# Bootstrap Central Limit Theorem for Chains of Infinite Order via Markov Approximations* 

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#### Abstract

We present a new approach to the bootstrap for chains of infinite order taking values on a finite alphabet. It is based on a sequential Bootstrap Central Limit Theorem for the sequence of canonical Markov approximations of the chain of infinite order. Combined with previous results on the rate of approximation this leads to a Central Limit Theorem for the bootstrapped estimator of the sample mean which is the main result of this paper.


Keywords: bootstrap, chains of infinite order, canonical Markov approximations, central limit theorem
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## 1. Introduction

It is well known that for processes with good mixing properties the empirical mean is asymptotically normally distributed (see for example [3]). But this result cannot be used in most practical circumstances, since to compute the limit variance of the empirical mean requires the knowledge of the auto-covariances of all orders, which are, in general, unknown.

[^0]In order to cope with this problem we introduce in the present paper a new procedure of bootstrap resampling for chains with finite alphabets whose transition probabilities depend on the whole past, but having good mixing properties. As in the standard bootstrap procedure, since the bootstrap variance can be directly estimated from the data, this method allows to construct asymptotic confidence intervals for the true mean.

The bootstrap resampling we construct uses the excursions of the chain between successive occurrences of the initial string of $k$ symbols as building blocks for the bootstrap sample. The bootstrap sample is obtained by concatenating randomly chosen blocks. These blocks are chosen uniformly and independently among the first $m_{k}$ excursion blocks. For chains which lose memory exponentially fast we prove a Central Limit Theorem for the empirical mean of the bootstrap sample, when the length $k$ of the initial reference string as well as the number of excursion blocks $m_{k}$ diverge with a suitable relation between them. This is the main result of the article.

The idea behind our procedure is that a typical large sample of the chain of infinite order behaves essentially as a sample of a Markov chain of order $k$ suitably chosen. The Markov property of the approximating chain implies that the successive excursion blocks are independent and identically distributed. This makes it possible to construct the bootstrap sample by simply concatenating randomly chosen blocks, exactly as proposed in the original paper by Efron [8] for the case of i.i.d. random variables.

This idea has already been exploited in the case of Markov chains in [1]. For chains of infinite order with different types of mixing conditions, different approaches to the bootstrap have been proposed in the papers by Carlstein [6] and Künsch [13] and thoroughly studied in the recent literature, see for example $[4,14,15,17,18]$.

Chains of infinite order seem to have been first studied by Onicescu and Mihoc [16] who called them chaînes à liaisons complètes. Their study was soon taken up by Doeblin and Fortet [7] who proved the first results on speed of convergence towards the invariant measure. The name "chains of infinite order" was coined by Harris [11]. Our proof is based on the upper bound on the rate of approximation of the chain of infinite order by the sequence of canonical Markov approximations presented in [10]. We also use the $\varphi$-mixing property of the chain of infinite order proven in [5]. We refer the reader to [12] for a complete survey, and to [9] for an elementary presentation of the subject from a constructive point of view.

The rest of the paper is organized as follows. In Section 2 we introduce the notation and the definitions and state the main results. In Section 3 we collect together a few technical results which will be used in the proof of the theorems. In Section 4 we prove a central limit theorem for the sequence of canonical approximating Markov chains. Finally in Section 5 we prove the main result
which is a bootstrap central limit theorem for the empirical mean of a chain of infinite order.

## 2. Notation, definitions and statement of the main result

Let $\left(X_{n}\right)_{n \in \mathrm{Z}}$ be a stationary process taking values on a finite alphabet $A$. We will use the shorthand notation

$$
p\left(x_{0} \mid x_{-1}, x_{-2}, \ldots\right)=\mathrm{P}\left(X_{0}=x_{0} \mid X_{-1}=x_{-1}, X_{-2}=x_{-2}, \ldots\right)
$$

to denote the regular version of the conditional probability of the process. To avoid long formulas, whenever convenient, we will use the notation $a_{0, l}$ to denote the sequence $\left(a_{0}, \ldots, a_{l}\right)$ of elements of $A$. We also use the notation $\left\{X_{n, n+l}=\right.$ $\left.a_{0, l}\right\}$ to denote the cylinder set $\left\{X_{n}=a_{0}, \ldots, X_{n+l}=a_{l}\right\}$. Following Harris [11], we call this process a chain of infinite order.

