# Simple classical Lie algebras in characteristic 2 and their gradations, I.

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## 1 Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $B$ be a Chevalley $\mathbb{Z}$-form of a finite dimensional complex simple Lie algebra. The Lie algebra $A = (B \otimes_{\mathbb{Z}} k)/Z$, where $Z$ is the centre of $B \otimes_{\mathbb{Z}} k$, is called a classical Lie algebra over $k$. This is a universal definition of classical Lie algebras over $k$ [Hu1]. Obviously, this definition is external with respect to the field $k$. If $p > 3$, then there exists an internal characterization of classical Lie algebras given by the following theorem.

**Theorem 1.1.** *[Pr] A Lie algebra $L$ over a field $k$ of characteristic $p > 3$ is classical if and only if $L$ has no elements $a$ such that $(\mathrm{ad}(a))^2 = 0$.*

**Definition 1.1.** *Let $L$ be an algebra and $S \cong sl_2(k)$ be a subalgebra of $Der\, L$ (the Lie algebra of derivations of $L$). We say that a pair $(L, S)$ is* **semisimple**

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*if* $L$, *as an* $S$*-module, is of the form*

$$L = \sum_i \oplus S_i \oplus \sum_j \oplus V_j$$

*where each* $V_j$ *is an irreducible two dimensional* $S$*-module each* $S_i$ *is a* $S$*-submodule of* $M_2(k)$*, where* $M_2(k)$ *is the space of 2 by 2 matrices with the following action of* $sl_2(k)$*:* $x \cdot y = xy - yx$.

Now we can formulate the following conjecture.

**Conjecture 1.1.** *A finite dimensional Lie algebra* $L$ *over a field* $k$ *is classical if and only if there exists a semisimple pair* $(L,\, S)$.

In this work we give the first steps towards proving this conjecture for the case where $k$ has characteristic 2. We note that in characteristic $p > 2$ the algebra $S$ is semisimple but if $k$ has characteristic 2 then $S$ is nilpotent.

Let $V$ be a $k$-vector space of finite dimension $n$ and $f$ be a symmetric non-degenerate bilinear form on $V$. Consider the $k$-space $\{X \in End(V) \,|\, f(vX, v) = 0,\ \forall\ v \in V\}$ and denote it by $B_\ell$ if $n = 2\ell + 1$, by $C_\ell$ ($D_\ell$) if $n = 2\ell$ and $f(v,v) = 0, \forall v \in V$ (resp. there exists $v \in V$ such that $f(v,v) = 1$). Note that $B_\ell$, $C_\ell$ and $D_\ell$ are Lie algebras.

From now on we assume that $char\ k = 2$. In Theorem 3.1 of this paper, we classify all semisimple pairs $(L,\, S)$ where $S$ belongs to a family of nilpotent algebras that we define in Section 3. As a corollary, we obtain a construction of the simple Lie algebras over a field of characteristic 2 of types $B_{2\ell}$, $C_{2\ell}$, $E_7$ and $E_8$ and some representations of these algebras.

## 2 Even sets

**Definition 2.1.** *Let* $I_n = \{1, \ldots, n\}$. *We call* $\mathfrak{a} \subset \mathcal{P}(I_n) = \{\sigma \,|\, \sigma \subseteq I_n\}$ *an* **even** *set if for all* $\sigma, \tau \in \mathfrak{a}$*, we have* $|\sigma| \equiv |\tau| \equiv 0$ *and* $|\sigma \cap \tau| \equiv 0 \mod 2$.

We note that $\mathcal{P}(I_n)$ is an elementary abelian group with the operation $\sigma \triangle \tau = (\sigma \setminus \tau) \cup (\tau \setminus \sigma)$.

**Lemma 2.1.** *If $\mathfrak{a}$ is an even set, then so is $< \mathfrak{a} >$ (the group generated by $\mathfrak{a}$).*

*Proof.* If $\mathfrak{a}$ is even and $\sigma, \tau \in \mathfrak{a}$, then $|\sigma|, |\tau|, |\sigma \cap \tau|, |\sigma \setminus \tau|, |\tau \setminus \sigma|$ are all even numbers. Hence $|\sigma \triangle \tau| \equiv 0 \mod 2$. Now let also $\varphi \in \mathfrak{a}$, then $|\varphi \setminus \sigma| \equiv 0$ and $|\varphi \setminus \tau| \equiv 0 \mod 2$. But as $\varphi \cap \sigma = ((\varphi \cap \sigma) \setminus \tau) \cup (\varphi \cap \sigma \cap \tau)$ and $\varphi \cap \tau = ((\varphi \cap \tau) \setminus \sigma) \cup (\varphi \cap \tau \cap \sigma)$, then $|\varphi \cap \sigma| + |\varphi \cap \tau| \equiv |(\varphi \cap \sigma) \setminus \tau| + |(\varphi \cap \tau) \setminus \sigma| = |\varphi \cap (\sigma \triangle \tau)| \equiv 0 \mod 2$. □

**Definition 2.2.** *A subset $\sigma \subseteq I_n$ is called* **$\mathfrak{a}$-even** *if $|\mu \cap \tau| \equiv 0 \mod 2$ for all $\tau \in \mathfrak{a}$. A subset $B \subseteq \mathcal{P}(I_n)$ is called an* **$\mathfrak{a}$-even set** *if all its elements are* **$\mathfrak{a}$-even**.

Now as an easy corollary of Lemma 2.1 we get:

**Lemma 2.2.** *If $\sigma \subseteq I_n$ is $\mathfrak{a}$-even, then $\sigma$ is $< \mathfrak{a} >$-even.*

For an even set $\mathfrak{a} \subset \mathcal{P}(I_n)$, we introduce a commutative algebra $\tilde{S} = \tilde{S}(\mathfrak{a})$ with basis $\{ e_i, h_i, f_i, h^\sigma \mid i \in I_n, \sigma \in < \mathfrak{a} > \setminus \emptyset \}$ and multiplication given by

$$\begin{aligned} & e_i f_i = h_i, \\ & e_i h^\sigma = e_i, \quad f_i h^\sigma = f_i, \qquad \text{for } i \in \sigma, \end{aligned} \tag{1}$$

and zero for all other cases. Define $h^\emptyset = 0$.

**Definition 2.3.** *A subset $H$ of $\mathcal{P}(I_n)$ is* **connected** *if, for every partition $I_n = I \cup J$, there is $\sigma \in H$ such that $\sigma \cap I \neq \emptyset$ and $\sigma \cap J \neq \emptyset$.*

# 3 Module Algebras

The definitions given in this section follow the ideas developed by Grishkov in [G1], [G2] where he describes a new way of writing a basis for Lie algebras by connecting them to a category of graded algebras.

Let $\mathfrak{a} \subset \mathcal{P}(I_n)$ be an even set. For $\mu \subseteq I_n$, denote $h_\mu = \sum_{i \in \mu} h_i$ and $h_\emptyset = 0$. It is easy to show that the algebra $\tilde{S}(\mathfrak{a})$ contains a central ideal $I$ generated by $\{h^\sigma + h^\tau + h^{\sigma \triangle \tau} + h_{\sigma \cap \tau} \mid \sigma, \tau \in <\mathfrak{a}>\}$. We denote

$$S(\mathfrak{a}) = S = \tilde{S}(\mathfrak{a})/I. \tag{2}$$

Note that $h^\sigma + h^\sigma + h^\emptyset + h_\sigma \in I$, so $h_\sigma \in I$. Hence, in $S(\mathfrak{a})$, $h_\sigma = 0$, since $h^\emptyset = 0$.

For every $\mathfrak{a}$-even set $\sigma$, define an $S$-module $\Lambda_\sigma$ whose basis is $\{(\sigma, \mu) \mid \mu \subseteq \sigma\}$ and the $S$-action is given by

$$\begin{aligned}
(\sigma, \mu)\, e_i &= (\sigma, \mu \cup i), \quad i \in \sigma \setminus \mu; \\
(\sigma, \mu)\, f_i &= (\sigma, \mu \setminus i), \quad i \in \mu; \\
(\sigma, \mu)\, h_i &= (\sigma, \mu), \quad i \in \sigma; \\
(\sigma, \mu)\, h^\varphi &= \left( \frac{|\sigma \cap \varphi|}{2} + |\varphi \cap \mu| \right) (\sigma, \mu), \quad \text{for } \varphi \in \mathfrak{a},
\end{aligned} \tag{3}$$

and, by Lemma 2.2, for all other cases the action is zero.

