, then the corresponding L-module $W = V \square \Lambda$ admits a L-invariant symmetric bilinear form as follows:

$$\tilde{f}(v \otimes x,\, w \otimes y) \;=\; f(v,\, w)\,(x,\, y), \qquad\qquad v,\, w \in V,\; x,\, y \in \Lambda$$

Moreover, $\tilde{f}$ is symplectic (orthogonal) if and only if the restriction of f to $V_0 \oplus V_\emptyset$ is symplectic (orthogonal). In our case, $V_0 \oplus V_\emptyset = 0$ hence this form is non degenerated and symplectic. As $\dim W = 4\ell$ and $\dim L = 8\ell^2 - 2\ell$, we have that $L = C_{2\ell}$.

2. and 3. In all this cases M has a module V with a basis $\{v_1, \ldots, v_\ell,\, \mu,\, \tau\}$ described in Theorem 2.2, with $v_i \in V_{(2i-1,2i)}$. The M-module V admits an M-invariant bilinear form given by $(v_i, v_j) = \delta_{ij}$ and $(\lambda, \mu) = 0$, $(\lambda, \lambda) = (\mu, \mu) = 1$, if M has multiplication rules defined by (16) or $(v_i, v_j) = \delta_{ij}$ and $(\lambda, \mu) = 1$, $(\lambda, \lambda) = (\mu, \mu) = 0$, if M has multiplication rules defined by (17).

In the first case, the corresponding L-invariant bilinear form on the L-module $W = V \square \Lambda$ is orthogonal and, in the second case, it is symplectic. As $\dim L = 8\ell^2 + 6\ell + 1$, then $L = D_{2\ell+1}$ in the first and $L = C_{2\ell+1}$, in the second case.

4. We prove this statement in the case $\mathcal{E}_8$. The case $\mathcal{E}_7$ is corollary of this.

By definition, a Lie algebra L over a field k of characteristic 2 is a Lie algebra of type E_8 if there exists a $\mathbf{Z}$-form $\mathcal{L}_{\mathbf{Z}}$ of the Lie algebra $\mathcal{L}$ over the field $\mathbf{C}$ of all complex numbers such that $L = \mathcal{L}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$.

Let $\mathcal{L}$ be the Lie algebra of type E_8 over $\mathbf{C}$] constructed in [G3] with a basis

$$\{\, e_1,\, f_1, \ldots,\, e_8,\, f_8,\, h_1, \ldots,\, h_8,\, (\sigma, \mu),\, \mu q \sigma \in \mathcal{E}_8 \,\}$$

and multiplication rules stated by Theorem 1 [G3].

Let $\mathcal{L}_{\mathbf{Z}}$ be a $\mathbf{Z}$- module with generators $\{\, e_i,\, f_i,\, h_i,\, i = 1, \ldots, 8,\, (\sigma, \mu),\, h^\sigma =$

$\frac{1}{2}(\sum_{i\in\sigma} h_i),\ \mu \subseteq \sigma \in \mathcal{E}_8\}$.

Note that $[\mathcal{L}_{\mathbf{Z}}, \mathcal{L}_{\mathbf{Z}}] \subseteq \mathcal{L}_{\mathbf{Z}}$, since for $\varphi \cap \psi = \emptyset, \varphi \cup \psi = \sigma$ we have, by Theorem 1 [G3], that

$$(\sigma,\varphi)(\sigma,\psi) \;=\; (-1)^{|\psi|+1}(\sum_{i\in\psi} h_i - \sum_{j\in\varphi} h_j)/2 \;=\; (-1)^{|\psi|+1}(h^\sigma - \sum_{j\in\varphi} h_j)\,,$$

$$(\sigma,\mu)h^\tau \;=\; \frac{1}{2}(|\ \mu\cap\tau\ |\ -\ |\ \overline{\mu}\cap\tau\ |)(\sigma,\mu),\ \text{ where } \overline{\mu} = \sigma\setminus\mu. \tag{23}$$

But $(|\ \mu\cap\tau\ |\ -\ |\ \overline{\mu}\cap\tau\ |)\ =\ (|\ \sigma\cap\tau\ |\ -2\ |\ \overline{\mu}\cap\tau\ |) \equiv |\ \sigma\cap\tau\ | \equiv 0 (mod\ 2)$. Hence $(\sigma,\mu)h^\tau \in \mathcal{L}_{\mathbf{Z}}$.

Now we prove that $L \cong \mathcal{L}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$. Define $\xi : L \longrightarrow \mathcal{L}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k$ given by $\xi(e_i) = e_i$, $\xi(f_i) = f_i$, $\xi(\sigma,\mu) = (\sigma,\mu)$, $\xi(h_i) = h_i$ $\xi(h^\sigma) = h^\sigma$. (Note that although the notation for the elements being the same, they are in two diferent algebras.)

To prove that ξ is an algebra isomorphism, it is enough to prove that $\xi((\sigma,\mu)h^\varphi) = [\xi(\sigma,\mu), \xi(h^\varphi)]$ (*). By (2),

$$\xi((\sigma,\,\mu)\,h^\varphi) = \left(\frac{|\sigma\cap\varphi|}{2} + |\varphi\cap\mu|\right)\,\xi((\sigma,\,\mu))$$

and by (23)

$$\xi((\sigma,\mu))\,\xi(h^\varphi) \;=\; \frac{1}{2}(|\ \mu\cap\varphi\ |\ -\ |\ \overline{\mu}\cap\varphi\ |)\xi((\sigma,u)).$$

But $\frac{1}{2}(|\ \mu\cap\varphi\ |\ -\ |\ \overline{\mu}\cap\varphi\ |)\ =\ -\frac{1}{2}(|\ \mu\cap\varphi\ |\ +\ |\ \overline{\mu}\cap\varphi\ |) +\ |\ \mu\cap\varphi\ | = -\frac{1}{2}(|\ \sigma\cap\varphi\ |\ +\ |\ \mu\cap\varphi\ | \equiv \frac{|\sigma\cap\varphi|}{2} + |\varphi\cap\mu|$ and the equality (*) holds.

□

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