We assume that $\left(X_{n}\right)_{n \in \mathrm{Z}}$ satisfies the following hypotheses.
$\mathrm{H}_{1}$

$$
\begin{equation*}
\min _{a \in A} \inf _{\left(\ldots, x_{-2}, x_{-1}\right) \in \mathcal{A}_{a}} p\left(a \mid x_{-1}, x_{-2}, \ldots\right)=\delta>0 \tag{2.1}
\end{equation*}
$$

where $\mathcal{A}_{a}=\left\{\left(\ldots, x_{-2}, x_{-1}\right): p\left(a \mid x_{-1}, x_{-2}, \ldots\right)>0\right\}$.
$\mathrm{H}_{2}$

$$
c=-\limsup _{l \rightarrow \infty} \frac{1}{l} \log \beta_{l}>0
$$

where

$$
\beta_{l}=\sup _{\substack{x_{i}=y_{i} \\ i=-l, \ldots, 0}}\left|p\left(x_{0} \mid x_{-1}, x_{-2}, \ldots\right)-p\left(y_{0} \mid y_{-1}, y_{-2}, \ldots\right)\right| .
$$

Let $f: A^{r} \rightarrow \mathbf{R}$ be a real observable of the chain, where $r$ is a fixed positive integer and denote

$$
\mu=\mathrm{E}\left(f\left(X_{1}, \ldots, X_{r}\right)\right)
$$

the average value of the observable $f$. We are interested in the fluctuations of an estimator of $\mu$. To simplify the presentation we can assume without loss of generality that $r=1$, namely the cylinder function $f$ through which we observe the chain depends only on one coordinate.

To avoid uninteresting pathologies we will assume that the following third hypothesis holds
$\mathrm{H}_{3}$

$$
\sigma^{2}=\operatorname{Var}\left(f\left(X_{0}\right)\right)+2 \sum_{j=1}^{+\infty} \operatorname{Cov}\left(f\left(X_{0}\right), f\left(X_{j}\right)\right)>0
$$

We recall that hypotheses $\mathbf{H}_{\mathbf{1}}$ and $\mathbf{H}_{\mathbf{2}}$ imply that the chain $\left(X_{n}\right)_{n \in \mathrm{Z}}$ is exponentially $\varphi$-mixing [5]. This last property implies that the series defining
$\sigma^{2}$ is convergent (see for instance Theorem 19.1 in [3]). However it is well known that this does not imply that $\sigma$ is strictly positive.

Our bootstrap procedure is defined as follows. For any positive integer $k$, the sequence $\left(R_{j}(k)\right)_{j \in \mathrm{~N}}$ of return times of the first string of length $k$ is defined by

$$
R_{i+1}(k)=\inf \left\{n>R_{i}(k):\left(X_{n}, \ldots, X_{n+k-1}\right)=\left(X_{1}, \ldots, X_{k}\right)\right\}
$$

with $R_{0}(k)=1$.
Let $\xi_{i}(k)$ be the block of values of the chain from $R_{i-1}(k)$ up to $R_{i}(k)-1$, namely

$$
\begin{equation*}
\xi_{i}(k)=\left(X_{R_{i-1}(k)}, \ldots, X_{R_{i}(k)-1}\right) \tag{2.2}
\end{equation*}
$$

We will make a uniform i.i.d. selection of the first $m$ blocks $\xi_{1}(k), \ldots, \xi_{m}(k)$ to construct a bootstrap sample of the chain. We will take $m=m_{k}$ as a diverging function of $k$ to be fixed later. This leads naturally to the construction of a sequence of bootstrap samples indexed by $k$.

The formal definition is as follows. For every $k$, let $I_{1}(k), \ldots, I_{m_{k}}(k)$ be $m_{k}$ independent random variables with uniform distribution in the set $\left\{1, \ldots, m_{k}\right\}$. The bootstrap blocks are defined as

$$
\xi_{l}^{*}(k)=\xi_{I_{l}(k)}(k),
$$

for $l=1, \ldots, m_{k}$. The bootstrap sample $X_{1}^{*}(k), \ldots, X_{R_{m_{k}}^{*}(k)}^{*}(k)$ is constructed by concatenating the bootstrap blocks $\xi_{1}^{*}(k), \ldots, \xi_{m(k)}^{*}(k)$. We observe that the return times of the bootstrap sample assume the values $R_{0}^{*}(k)=1$ and for $l=1, \ldots, m_{k}$

$$
R_{l}^{*}(k)=R_{l-1}^{*}(k)+R_{I_{l}(k)+1}(k)-R_{I_{l}(k)}(k)
$$

We consider the following sequence of estimators for $\mu$

$$
\begin{equation*}
\hat{\mu}_{k}=\frac{1}{R_{m_{k}}(k)-1} \sum_{n=1}^{R_{m_{k}}(k)-1} f\left(X_{n}\right) \tag{2.3}
\end{equation*}
$$