To prove that this is the right definition for this $S$-module, it is sufficient to show that $(\sigma, \mu)I = 0$. Indeed, for $\varphi, \tau, \varphi \triangle \tau \in \mathfrak{a}$, we have

$$\begin{aligned}
&(\sigma, \mu)\, (h^\varphi + h^\tau + h^{\tau \triangle \varphi} + \sum_{i \in \varphi \cap \tau} h_i) = \\
&(\sigma, \mu) \quad ( (|\varphi \cap \sigma| + |\sigma \cap \tau| + |(\tau \triangle \varphi) \cap \sigma|)/2 + |\varphi \cap \mu| + |\tau \cap \mu| + |(\tau \triangle \varphi) \cap \mu| \\
&+ \quad |\sigma \cap \varphi \cap \tau|) = ((|\varphi \cap \sigma \cap \tau| + |(\varphi \setminus \tau) \cap \sigma|)/2 + (|\sigma \cap \tau \cap \varphi| + \\
&+ \quad |(\tau \setminus \varphi) \cap \sigma|)/2 + (|(\tau \setminus \varphi) \cap \sigma| + |(\varphi \setminus \tau) \cap \sigma|)/2 + |\varphi \cap \mu \cap \tau| + \\
&+ \quad |(\varphi \setminus \tau) \cap \mu| + |\tau \cap \mu \cap \varphi| + |(\tau \setminus \varphi) \cap \mu| + |(\tau \setminus \varphi) \cap \mu| \\
&+ \quad |(\varphi \setminus \tau) \cap \mu| + |\sigma \cap \varphi \cap \tau|)\, (\sigma, \mu) \\
&= \quad (|\varphi \cap \sigma \cap \tau| + |(\varphi \setminus \tau) \cap \sigma| + |(\tau \setminus \varphi) \cap \sigma| + |\sigma \cap \varphi \cap \tau|)\, (\sigma, \mu) \\
&= \quad |(\varphi \triangle \tau) \cap \sigma|\, (\sigma, \mu) = 0, \quad \text{since } \sigma, \varphi \triangle \tau \in \mathfrak{a}.
\end{aligned}$$

Then $(\sigma, \mu)\, I = 0$, as required.

Now let $\Delta = \{0\} \cup \mathfrak{a}$.

**Definition 3.1.** *An algebra* $A$ *is called a* **$\Delta$-algebra** *if* $A = \sum_{\alpha\in\Delta} \oplus A_\alpha$ *and, for every* $\alpha \neq \beta \in \mathfrak{a}$, *we have* $A_\alpha A_\beta \subseteq A_{\alpha\Delta\beta}$, $A_0^2 \subseteq A_0$, $A_0 A_\alpha \subseteq A_\alpha$, $A_\alpha A_\alpha \subseteq A_0 + A_\emptyset$ *and* $A_0 A_\emptyset = 0$.

Define a commutative $\Delta$-graded algebra $\Lambda$ as follows. As a $k$-space, $\Lambda$ is

$$\Lambda = \Lambda_0 \bigoplus \sum_{\sigma\in\mathfrak{a}} \oplus \Lambda_\sigma, \quad \text{where } \Lambda_0 = S(\mathfrak{a}). \tag{4}$$

Moreover, $S = S(\mathfrak{a})$ is a subalgebra of $\Lambda$ and, by (3), each $\Lambda_\sigma$ is an $S$-module. For $\sigma \neq \tau \in \mathfrak{a}$, the multiplication is given by

$$(\sigma,\, \mu)\,(\tau,\, \varphi) = (\sigma\Delta\tau,\, (\mu\setminus\tau)\cup(\varphi\setminus\sigma)), \ \text{ if } \mu\cap\varphi = \emptyset,\ \mu\cup\varphi \supset \sigma\cap\tau\,. \tag{5}$$

$$(\sigma,\, \mu)\,(\sigma,\, \varphi) = \begin{cases} e_i, & \mu\cap\varphi = i,\ \mu\cup\varphi = \sigma, \\ f_i, & \mu\cap\varphi = \emptyset,\ \mu\cup\varphi = \sigma\setminus i, \\ h^\sigma + h_\varphi + (\emptyset,\emptyset), & \mu\cap\varphi = \emptyset,\ \mu\cup\varphi = \sigma\,, \end{cases} \tag{6}$$

and all other products are zero.

Note that in the last case of (6), since $h_\varphi + h_\mu = h_\sigma = 0$, then

$$(\sigma,\, \mu)\,(\sigma,\, \varphi) = h^\sigma + h_\varphi + (\emptyset,\emptyset) = (\sigma,\, \varphi)\,(\sigma,\, \mu) = h^\sigma + h_\mu + (\emptyset,\emptyset).$$

**Proposition 3.1.** *Let* $\mathfrak{a} \subset \mathcal{P}(I_n)$ *be an even set and* $\Lambda$ *be the algebra defined before. Then*

$$Z(\Lambda) = \{h_\mu \mid \mu \subseteq I_n,\ \mu \textit{ is } \mathfrak{a} - \textit{even}\}.$$

*Proof.* Let $h = \sum_{i=1}^{n} \alpha_i\, h_i \in Z(\Lambda)$ with $\alpha_1, \ldots, \alpha_n \in k$. Then $[(\sigma,\sigma), h] = \left(\sum_{i\in\sigma} \alpha_i\right)(\sigma,\sigma) = 0$ so that

$$\sum_{i\in\sigma} \alpha_i = 0, \qquad \forall\, \sigma \in \mathfrak{a}. \tag{7}$$

Let us consider (7) as a linear system with coefficients in the field $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$. Then there exists a basis $\{v_1, \ldots, v_m\}$ of solutions of this system defined over $\mathbf{F}_2$. This means that

$$v_i = (v_{i1}, \ldots, v_{in}), \qquad v_{ij} \in \mathbf{F}_2,\ i = 1, \ldots, m.$$

For each $i$, denote $\mu_i = \{j \in I_n \mid v_{ij} = 1\}$ and $h_\mu = \sum_{i\in\mu} h_i$. From (7), it follows that $\sum_{j\in\sigma} v_{ij} = \sum_{j\in\sigma\cap\mu_i} v_{ij} = |\ \sigma \cap \mu_i\ | = 0$. Hence $\mu_1, \ldots, \mu_m$ are all $\mathfrak{a}$-even sets and $h_{\mu_1}, \ldots, h_{\mu_m}$ is a basis of $Z(\Lambda)$. □

Recall the definition of the product of two $\Delta$-algebras. Let $A = \sum_{\alpha\in\Delta} \oplus A_\alpha$ and $B = \sum_{\alpha\in\Delta} \oplus B_\alpha$ be two $\Delta$-algebras. Then $A\Box B = \sum_{\alpha\in\Delta} \oplus A_\alpha \otimes B_\alpha$ is a $\Delta$-algebra with multiplication $[\cdot, \cdot]$ given by

$$[a_\alpha \otimes b_\alpha, a_\beta \otimes b_\beta] = \sum_{\gamma\in\Delta} c_\gamma \otimes d_\gamma, \qquad \text{if } a_\alpha a_\beta = \sum_{\gamma\in\Delta} c_\gamma,\ b_\alpha b_\beta = \sum_{\gamma\in\Delta} d_\gamma.$$

**Proposition 3.2.** *Let $\mathfrak{a}$ be an even set, $\Lambda = \Lambda(\mathfrak{a})$ and $\Delta = \{0\}\cup\mathfrak{a}$. Let $M = M_0 \oplus \sum_{\sigma\in\mathfrak{a}} \oplus M_\sigma$ be a commutative $\Delta$-algebra. Then the algebra $L = \Lambda\Box M$ is a Lie algebra if and only if $M$ satisfies the following $\Delta$-identities:*

$$a_\sigma b_\tau \cdot c_\lambda = 0, \qquad |\sigma\cap\lambda\cap\tau| > 1,\ \sigma\neq\tau\neq\lambda\neq\sigma\neq\tau\triangle\lambda, \tag{8}$$

$$a_\sigma b_\tau \cdot c_\lambda = a_\sigma \cdot b_\tau c_\lambda, \qquad |\sigma\cap\tau\cap\lambda| = 1, \sigma\neq\tau\neq\lambda\neq\sigma\neq\tau\triangle\lambda, \tag{9}$$

$$a_\sigma b_\tau \cdot c_\lambda + b_\tau c_\lambda \cdot a_\sigma + c_\lambda a_\sigma \cdot b_\tau = 0, |\sigma\cap\tau\cap\lambda| = 0, \sigma\neq\tau\neq\lambda\neq\sigma\neq\tau\triangle\lambda, \tag{10}$$

$$(a_\sigma b_\sigma)_0 c_\tau = (a_\sigma b_\sigma)_\emptyset c_\tau = 0, \qquad \sigma\neq\tau,\ |\sigma\cap\tau| > 2, \tag{11}$$

$$(a_\sigma b_\sigma)_\emptyset c_\tau = 0,\ (a_\sigma b_\sigma)_0 c_\tau = a_\sigma c_\tau \cdot b_\sigma, \qquad \sigma\neq\tau,\ |\sigma\cap\tau| = 2, \tag{12}$$

$$a_\sigma c_\tau \cdot b_\sigma = a_\sigma \cdot c_\tau b_\sigma + (a_\sigma b_\sigma)_\emptyset c_\tau, \qquad |\sigma\cap\tau| = 0, \tag{13}$$

$$(a_\sigma b_\tau \cdot c_\lambda)_0 = (a_\sigma \cdot b_\tau c_\lambda)_0, \qquad \lambda = \sigma\triangle\tau, \tag{14}$$

$$(ab)_0 c + \left(\frac{|\tau|}{2} + 1\right)(ca)_0 b + (ca)_\emptyset b = 0, \qquad a, b, c \in M_\tau,\ |\tau| > 2, \tag{15}$$

$$(ab)_0 c = (cb)_0 a,\ \ (ca)_\emptyset b = 0, \qquad a, b, c \in M_\tau,\ |\tau| = 4, \tag{16}$$

$$(ab)_0 c + (bc)_0 a = (ac)_\emptyset b, \qquad a, b, c \in M_\tau,\ |\tau| = 2, \tag{17}$$

$$(a_0 x)y = a_0(xy), \qquad \forall\ x, y \in M, \tag{18}$$

$$(a_\sigma b_\sigma)_0 c_\tau = 0, \qquad |\sigma| > 4, \tag{19}$$

$$(a_\sigma b_\tau \cdot c_\lambda)_\emptyset \,+\, (b_\tau c_\lambda \cdot a_\sigma)_\emptyset \,+\, (c_\lambda a_\sigma \cdot b_\tau)_\emptyset \,=\, 0, \qquad \sigma = \lambda \triangle \tau, \quad (20)$$

$$(ab)_\emptyset c \,+\, (ac)_\emptyset b \,=\, (ab)_0 c \,+\, (ac)_0 b,, \qquad a, b, c \in M_\sigma, |\sigma| = 2, \quad (21)$$

$$a_\emptyset \cdot (b_\sigma c_\tau) \,=\, (a_\emptyset b_\sigma) \cdot c_\tau\,, \qquad \sigma \neq \tau,\ \sigma \neq \emptyset\ \ or\ \tau \neq \emptyset, \quad (22)$$

$$(a_\emptyset b_\sigma \cdot c_\sigma)_\emptyset \,+\, (b_\sigma c_\sigma)_\emptyset \cdot a_\emptyset \,+\, (c_\sigma a_\emptyset \cdot b_\sigma)_\emptyset \,=\, 0 \qquad \forall\ \sigma, \quad (23)$$

$$a_\sigma b_0 \cdot c_0 \,=\, a_\sigma \cdot b_0 c_0\,, \qquad \sigma \neq \emptyset, \quad (24)$$

$$(a_\sigma b_\sigma)_0 \cdot c_0 \,=\, (a_\sigma c_0 \cdot b_\sigma)_0\,, \qquad \sigma \neq \emptyset, \quad (25)$$

$$(a_\sigma c_0 \cdot b_\sigma)_\emptyset \,=\, (b_\sigma c_0 \cdot a_\sigma)_\emptyset\,, \qquad \sigma \neq \emptyset, \quad (26)$$

$$(a_\tau\, x)\, y \,+\, (a_\tau\, y)\, x \,+\, a_\tau\, (x\, y) \,=\, 0 \qquad x,\, y \,\in\, M_\emptyset, \quad (27)$$

$$(a_\emptyset b_\sigma \cdot c_\sigma)_0 \,=\, (a_\emptyset c_\sigma \cdot b_\sigma)_0\,, \qquad \sigma \neq \emptyset. \quad (28)$$

*Proof.* Suppose that $L = \Lambda \Box M$ is a Lie algebra.

Let $a = (\sigma, \mu) \otimes a_\sigma,\ b = (\tau, \varphi) \otimes b_\tau,\ c = (\lambda, \psi) \otimes c_\lambda$ be elements in $L$. Set $t_1 = [[a, b], c],\ t_2 = [[b, c], a],\ t_3 = [[a, c], b]$. By Jacobi's identity we must have $t_1 + t_2 + t_3 = 0$. Thus, if $\mu = \sigma,\ \varphi = \emptyset,\ \psi = (\lambda \setminus \sigma) \cup i,\ i \in \sigma \cap \tau \cap \lambda$, then we have $t_1 = [(\sigma \triangle \tau, \sigma \setminus \tau) \otimes a_\sigma b_\tau, (\lambda, (\lambda \setminus \sigma) \cup i)) \otimes c_\lambda] \,=\, (\sigma \triangle \tau \triangle \lambda,\ ((\sigma \triangle \lambda) \setminus \tau) \cup i) \,\otimes\, (a_\sigma b_\tau) c_\lambda \,=\, 0$, because $t_2 = [[(\sigma, \sigma) \otimes a_\sigma, (\lambda, (\lambda \setminus \sigma) \cup i) \otimes c_\lambda], (\tau, \emptyset) \otimes b_\tau] \,=\, 0$ since $\sigma \cap ((\lambda \setminus \sigma) \cup i) \,=\, i \neq \emptyset$ as $i \in \sigma$ (so identity (5) does not apply) and $t_3 = [[(\tau, \emptyset) \otimes b_\tau, (\lambda, (\lambda \setminus \sigma) \cup i) \otimes c_\lambda], (\sigma, \sigma) \otimes a_\sigma] \,=\, 0$, as there exists $j \in (\sigma \cap \tau \cap \lambda) \setminus i$ and $j \notin (\lambda \setminus \sigma) \cup i$. Therefore, $(a_\sigma b_\tau) c_\lambda = 0$ and this proves (8).

Now if $\mu = \sigma,\ \varphi = \emptyset$ and $\psi = (\lambda \setminus \sigma) \cup i,\ i = \sigma \cap \tau \cap \lambda$, then we have $t_1 = [(\sigma \triangle \tau, \sigma \setminus \tau) \otimes a_\sigma b_\tau, (\lambda, (\lambda \setminus \sigma) \cup i)) \otimes c_\lambda] \,=\, (\sigma \triangle \tau \triangle \lambda, ((\sigma \triangle \lambda) \setminus \tau) \cup i) \otimes a_\sigma b_\tau \cdot c_\lambda$, $t_2 = [[(\tau \triangle \lambda, ((\lambda \setminus \sigma) \cup i) \setminus \tau)) \otimes b_\tau c_\lambda, (\sigma, \sigma) \otimes a_\sigma] \,=\, (\tau \triangle \lambda \triangle \sigma, ((\sigma \triangle \lambda) \setminus \tau) \cup i) \otimes b_\tau c_\lambda \cdot a_\sigma$ and $t_3 = [[(\sigma, \sigma) \otimes a_\sigma, (\lambda, (\lambda \setminus \sigma) \cup i) \otimes c_\lambda], (\tau, \emptyset) \otimes b_\tau] \,=\, 0$, since $i \in \sigma$. This proves (9).

If $\mu = \sigma,\ \varphi = \emptyset,\ \psi = \tau \cap \lambda$ then $t_1 \,=\, (\sigma \triangle \tau \triangle \lambda, \sigma \setminus (\tau \triangle \lambda)) \otimes a_\sigma b_\tau \cdot c_\lambda$, $t_2 = (\sigma \triangle \tau \triangle \lambda, \sigma \setminus (\tau \triangle \lambda)) \otimes b_\tau c_\lambda \cdot a_\sigma$, $t_3 = (\sigma \triangle \tau \triangle \lambda, \sigma \setminus (\tau \triangle \lambda)) \otimes a_\sigma c_\lambda \cdot b_\tau$. Hence

$t_1 + t_2 + t_3 = 0$ implies (10).

For $\mu = \tau = \lambda = \sigma$, $\varphi = \psi = \emptyset$, we have $t_1 = [h^\sigma + (\emptyset, \emptyset) \otimes (ab)_0, (\sigma, \emptyset) \otimes c] = (\sigma, \emptyset) \otimes (ab)_0 c + (ab)_\emptyset c$; $t_2 = [h^\sigma + (\emptyset, \emptyset) \otimes (ac)_0, (\sigma, \emptyset) \otimes b] = (\sigma, \emptyset) \otimes (ac)_0 b + (ac)_\emptyset b$ and $t_3 = [[(\sigma, \emptyset) \otimes b, (\sigma, \emptyset) \otimes c], (\sigma, \sigma) \otimes a] = 0$. This proves (21).

For (23) and (28), by (5) and (6), $t_1 = [[(\emptyset, \emptyset) \otimes a_\emptyset, (\sigma, \sigma) \otimes b_\sigma], (\sigma, \emptyset) \otimes c_\sigma] = ((a_\emptyset b_\sigma) c_\sigma)_0 \otimes h^\sigma + ((a_\emptyset b_\sigma) c_\sigma)_\emptyset \otimes (\emptyset, \emptyset)$, $t_2 = [[(\emptyset, \emptyset) \otimes a_\emptyset, (\sigma, \emptyset) \otimes c_\sigma], (\sigma, \sigma) \otimes b_\sigma] = ((a_\emptyset c_\sigma) b_\sigma)_0 \otimes h^\sigma + ((a_\emptyset c_\sigma) b_\sigma)_\emptyset \otimes (\emptyset, \emptyset)$ and $t_3 = [[(\sigma, \sigma) \otimes b_\sigma, (\sigma, \emptyset) \otimes c_\sigma], (\emptyset, \emptyset) \otimes a_\emptyset] = (b_\sigma c_\sigma)_\emptyset a_\emptyset \otimes (\emptyset, \emptyset)$. Hence, as our algebra is $\Delta$-graded, we get $(a_\emptyset b_\sigma \cdot c_\sigma)_\emptyset + (b_\sigma c_\sigma)_\emptyset \cdot a_\emptyset + (c_\sigma a_\emptyset \cdot b_\sigma)_\emptyset = 0$ and $(a_\emptyset b_\sigma \cdot c_\sigma)_0 = (a_\emptyset c_\sigma \cdot b_\sigma)_0$, as required.