Its bootstrap counterpart is given by

$$
\begin{equation*}
\mu_{k}^{*}=\frac{1}{R_{m_{k}}^{*}(k)-1} \sum_{n=1}^{R_{m_{k}}^{*}(k)-1} f\left(X_{n}^{*}(k)\right) . \tag{2.4}
\end{equation*}
$$

Let

$$
\sigma_{k}^{*}=\sqrt{\frac{\operatorname{Var}\left(\sum_{n=1}^{R_{m_{k}}^{*}(k)-1}\left(f\left(X_{n}^{*}(k)\right)-\hat{\mu}_{k}\right) \mid X_{1}, \ldots, X_{R_{m_{k}}-1}\right)}{R_{m_{k}}^{*}(k)-1}},
$$

where Var denotes the variance. Observe that $\sigma_{k}^{*}$ is a function of the sample $X_{1}, \ldots, X_{R_{m_{k}}(k)}$ and therefore the above variance is taken with respect to the independent random variables $I_{1}(k), \ldots, I_{m_{k}}(k)$.

In the statement of our theorems the number of blocks used in the bootstrap sample is

$$
m_{k}(\alpha)=\left[e^{\alpha k}\right]
$$

where $\alpha$ is a positive real number to be suitably chosen later and [•] denotes the integer part. We will often write $m_{k}$ instead of $m_{k}(\alpha)$.

Theorem 2.1. Let $\left(X_{n}\right)_{n \in \mathrm{Z}}$ be a chain of infinite order satisfying hypotheses $\mathbf{H}_{\mathbf{1}}, \mathbf{H}_{\mathbf{2}}$ and $\mathbf{H}_{\mathbf{3}}$ and such that $c>18 \ln (1 / \delta)$, where $\delta$ and $c$ are the constants appearing in $\mathbf{H}_{\mathbf{1}}$ and $\mathbf{H}_{\mathbf{2}}$, respectively. Then, for any $\alpha \in(5 \ln (1 / \delta), c-\ln (1 / \delta))$, for $m_{k}=\left[e^{\alpha k}\right]$, and for almost all realizations of the chain $\left(X_{n}\right)_{n \in \mathrm{Z}}$, we have

$$
\begin{equation*}
\frac{\sqrt{R_{m_{k}}^{*}(k)-1}}{\sigma_{k}^{*}}\left(\mu_{k}^{*}-\hat{\mu}_{k}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \tag{2.5}
\end{equation*}
$$

as $k$ tends to $+\infty$, where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and $\mathcal{N}(0,1)$ denotes the standard normal distribution.

The proof of Theorem 2.1 is based on the following sequential bootstrap procedure which is interesting by itself. Let $\left(X_{n}^{(k)}\right)_{n \in \mathrm{Z}}, k=1,2, \ldots$ be a sequence of stationary irreducible aperiodic Markov chains of order $k=1,2, \ldots$, respectively, taking values in the same finite alphabet $A$ with transition probabilities denoted by

$$
p^{(k)}\left(a \mid b_{-k,-1}\right)=\mathrm{P}\left(X_{0}=a \mid X_{-k,-1}^{(k)}=b_{-k,-1}\right) .
$$

We may assume, without loss of generality, that the Markov chains $\left(X_{n}^{(k)}\right)_{n \in \mathrm{Z}}$, for $k=1,2, \ldots$ are all defined on the same probability space (see for instance [9]).

We define

$$
\delta^{(k)}=\min _{a \in A} \inf _{\left(x_{-k}, \ldots, x_{-1}\right) \in \mathcal{A}_{a}^{(k)}} p^{(k)}\left(a \mid x_{-1}, \ldots, x_{-k}\right)
$$

and

$$
\begin{equation*}
\underline{\delta}=\inf \left\{\delta^{(k)}: k \geq 1\right\} \tag{2.6}
\end{equation*}
$$

where $\mathcal{A}_{a}^{(k)}=\left\{\left(x_{-k}, \ldots, x_{-1}\right): p^{(k)}\left(a \mid x_{-1}, \ldots, x_{-k}\right)>0\right\}$.
For each $k$ we define recursively the sequence of return times $\left(R_{j}^{(k)}\right)_{j \in \mathrm{~N}}$ by $R_{0}^{(k)}=1$, and for $i \geq 1$

$$
\begin{equation*}
R_{i}^{(k)}=\inf \left\{n>R_{i-1}^{(k)}:\left(X_{n}^{(k)}, \ldots, X_{n+k-1}^{(k)}\right)=\left(X_{1}^{(k)}, \ldots, X_{k}^{(k)}\right)\right\} . \tag{2.7}
\end{equation*}
$$