For (25) and (26), by (3) and (6), for $i \in \sigma$, $t_1 = [[(\sigma, \sigma) \otimes a_\sigma, (\sigma, i) \otimes b_\sigma], f_i \otimes c_0] = ((a_\sigma b_\sigma)_0 c_0)_0 \otimes h_i$; $t_2 = [[(\sigma, \sigma) \otimes a_\sigma, f_i \otimes c_0], (\sigma, i) \otimes b_\sigma] = ((a_\sigma c_0) b_\sigma)_0 \otimes (h^\sigma + h_i) + ((a_\sigma c_0) b_\sigma)_\emptyset \otimes (\emptyset, \emptyset)$ and $t_3 = [[(\sigma, i) \otimes b_\sigma, f_i \otimes c_0], (\sigma, \sigma) \otimes a_\sigma] = ((b_\sigma c_0) a_\sigma)_0 \otimes (h^\sigma + \sum_{i \in \sigma} h_i) + ((b_\sigma c_0) a_\sigma)_\emptyset \otimes (\emptyset, \emptyset)$ and the identities follow because of the grading of the algebra. The proof of the other identities are left as an easy exercise to the reader. □

**Lemma 3.1.** *Let $\mathfrak{a} \subset \mathcal{P}(I_n)$ be an even set such that $\emptyset \notin \mathfrak{a}$ and $\Delta = \{0\} \cup \mathfrak{a}$. Let $\mathcal{M}$ be the variety of $\Delta$-algebras satisfying identities* (8) *to* (28)*. Let $M \in \mathcal{M}$ be a simple algebra (containing no graded ideals). If $\mathfrak{M} = \{\sigma \in \mathfrak{a} \mid M_\sigma \neq 0\}$, then*

*(a) for all $\sigma \in \mathfrak{M}$, we have $|\sigma| = 2$ or 4.*

*(b) If $\sigma \neq \tau \in \mathfrak{M}$ and $\sigma \cap \tau \neq \emptyset$ then $\sigma \Delta \tau \in \mathfrak{M}$.*

*(c) $\mathfrak{M}$ is connected.*

*Proof.* Note that, by (18), $M_0$ is commutative and associative. Moreover $M_0$ is in the associative centre of the algebra $M$. (By definition, the associative centre of an algebra $A$ is the set $C(A) = \{a \in A \mid (a, A, A) = (A, a, A) = (A, A, a) = 0\}$, where $(x, y, z) = xy \cdot z - x \cdot yz$.) If $M_0$ is not semisimple, then $M_0$ contains an element $a$ such that $a^2 = 0$. But in this case, $aM$ is a nilpotent ideal. If $M_0$ is semisimple, but not simple, then there exist two orthogonal idempotent elements $e_1$ and $e_2$ such that $e_i\, e_j = \delta_{ij} e_i$. In this case, $e_i M$ are proper ideals of $M$. Hence $M_0$ is simple and $M_0 = k\, s$ with $s^2 = s$.

We define a symmetric bilinear form $(\ ,\ )$ on the algebra $M$ as follows:

$$\begin{aligned} &(s,\, s) \;=\; 1, \\ &(a_\sigma,\, b_\sigma) \;=\; \alpha, \;\; \text{if } (a_\sigma\, b_\sigma)_0 \;=\; \alpha\, s, \\ &(a_\sigma,\, b_\tau) \;=\; 0, \;\; \text{for } \sigma \neq \tau\,. \end{aligned} \tag{29}$$

By (14) and (18), this form is invariant $(xy,\, z) = (x,\, yz)$ and non trivial, hence it is non-degenerate. But $(a_\sigma,\, b_\sigma) = 0$ if $|\sigma| > 4$, by (19). Therefore, $M_\sigma = 0$ when $|\sigma| > 4$. Thus, (a) is proved.

Now if $\sigma,\, \tau \in \mathfrak{M}$ and $\sigma \cap \tau \neq \emptyset$, $\sigma \neq \tau$, then by (12) $a_\sigma\, b_\tau \cdot c_\sigma = a_\sigma\, c_\sigma \cdot b_\tau$, since $|\sigma|,\, |\tau| \leq 4$ and $|\sigma \cap \tau| = 2$. If $M_\sigma\, M_\tau = 0$ and $a_\sigma\, c_\sigma = \alpha s + k_\emptyset$, then $0 = a_\sigma\, b_\tau \cdot c_\sigma = a_\sigma\, c_\sigma \cdot b_\tau = \alpha s\, b_\tau + k_\emptyset b_\tau$, that is, $\alpha s\, b_\tau = k_\emptyset b_\tau$. As the bilinear form is non-degenerate, $\alpha \neq 0$ for all $b_\tau \in M_\tau$, but by (12) $(a_\sigma\, c_\sigma)_\emptyset \cdot b_\tau = 0$. Hence $k_\emptyset b_\tau = 0$, a contradiction. Therefore, $M_\sigma\, M_\tau \neq 0$ and $\sigma \triangle \tau \in \mathfrak{M}$. This proves (b). Part (c) is obvious as $M$ is a simple algebra. □

We observe that if $M$ is simple, then $L = \Lambda \Box M$ is not necessarily a simple algebra, but $L/Z(L)$ is a simple algebra, where $Z(L)$ is the center of $L$.

In order to define a symmetric invariant bilinear form on $L = \Lambda \Box M$, we need the corresponding form on the algebra $\Lambda$ as follows.

**Proposition 3.3.** *For the $\Delta$-graded algebra $\Lambda = \Lambda_0 \bigoplus \sum_{\sigma \in \mathfrak{a}} \oplus \Lambda_\sigma$ defined in* (4), *where $\Lambda_0 = S(\mathfrak{a})$, a symmetric bilinear form is given by*

$$\begin{aligned} ((\emptyset, \emptyset), (\emptyset, \emptyset)) \;&=\; ((\sigma,\, \mu), (\sigma,\, \overline{\mu})) \;=\; (e_i,\, f_i) \;=\; 1, \;\; \textit{for } \mu \subseteq \sigma, \\ (h^\sigma,\, h^\tau) \;&=\; \frac{|\sigma \cap \tau|}{2}, \\ (h^\sigma,\, h_i) \;&=\; |\sigma \cap i| \end{aligned} \tag{30}$$

*and in all other cases the bilinear form is zero, is invariant. The kernel of this bilinear form is* $N(\Lambda) = Z(\Lambda)$.

Thus if the algebra $M$ has a bilinear form then $L = \Lambda \square M$ also does and it is given by

$$(\sum \lambda_i \otimes m_i, \sum \mu_j \otimes n_j) = \sum (\lambda_i, \mu_j)\,(m_i, n_j)\,. \tag{31}$$

Moreover, the kernel $N(\Lambda \square M) = N(\Lambda)\square M + \Lambda \square N(M)$.

For each subset $P \subseteq \mathcal{P}(I_n)$ denote $\overline{P}(I_n) = \{\sigma \subseteq I_n \mid I_n \setminus \sigma \in P\}$.

**Lemma 3.2.** *Under the same hypotheses of Lemma 3.1, for* $\mathfrak{a} \subset \mathcal{P}(I_n)$ *and* $M \in \mathcal{M}$, $\mathfrak{M} = \{\sigma \in \mathfrak{a} \mid M_\sigma \neq 0\}$ *is one of the following sets:*

*(i)* $\{(2i-1,\, 2i,\, 2j-1,\, 2j) \mid 1 \leq i < j \leq \ell\} = \mathcal{C}_{2\ell}\,,$

*(ii)* $\{(2i-1,\, 2i,\, 2j-1,\, 2j),\ (2i-1,\, 2i) \mid 1 \leq i < j \leq \ell\} = \mathcal{B}_{2\ell}\,,$

*(iii)* $\{\,(1234), (1256), (1357), (3456), (2457), (2367), (1467)\,\} = \mathcal{E}_7\,,$

*(iv)* $\mathcal{E}_7 \cup \{\,\overline{\sigma} \mid \sigma \in \mathcal{E}_7,\ \overline{\sigma} = I_8 \setminus \sigma\,\} = \mathcal{E}_8\,.$

*Proof.* Recall that $\mathfrak{a}$ and $\mathfrak{M}$ are subsets of $\mathcal{P}(I_n)$. Since $\mathfrak{M}$ is connected, Lemma 3.1 holds. Set $\mathfrak{M}_2 = \{\sigma \in \mathfrak{M} \mid |\sigma| = 2\}$ and $\mathfrak{M}_4 = \{\sigma \in \mathfrak{M} \mid |\sigma| = 4\}$. We use induction on $n \geq 2$. For $n = 2$, $\mathfrak{M} = \{(12)\} = \mathcal{B}_2$. Let $n > 2$.

1) Suppose that $\mathfrak{M}_2 \neq \emptyset$.

First we claim that if $I = \bigcup_{\tau \in \mathfrak{M}_2} \tau \neq I_n$, then $I_n = I \cup J$ (with $J = I_n \setminus I$) and, for every $\sigma \in \mathfrak{M}_4$, $\sigma \subseteq I$ or $\sigma \subseteq J$. Indeed, if $\sigma \not\subseteq I$ but $\sigma \cap I \neq \emptyset$, then there exists $\tau \in \mathfrak{M}_2$ such that $\sigma \cap \tau \neq \emptyset$ meaning that $\tau \subset \sigma$. Thus, $\sigma \triangle \tau = \sigma \setminus \tau \in \mathfrak{M}_2$ and $\sigma \setminus \tau \subseteq J$, contradicting the definition of $I$.

However, $I = \bigcup_{\tau \in \mathfrak{M}_2} \tau \neq I_n$ implies that $\mathfrak{M}$ is not connected, a contradiction. Hence, $I_n = \bigcup_{\tau \in \mathfrak{M}_2} \tau\,.$

Now it is clear that $\tau \cap \mu = \emptyset$, for all $\tau \neq \mu \in \mathfrak{M}_2$. Therefore, in this case, we have $\mathfrak{M} = \mathcal{B}_{2\ell}\,.$

2) For $\mathfrak{M} = \mathfrak{M}_4$, if $n > 4$ then $n \geq 6$ and, without loss of generality, we have $\mathfrak{N} = \{(1234), (1256), (3456)\} \subseteq \mathfrak{M}_4$. (For $n = 4$, $\mathfrak{M} = \{(1234)\}$.)

A subset $\mathcal{X} \subset \mathcal{P}(I_n)$ is called $n$-**maximal** if, for every $\xi \in I_n$ such that $|\xi| \equiv 0 \mod 2$, or $\xi \in \mathcal{X}$ or there is $\eta \in \mathcal{X}$ such that $|\xi \cap \eta| \equiv 1 \mod 2$.

It is clear that $\mathfrak{N}$ is 6-maximal and, in this case, $\mathfrak{N} = \mathfrak{M} = \mathcal{C}_{2\cdot 3}$. Let $n = 7$. Then $\mathfrak{N}$ is not 7-maximal and there are exactly four other elements $\varphi \in \mathcal{P}(I_7)$ such that $|\varphi| = 4$ and $|\varphi \cap \sigma| \equiv 0 \mod 2$, for all $\sigma \in \mathfrak{N}$, namely, $\varphi \in \{(1357), (2457), (2367), (1467)\}$. Hence, for $n = 7$, $\mathfrak{M} = \mathcal{E}_7$.

Let $n = 8$. Suppose that $\mathcal{E}_7 \subset \mathfrak{M}$ and that $\{\sigma \subset I_8 \,|\, \sigma \notin \mathcal{E}_7,\ |\sigma \cap \mu| \equiv 0 \mod 2$ for all $\mu \in \mathcal{E}_7\} = \overline{\mathcal{E}_7} = \{\overline{\sigma} \,|\, \sigma \in \mathcal{E}_7, \overline{\sigma} = I_8 \setminus \sigma\}$. Hence, $\overline{\mathcal{E}_7} \cap \mathfrak{M} \neq \emptyset$ and $\mathfrak{M} = \mathcal{E}_8 = \overline{\mathcal{E}_7} \cup \mathcal{E}_7$. If $\mathcal{E}_7 \not\subset \mathfrak{M}$, then $\mathfrak{M} = C_8$.

Let $n > 8$. If $\mathcal{E}_7 \subseteq \mathfrak{M}$ then $\mathcal{E}_8 \subseteq \mathfrak{M}$. Now let $\sigma \in \mathfrak{M}$ be such that $\sigma \cap I_8 \neq \emptyset$ and $\sigma \not\subset I_8$. As $|\sigma \cap \tau| \equiv 0 \mod 2$ for all $\tau \in \mathcal{E}_8$, then $\sigma \cap I_8$ is $\mathcal{E}_8$-even. But for all $\sigma \subset I_8$, $\sigma$ is $\mathcal{E}_8$-even if and only if $\sigma \in \mathcal{E}_8$, contradicting with $\sigma \not\subset I_8$. Hence, $\mathcal{E}_7 \not\subseteq \mathfrak{M}$. From this we have that, for all $\sigma \neq \tau \in \mathfrak{M}$ (with $\sigma \cap \tau \neq \emptyset$) and all $\psi \in \mathfrak{M} \setminus \{\sigma, \tau, \sigma \triangle \tau\}$, $\psi \cap \sigma = \emptyset$ or $\psi \cap \tau = \emptyset$ or $\psi \cap (\sigma \triangle \tau) = \emptyset$. This yields $\mathfrak{M} = \mathcal{C}_{2\ell}$.

□

**Theorem 3.1.** *Let $M$ be a $\Delta$-algebra as in Lemma 3.2. Then $M$ has basis $\{s, a_\sigma \,|\, s \in M_0, \sigma \in \mathfrak{M}\}$, with multiplication rules given by*

$$s^2 = s, \quad s\, a_\sigma = a_\sigma, \quad a_\sigma^2 = s, \quad \textit{for } s \in M_0,\ \sigma \in \mathfrak{M},$$

$$a_\sigma\, a_\tau = a_{\sigma \triangle \tau}, \qquad \textit{if } \sigma \triangle \tau \in \mathfrak{M},$$

*and all other products are zero.*

*Proof.* Let $(\ ,\ )$ be the non-degenerate symmetric bilinear form which was introduced in Lemma 3.1. Recall that $\emptyset \notin \mathfrak{a}$.

Let $\sigma \in \mathfrak{M}_4$. If $a \in M_\sigma$ with $a^2 = 0$, then there exists $b \in M_\sigma$ such that

$(a,\, b) = 1$ (that is, $a\, b = s$). Thus by (16), $0 = a^2 \cdot b = ab \cdot a = s \cdot a = a$. Hence $\dim M_\sigma = 1$, as any vector space $V$, with $\dim V > 1$, contains a vector $v$ such that $(v,\, v) = 0$. Denote $a = a_\sigma$ if $a^2 = s$.

Let $\sigma \in \mathfrak{M} \setminus \mathfrak{M}_4$, $|\sigma| = 2$. Suppose that $a^2 = 0$ for every $a \in M_\sigma$. Since the bilinear form is non-degenerate, there exists $b \in M_\sigma$ such that $a\, b = s$. However, by identity (17), $a = ab \cdot a = b \cdot a^2 = 0$, a contradiction. Thus, there exists $a \in M_\sigma$ such that $a^2 = s$ and, for $b \in M_\sigma$, by (17), we have $b = a^2 \cdot b = ab \cdot a = \alpha\, a$, with $\alpha \in k$ (as $\emptyset \notin \mathfrak{a}$). Hence, $M_\sigma = k\, a$, for $a^2 = s$.

Now fix an element $a_\sigma \in M_\sigma$, $\sigma \in \mathfrak{M}$, such that $a_\sigma^2 = s$. For $\sigma \neq \tau \in \mathfrak{M}$, with $\sigma \cap \tau \neq \emptyset$, we have by (14), $(a_\sigma a_\tau)_0^2 = (a_\sigma a_\tau \cdot (a_\sigma a_\tau))_0 = (a_\sigma(a_\tau(a_\sigma a_\tau)))_0 = (a_\sigma(a_\tau^2 a_\sigma))_0 = (a_\sigma a_\sigma)_0 = s$, as by (12), $a_\tau(a_\sigma a_\tau) = a_\tau^2 a_\sigma$. Hence, $a_\sigma a_\tau = a_{\sigma \triangle \tau}$.

Suppose that $\sigma,\, \tau \in \mathfrak{M} \setminus \mathfrak{M}_4$ are such that $\sigma \cap \tau = \emptyset$. Then $\sigma \cup \tau = \mu \in \mathfrak{M}_4$ and $a_\sigma a_\mu = a_\tau$. Hence, by (12), $a_\tau a_\sigma = a_\sigma a_\mu \cdot a_\sigma = a_\sigma^2 a_\mu = a_\mu$. $\square$

# 4 Representations

Let $\mathfrak{a}$ be an even set (with $\emptyset \notin \mathfrak{a}$) and $\Delta = \{0\} \cup \mathfrak{a}$. Let $M = M_0 \oplus \sum_{\sigma \in \mathfrak{a}} \oplus M_\sigma$ be a simple $\Delta$-algebra and $B \subseteq \mathcal{P}(I_n)$ be an $\mathfrak{a}$-even set such that $B \triangle \mathfrak{a} \subseteq B$.

**Definition 4.1.** *A $k$-space $V = V_0 \oplus \sum_{\mu \in B} \oplus V_\mu$ is called a $M$-**module** if there exist a linear map $m : V \times M \longrightarrow V$ such that $V_\mu M_\sigma \subseteq V_{\mu \triangle \sigma}$, for $\mu \neq \sigma$, $V_\sigma M_\sigma \subseteq V_0 + V_\emptyset$, $V_\emptyset M_0 = 0$, and the algebra $\tilde{M} = M \oplus V$, with multiplication*

$$(m_1 + v_1) \cdot (m_2 + v_2) = m_1 m_2 + v_1 m_2 + v_2 m_1,$$

*satisfies the identities* (3)- (28).

In this section we study the irreducible $M$-modules.

Suppose that $V$ is irreducible as an $M$-module. Set $V' = V_0 \oplus \sum_{\emptyset \neq \mu \in B} \oplus V_\mu$. We start with some general rules and remarks on the $M$-action.

**Rule A.** $M_0$ acts as identity on $V'$ and trivially on $V_\emptyset$.

Indeed, let $W = \{v_\sigma \mid \sigma \neq \emptyset,\ v_\sigma \cdot s = 0\}$. Clearly $W$ is a submodule of $V$, as for any $v_\sigma \in W$, for all $\tau \in \mathfrak{M}$, $(v_\sigma a_\tau)s = (v_\sigma \cdot s)a_\tau = 0$, using identity (18). Hence, $W = 0$, as $V$ is irreducible, and $v_\sigma \cdot s \neq 0$, for all $\sigma \neq \emptyset$. Letting $w_\sigma = v_\sigma \cdot s$ we get, by (24), $w_\sigma \cdot s = v_\sigma \cdot s^2 = v_\sigma \cdot s = w_\sigma$. If $w_\sigma \neq v_\sigma$, then $(w_\sigma - v_\sigma) \cdot s = 0$, a contradiction. Hence, $v_\sigma \cdot s = v_\sigma$, for all $\sigma \neq \emptyset$, as required. We have $V_\emptyset M_0 = 0$, by definition of $M$. Thus, Rule A is proved. □

Now let $\mu \in B$, $\sigma \in \mathfrak{M}$ be such that $\mu \neq \sigma$ and choose $0 \neq v_\mu \in V$.

**Rule B.** If $\mu \cap \sigma \neq \emptyset$ then $|\mu \cap \sigma| \leq 2$ and $v_\mu \cdot a_\sigma \neq 0$.

Indeed, if $|\mu \cap \sigma| > 2$ then, by identity (11), $v_\mu = v_\mu \cdot a_\sigma^2 = (v_\mu a_\sigma)a_\sigma = 0$, a contradiction. For $|\mu \cap \sigma| = 2$, by identity (12), $(v_\mu a_\sigma)a_\sigma = v_\mu \cdot a_\sigma^2 = v_\mu \neq 0$, hence $v_\mu a_\sigma \neq 0$. □

NOTATION: In the sequel we use the following notation for $\sigma \in \mathfrak{M}$:

$$V_\sigma = V_i, \quad v_\sigma = v_i, \quad a_\sigma = a_i \quad \text{when } \sigma = (2i-1,\, 2i),$$
$$V_\sigma = V_{ij}, \quad v_\sigma = v_{ij}, \quad a_\sigma = a_{ij}, \quad \text{when } \sigma = (2i-1,\, 2i,\, 2j-1,\, 2j).$$

We define a conjugation on $I_{2n}$ and on $\mathcal{P}(I_n)$ by

$$\bar{i} = \begin{cases} 2j, & \text{if } i = 2j-1. \\ 2j-1, & \text{if } i = 2j. \end{cases}$$

It is clear that conjugation is an involution.

**Theorem 4.1.** *Let $M$ be a simple $\Delta$-algebra as defined above and $V$ be an irreducible $M$-module. Then:*

*1. if $\mathfrak{M} = \mathcal{C}_{2\ell}$ then 1.1. $V = < v_0,\ v_{ij} \mid 1 \leq i \neq j \leq \ell >$, where $0 \neq v_0 \in V_0$ and $v_{ij} = v_0 \cdot a_{ij}$, for all $1 \leq i \neq j \leq \ell$ (adjoint module) or*

*1.2. $V = < v_i \mid 1 \leq i \leq \ell >$, where $0 \neq v_1 \in V_1$ and $v_j = v_1 \cdot a_{1j}$ for $2 \leq j \leq \ell$ (standard module) or*

*1.3.* $V = < v_\lambda \in Sp \,|\, \sharp\{i_j \in \lambda \,|\, i_j \equiv 0 \mod 2\} \equiv 0 \mod 2 >$, *where* $Sp = \{v_\lambda \,|\, \lambda = (i_1\, i_2 \cdots i_\ell), \text{ with } i_j \in \{2j-1, 2j\}, \text{ for } 1 \leq j \leq \ell\}$ *and* $v_\lambda \cdot a_\sigma = v_{\lambda\triangle\sigma}$ *for all* $\sigma \in \mathcal{C}_{2\ell}$ *(spinor module).*

*2. If* $\mathfrak{M} = \mathcal{B}_{2\ell}$ *then*

*2.1.* $V = < v_0,\, v_i,\, v_{ij} \,|\, 1 \leq i \neq j \leq \ell >$, *where* $0 \neq v_0 \in V_0$, $v_i = v_0 \cdot a_i$ *and* $v_{ij} = v_0 \cdot a_{ij}$, *for all* $1 \leq i \neq j \leq \ell$ *(adjoint module) or*

*2.2.* $V = < v_i, \lambda \,|\, 1 \leq i \leq \ell >$, *where* $0 \neq v_1 \in V_1$, $v_j = v_1 \cdot a_{1j}$ *for* $2 \leq j \leq \ell$ *and* $\lambda = (v_i\, a_i)_\emptyset$ *(standard module).*

*3. If* $\mathfrak{M} = \mathcal{E}_7$ *then* $V = < v_\mu \,|\, \mu \in \overline{\mathcal{E}_7}(I_7) >$, *where* $v_\mu\, a_\sigma = v_{\mu\triangle\sigma}$ *for all* $\mu \in \overline{\mathcal{E}_7}(I_7)$, $\sigma \in \mathcal{E}_7$.

*4. If* $\mathfrak{M} = \mathcal{E}_8$ *then* $V = < v_0, v_\sigma \,|\, \sigma \in \mathcal{E}_8 >$ *where* $0 \neq v_0 \in V_0$, $v_0 \cdot a_\sigma = v_\sigma$ *and* $v_\sigma \cdot a_\mu = v_{\sigma\triangle\mu}$, *for all* $\sigma, \mu \in \mathcal{E}_8$ *(standard module).*

The proof of this theorem is given in the following subsections.

## 4.1 $M$-modules for $\mathfrak{M} = \mathcal{C}_{2\ell}$

For $\mathfrak{M} = \mathcal{C}_{2\ell} = \{(2i-1,\, 2i,\, 2j-1,\, 2j) \,|\, 1 \leq i < j \leq \ell\}$, by Theorem 3.1, a simple $\Delta$-algebra $M$ has a basis $\{s,\, a_\sigma \,|\, s \in M_0,\, \sigma \in \mathfrak{M}\}$. Let $V$ be an irreducible $M$-module and $B = \{\mu \subseteq I_n \,|\, V_\mu \neq 0\}$.

**Case I**: Suppose that $\mathfrak{M} \cap B \neq \emptyset$.

Let $\sigma \in \mathfrak{M} \cap B$ and choose $0 \neq v_\sigma \in V_\sigma$. Then, by (16), $(v_\sigma a_\sigma)_0 a_\sigma = v_\sigma (a_\sigma a_\sigma)_0 = v_\sigma \neq 0$ Hence, there exists $0 \neq v_0 \in V_0$. Define

$$v_0 \cdot a_{ij} \overset{def}{=} v_{ij}, \quad \text{for all } 1 \leq i \neq j \leq \ell. \tag{32}$$

In this way, we have by (18),

$$\begin{aligned} v_{ij} \cdot a_{jk} &= (v_0 a_{ij}) a_{jk} = v_0 (a_{ij} a_{jk}) = v_0 a_{ik} = v_{ik}, \quad \text{for } 1 \leq i \neq j \neq k \leq \ell, \\ v_{ij} \cdot a_{pk} &= (v_0 a_{ij}) a_{pk} = v_0 (a_{ij} a_{pk}) = 0, \quad \text{for } \{i,j\} \cap \{p,k\} = \emptyset. \end{aligned}$$

Therefore, $\{v_0,\ v_{ij}\,|\,1 \le i \ne j \le \ell\}$ is a basis of the $M$-module $V$.

**Case II**: Suppose that $\mathfrak{M} \cap B = \emptyset$ (implying $V_0 = 0$ and $V_\emptyset = 0$).

II.1) Suppose that there exists $\mu \in B$ with $|\mu| = 2$.

a) If $\mu = (12) \in B$, then let $0 \ne v_1 \in V_1$. Define

$$v_1 \cdot a_{1j} \stackrel{def}{=} v_j \ne 0, \quad \text{for } 2 \le j \le \ell\,. \tag{33}$$

Now for $1 \le i < j \le k \le \ell$, using identity (8), we have $v_i \cdot a_{jk} = v_i(a_{ij}a_{jk}) = 0$. From this and by (10), $v_i \cdot a_{ij} = (v_1 a_{1i})a_{ij} = v_1(a_{1i}a_{ij}) + (v_1 a_{ij})a_{1i} = v_1 a_{1j} = v_j$. Hence, $\{v_i\,|\,1 \le i \le \ell\}$ is a basis of the $M$-module $V$.

b) For $\ell > 2$, by definition of $B$, $\mu = (13) \notin B$. For $\ell = 2$, this is simply a renumbering of case a).

II.2) Suppose that $|\mu| > 2$, for all $\mu \in B$.

We claim that $(2i-1, 2i) \not\subset \mu$, for all $1 \le i \le \ell$. Indeed, without loss of generality, suppose that $(1,2) \subseteq \mu$. If $\sigma_{1i} = (1, 2, 2i-1, 2i)$, then, by Rule B, $|\,\sigma_{1i} \cap \mu\,| \le 2$. Note that $\sigma_{1i} \ne \mu$, as $\mathfrak{M} \cap B = \emptyset$. Now if, for all $i = 2, \ldots, \ell$, $|\,\sigma_{1i} \cap \mu\,| = 2$,then $\mu = (1,2)$ a contradiction and the claim is proved.

Now let $2i-1 \in \mu$ (or $2i \in \mu$). Then $|\,\sigma_{ij} \cap \mu\,| = 2$ implies $|\,(2j-1, 2j) \cap \mu\,| = 1$, for all $j = 1, \ldots, \ell$. Hence, after a suitable renumbering, we may suppose that $\mu = (1357 \cdots 2\ell - 1) \in B$. Thus, for all $\sigma \in \mathfrak{M}$,

$$v_\mu \cdot a_\sigma = v_{\mu \triangle \sigma}, \quad \text{as } |\mu \cap \sigma| = 2\,. \tag{34}$$

Therefore, $V$ must be contained in:

$$Sp = \{v_\lambda\,|\,\lambda = (i_1\, i_2\, \cdots\, i_\ell), \text{ with } i_j \in \{2j-1,\, 2j\}, \text{ for } 1 \le j \le \ell\}\,.$$

Now consider the following subspaces of $Sp$:

$$\begin{aligned} Sp_+ &= \{\, v_\lambda \in Sp\,|\,\sharp\{i_j \in \lambda\,|\,i_j \equiv 0 \mod 2\} \equiv 0 \mod 2\,\}\,, \\ Sp_- &= \{\, v_\lambda \in Sp\,|\,\sharp\{i_j \in \lambda\,|\,i_j \equiv 0 \mod 2\} \equiv 1 \mod 2\,\}\,. \end{aligned}$$

Note that $Sp_+$ and $Sp_-$ are invariant and irreducible under the action of $M$, for $\mathfrak{M} = \mathcal{C}_{2\ell}$, as by (34), $a_\sigma$ exchanges pairs of even or odd numbers in each $\lambda$.

Moreover, there is an isomorphism $\Phi : Sp_+ \longrightarrow Sp_-$ such that $(i_1\, i_2\, \cdots\, i_\ell) \longmapsto (\overline{i_1}\, i_2\, \cdots\, i_\ell)$ which commutes with the $M$-action.

## 4.2 Representations for $\mathfrak{M} = \mathcal{B}_{2\ell}$

For $\mathfrak{M} = \mathcal{B}_{2\ell} = \{(2i-1,\, 2i,\, 2j-1,\, 2j),\ (2i-1, 2i) \,|\, 1 \le i < j \le \ell\}$, by Theorem 3.1, a simple $\Delta$-algebra $M$ has a basis $\{s,\, a_\sigma,\, b_\tau \,|\, s \in M_0,\, \sigma \in \mathfrak{M}_4,\, \tau \in \mathfrak{M} \setminus \mathfrak{M}_4\}$. Let $V$ be an irreducible $M$-module.

**Case I**: Suppose $\mathfrak{M} \cap B \neq \emptyset$.

I.1) If there exists $0 \neq v_0 \in V_0$, then define, for $1 \le i \neq j \neq k \le \ell$,

$$v_i \overset{def}{=} v_0 \cdot a_i, \qquad\qquad v_{ij} \overset{def}{=} v_0 \cdot a_{ij}\ .$$

Thus, using the identities of Proposition 3.2, we get:

i) by (25), $(v_i a_i)_0 = ((v_0 a_i) a_i)_0 = v_0 (a_i a_i)_0 = v_0 s = v_0$.

ii) By (26), $(v_i a_i)_\emptyset = ((v_0 a_i) a_i)_\emptyset = v_0 (a_i a_i)_\emptyset = 0$.

iii) By (18), we have for $1 \le i \neq j \neq p \neq k \le \ell$

$$\begin{aligned} v_i \cdot a_j &= (v_0 a_i) a_j = v_0 a_{ij} = v_{ij}\,, \\ v_i \cdot a_{ij} &= (v_0 a_i) a_{ij} = v_0 (a_i a_{ij}) = v_0 a_j = v_j\,, \\ v_{ij} \cdot a_i &= (v_0 a_{ij}) a_i = v_0 (a_{ij} a_i) = v_0 a_j = v_j\,, \\ v_{ij} \cdot a_{jk} &= (v_0 a_{ij}) a_{jk} = v_0 (a_{ij} a_{jk}) = v_0 a_{ik} = v_{ik}\,, \\ v_{ij} \cdot a_{pk} &= (v_0 a_{ij}) a_{pk} = v_0 (a_{ij} a_{pk}) = 0\,. \end{aligned} \tag{35}$$

Note that $v_{ij} = 0$ implies $v_j = 0$, hence $v_0 = 0$, a contradiction. Therefore, $\{v_0,\, v_i,\, v_{ij} \,|\, 1 \le i \neq j \le \ell\}$ is a basis of the $M$-module $V$.

I.2) For $V_0 = 0$, let $\sigma \in \mathfrak{M} \cap B$ be such that $|\ \sigma\ | = 2$. Without loss of generality, let $v_1 \in V_1$. By Rule B, define $v_i \overset{def}{=} v_1\, a_{1i} \neq 0$, for $1 \le i \le \ell$. Moreover, $(v_1 a_1)_0 = 0$ and $(v_1 a_1)_\emptyset = \lambda_1$.

Again without loss of generality, if $v_1 a_2 \overset{def}{=} v_{12} \neq 0$, then, by (12), $v_{12} \cdot a_1 = v_{12} \cdot (a_{12} \cdot a_2) = (v_{12}\, a_{12})_0 \cdot a_2) = 0$, as $V_0 = 0$. But this contradicts Rule B. Therefore, this case does not occur.

Thus, we can suppose that $v_i a_j = 0$, for all $1 \le i \ne j \le \ell$. By Rule B, we can define $v_i a_{ij} \overset{def}{=} v_j \ne 0$. We also have $(v_i a_i)_\emptyset = \lambda_i$. By (12), $\lambda_i a_{pq} = 0$, for $i \notin \{p, q\}$. By (13), $\lambda_i a_j = (v_i a_i)_\emptyset\, a_j = v_i a_j \cdot a_i + v_i \cdot a_i\, a_j = v_i \cdot a_{ij} = v_j$. Hence $\lambda = \lambda_i$ for all $1 \le i \le \ell$ and $\lambda\, a_i = v_i$. Therefore, $\{v_i,\ \lambda \,|\, 1 \le i \le \ell\}$ is a basis of the $M$-module $V$.

**Case II**: Suppose $\mathfrak{M} \cap B = \emptyset$ (hence $V_0 = 0$ and $V_\emptyset = 0$).

In this case, any $\mu \in B$ is such that $|\mu| > 2$ and $(2i-1, 2i) \not\subset \mu$. Hence, as in Case II.2 of Section 4.1, $\mu = (1357 \cdots 2\ell - 1) \in B$ but $(12) \in \mathcal{B}_{2l}$ and $|(12) \cap \mu| = 1$, contradiction. So this case does not occur.

## 4.3 Representations for $\mathfrak{M} = \mathcal{E}_7$

Let $\mathfrak{M} = \{\,(1234), (1256), (3456), (1357), (2457), (2367), (1467)\,\}$ and $V$ be an irreducible $M$-module.

**Case I**: Suppose that there exists $\mu \in \mathfrak{M} \cap B$.

I.1) For $V_0 \ne 0$, choose $0 \ne v_0 \in V_0$ and define $v_0 \cdot a_\sigma \overset{def}{=} v_\sigma$, for $\sigma \in \mathfrak{M}$. Hence, for any $\tau \in \mathfrak{M}$, by (18), we have

$$v_\sigma \cdot a_\tau = (v_0 a_\sigma) a_\tau = v_0 (a_\sigma a_\tau) = v_0 a_{\sigma \triangle \tau} = v_{\sigma \triangle \tau}.$$

Therefore, in this case, $\{v_0,\, v_\sigma \,|\, \sigma \in \mathcal{E}_7\}$ is a basis of the $M$-module $V$.

II.2) If $V_0 = 0$ then, by (12), $0 = (v_\sigma a_\sigma)_0 a_\tau = (v_\sigma a_\tau) a_\sigma = v_{\sigma \triangle \tau} a_\sigma$, contradicting Rule B. So this case does not occur.

**Case II**: Suppose that $\mathfrak{M} \cap B = \emptyset$.

First we claim that if $\mu \in B$, then $|\mu| = 3$.

i) If $\mu = (ij)$ then $|\mu \cap \sigma| = 0$ or $2$, for all $\sigma \in \mathfrak{M}$. Thus $\mu \subseteq (1234) \cap (1256) \cap (1357) = \{1\}$, a contradiction. Hence, $|\mu| \ge 3$.

ii) Without loss of generality, we can suppose that $1 \in \mu$. Then, as $\mathfrak{M} \cap B = \emptyset$, $|(1234) \cap \mu| = (1j)$, $j \ne 1$. If $j = 2$ then $3, 4 \notin \mu$ and $|(1256) \cap \mu| = (12)$, implying that $5, 6 \notin \mu$. Hence, $|\mu| \le 3$. We argue analogously for $j = 3$ or

$j = 4$. Hence, by i) and ii) the claim is proved.

Now simple calculations show that $B = \overline{\mathcal{E}_7}(I_7)$ and, therefore, $V$ has a basis $\{v_\mu \mid \mu \in \overline{\mathcal{E}_7}(I_7)\}$, with $M$-action given by $v_\mu a_\sigma = v_{\mu \triangle \sigma}$.

### 4.4 Representations for $\mathfrak{M} = \mathcal{E}_8$

Let $\mathfrak{M} = \mathcal{E}_8 = \mathcal{E}_7 \cup \overline{\mathcal{E}_7}(I_8)$. Note that for every $\tau \in I_8$, we have or $\tau \in \mathcal{E}_8$ or $|\sigma \cap \tau| \equiv 1 \mod 2$ for some $\sigma \in \mathcal{E}_8$. This means that $B \subset \mathfrak{M} = \mathcal{E}_8$.

1) If there exists $0 \neq v_0 \in V_0$ then, as in Case I of Section 4.3, define $v_0 \cdot a_\sigma \overset{def}{=} v_\sigma$, for $\sigma \in \mathfrak{M}$. Hence, for any $\mu \in \mathfrak{M}$, by (18), we have

$$v_\sigma \cdot a_\mu = (v_0 a_\sigma) a_\mu = v_0 (a_\sigma a_\mu) = v_0 a_{\sigma \triangle \mu} = v_{\sigma \triangle \mu}.$$

Therefore, in this case, $\{v_0, v_\sigma \mid \sigma \in \mathcal{E}_8\}$ is a basis of the $M$-module $V$.

2) If $V_0 = 0$ then, by (12), $0 = (v_\sigma a_\sigma)_0 a_\tau = (v_\sigma a_\tau) a_\sigma = v_{\sigma \triangle \tau} a_\sigma$, contradicting Rule B. So this case does not occur.

## 5 Conclusion and final comments.

It is not difficult to prove that the Lie algebras obtained from the $\triangle$-algebras in the Theorem 3.1 by multiplying them by the corresponding algebra $\Lambda$ (see (4)) are the classical Lie algebras or its central extensions. Moreover, as simple $\triangle$-algebras have invariant bilinear forms then the corresponding Lie algebras also admit such forms.

In order to calculate the center of a Lie algebra $L = M \Box \Lambda$ corresponding to a given $\Delta$-algebra $M \in \mathcal{M}$, we need the following result.

**Proposition 5.1.** *Let $M$ be a $\Delta$-algebra in $\mathcal{M} = \mathcal{M}(\mathfrak{a})$, where $\mathfrak{a}$ is an even set as before. Let $Ann_M(M) = \{x \in M \mid xM = 0\}$ and $L = M(\mathfrak{a}) \Box \Lambda$ be the* *corresponding Lie algebra. Then*

$$Z(L) \;=\; \Lambda\Box Ann_M(M) \;+\; Z(\Lambda)\otimes M_0\,,$$

*where* $Z(\Lambda) \;=\; \{h_\mu \,|\, \mu \subseteq I_n,\ \mu \textit{ is } \mathfrak{a}-\textit{even}\}$.

*Proof.* All statements of this proof are obvious or follow from Proposition 3.1. $\Box$

For example, the Lie algebra $L = M(\mathcal{E}_8)\Box\Lambda$ has a basis

$$\{\, e_1,\ f_1, \ldots,\ e_7,\ f_7,\ h_1,\ h_2,\ h_5,\ h^{\sigma_1},\ h^{\sigma_2},\ h^{\sigma_3};\ (\mu,\sigma)\ |\ \mu \subseteq \sigma \in \mathcal{E}_8\},$$

where $\{\, e_1,\ f_1, \ldots,\ e_7,\ f_7,\ h_1,\ h_2,\ h_5,\ h^{\sigma_1},\ h^{\sigma_2},\ h^{\sigma_3}\}$ is a basis of the algebra $S = \tilde{S}(\mathcal{E}_8)/I$ defined by (3) and $\sigma_1 = (1234),\, \sigma_2 = (1256),\, \sigma_3 = (1357)$. Recall that in $S$ we have

$$h^\sigma + h^\mu + h^{\sigma\triangle\mu} + h_{\sigma\cap\mu} = 0, \tag{36}$$

for every $\sigma, \mu \in \mathcal{E}_8$, where $h_\tau \;=\; \sum\limits_{i\in\tau} h_i$. The multiplication in this basis of $L$ is given by formulas (3), (5) and (6). For example,

$$[(1357, 13), (3478, 7)] = (1458, 1),$$

$$[(1256, 2), (1256, 156)] = h^\sigma + h_1 + h_5 + h_6 = h^\sigma + h_2,$$

since, by (36), $h_6 \;=\; h_\sigma + h_1 + h_2 + h_5 = h_1 + h_2 + h_5$, where $\sigma = (1256)$.

Note that $B \;=\; \{\mu \in I_7 \,|\, \mu \text{ is } \mathcal{E}_7 - \text{even}\,\} \;=\; \mathcal{E}_7 \cup \overline{\mathcal{E}_7}(I_7)$. From this and by Proposition 5.1 we have that $Z(L) \;=\; k\,h$, where $h \;=\; h_2 + h_3 + h_5$.

It is clear that $\dim L \;=\; 133$, $\dim L/Z(L) \;=\; 132$ and $L/Z(L)$ is a simple Lie algebra of type $E_7$. Note that for the $M(\mathcal{E}_7)$-module $V$ constructed in Theorem 4.1, with basis $\{v_\mu \,|\, \mu \in \overline{\mathcal{E}_7}(I_7)\}$ we can construct the $L$-module $W \;=\; V\Box\Lambda$ of dimension 56. But $W$ is not a $L/Z(L)$-module, since $V\,h \;=\; V \neq 0$.

The modules $V \;=\; V_0 \oplus \sum_{\sigma\in B} \oplus V_\sigma$ of $\triangle$-algebras which we constructed in Section 4 correspond to the following modules over the corresponding simple Lie algebra $L$.

a) If $L$ if of type $C_{2\ell}$ or $B_{2\ell}$ and $|\sigma| = 2$, for all $\sigma \in B$, then $V$ corresponds to the standard module.

b) If $B = \mathcal{C}_{2\ell}$ or $\mathcal{B}_{2\ell}$, then $V$ corresponds to the adjoint module.

c) If $L$ is of type $C_{2\ell}$ and $|\sigma| = \ell$, for all $\sigma \in B$, then $V$ corresponds to a spinor module.

This gives us a useful construction of spinor modules over a simple Lie algebra $L$ of type $C_{2\ell}$. In this case such a Lie algebra has basis

$$\{e_i, f_i, h_j, h^k, (\sigma, \mu) \mid i = 1, ..., 2\ell;\ j = 1, 2, 3, 5, ..., 2\ell - 1; k = 2, ..., \ell; \mu \subseteq \sigma \in \mathcal{C}_{2\ell}\},$$

with multiplication rules given by

$$[e_i,\ f_i] = h_i,\ \text{ where } h_i = h_1 + h_2 + h_{2p-1},\ \text{ if } i = 2p > 2\,,$$

$$[e_i,\ h^j] = e_i,\ \ [f_i, h^j] = f_i,\ \text{ if } i \in \{1, 2, 2j-1, 2j\}\,,$$

$$[(\sigma, \mu),\ h^i] = (\sigma, \mu),\ \text{if}\, \sigma \cap (1, 2, 2i-1, 2i) \neq \emptyset \text{ and } |\mu \cap (1, 2, 2i-1, 2i)| = 0 \text{ or } 2.$$

$$[(\sigma, \mu), h_i] = (\sigma, \mu),\ \text{if } i \in \sigma.$$

$$[(\sigma, \mu), (\sigma, \sigma \setminus \mu)] = \begin{cases} h^j + h_\mu, & \text{for } \sigma = (1, 2, 2j-1, 2j); \\ h^j + h^k + h_\mu, & \text{for } \sigma = (2j-1, 2j, 2k-1, 2k). \end{cases}$$

Here, as above, $h_{2p} = h_1 + h_2 + h_{2p-1}$, if $p > 1$. The other products are given by the formulas (5), (6) or are equal to zero. Moreover, $Z(L) = k\, h$, where $h = \sum_{i=1}^{\ell} h_{2i-1}$.

The spinor module $W$ has a basis

$$\{(\beta, \alpha) \mid \alpha \subseteq \beta \in \mathcal{M}_\ell\}$$

where

$$\mathcal{M}_\ell = \{\mu = (i_1, ..., i_l) \mid i_j \in \{2j-1,\ 2j\} \text{ and } |\{i \in \mu \mid i \equiv 0\, (mod\, 2)\}| \equiv 0\, (mod\, 2)\}.$$

The action of $L$ is given by

$$\begin{aligned}(\beta,\alpha)e_i &= (\beta,\alpha\cup i), \text{ if } i\in\beta\setminus\alpha,\\ (\beta,\alpha)f_i &= (\beta,\alpha\setminus i), \text{ if } i\in\alpha,\\ (\beta,\alpha)h_i &= (\beta,\alpha), \text{ if } i\in\beta,\\ (\beta,\alpha)h^i &= (\beta,\alpha), \text{ if } (1,2,2i-1,2i)\cap\alpha=0 \text{ or } 2,\\ (\beta,\alpha)(\sigma,\mu) &= (\beta\triangle\sigma,\alpha\setminus\sigma\cup\mu\setminus\beta), \text{ if } \beta\cap\sigma\subseteq\alpha\cup\mu \text{ and } \mu\cap\alpha=\emptyset.\end{aligned} \tag{37}$$

All the other products are zero. Observe that, for any (non adjoint) irreducible $M$-module $V$, we have $(V\square\Lambda)\,Z(L)\neq 0$.

Note that Theorem 3.1 is not true if we omit the condition $\emptyset\notin\mathfrak{a}$. But we can formulate the following conjecture.

**Conjecture 5.1.** *Let $M$ be an arbitrary simple finite dimensional $\triangle$-algebra which satisfies all identities (8)-(28) and $M_\emptyset^2=0$. Then the corresponding Lie algebra $L=M\square\Lambda$ is a simple Lie algebra of type $B_{2\ell}$, $C_\ell$, $D_{2\ell+1}$, $E_7$ or $E_8$.*

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