Let $\xi_{i}^{(k)}$ be the block of values of the chain $\left(X_{n}^{(k)}\right)_{n \in \mathrm{Z}}$ from $R_{i-1}^{(k)}$ up to $R_{i}^{(k)}-1$, namely

$$
\xi_{i}^{(k)}=\left(X_{R_{i-1}^{(k)}}^{(k)}, \ldots, X_{R_{i}^{(k)}-1}^{(k)}\right)
$$

We construct a bootstrap sample of the Markov chain $\left(X_{n}^{(k)}\right)_{n \in \mathrm{Z}}$ by performing an i.i.d. selection of the blocks $\xi_{l}^{(k)}$. The formal definition is the following. For every $k$, let $I_{1}(k), \ldots, I_{m_{k}}(k)$ be $m_{k}$ independent random variables with uniform distribution in the set $\left\{1, \ldots, m_{k}\right\}$. The bootstrap blocks are defined by

$$
\xi_{l}^{(k) *}=\xi_{I_{l}(k)}^{(k)},
$$

for $l=1, \ldots, m_{k}$. The bootstrap sample $X_{l}^{(k) *}, l=1, \ldots, R_{m_{k}}^{(k) *}$, is constructed by concatenating the blocks $\xi_{1}^{(k) *}, \ldots, \xi_{m_{k}}^{(k) *}$. We observe that the return times of the bootstrap sample assume the values $R_{0}^{(k) *}=1$ and for $l=1, \ldots, m_{k}$

$$
R_{l}^{(k) *}=R_{l-1}^{(k) *}+R_{I_{l}(k)+1}^{(k)}-R_{I_{l}(k)}^{(k)} .
$$

We consider the following estimator for $\mu^{(k)}=\mathrm{E}\left(f\left(X_{1}^{(k)}\right)\right)$

$$
\begin{equation*}
\hat{\mu}^{(k)}=\frac{1}{R_{m_{k}}^{(k)}-1} \sum_{n=1}^{R_{m_{k}}^{(k)}-1} f\left(X_{n}^{(k)}\right) . \tag{2.8}
\end{equation*}
$$

Its bootstrap counterpart is given by

$$
\begin{equation*}
\mu^{(k) *}=\frac{1}{R_{m_{k}}^{(k) *}-1} \sum_{n=1}^{R_{m_{k}}^{(k) *}-1} f\left(X_{n}^{(k) *}\right) \tag{2.9}
\end{equation*}
$$

We define

$$
\begin{equation*}
\sigma^{(k) *}=\sqrt{\frac{\operatorname{Var}\left(\sum_{n=1}^{R_{m}^{(k) *}-1}\left(f\left(X_{n}^{(k) *}\right)-\hat{\mu}^{(k)}\right) \mid X_{1}^{(k)}, \ldots, X_{R_{m_{k}}^{(k)}-1}^{(k)}\right)}{R_{m_{k}}^{(k) *}-1}} \tag{2.10}
\end{equation*}
$$

Recall that, as before, this variance is with respect to the independent random variables $I_{1}(k), \ldots, I_{m_{k}}(k)$.

Theorem 2.2. Let $\left(X_{n}^{(k)}\right)_{n \in \mathrm{Z}}, k=1,2, \ldots$, be a sequence of stationary, irreducible, and aperiodic Markov chains of order $k=1,2, \ldots$, respectively, taking values in the same finite alphabet $A$ and satisfying the following hypotheses

$$
\begin{equation*}
\underline{\delta}>0 \tag{2.11}
\end{equation*}
$$

where $\underline{\delta}$ is defined in (2.6), and

$$
\begin{equation*}
\liminf _{k \rightarrow+\infty} E\left(\left(\sum_{n=1}^{R_{1}^{(k)}-1}\left(f\left(X_{n}^{(k)}\right)-\mu^{(k)}\right)\right)^{2}\right)>0 \tag{2.12}
\end{equation*}
$$

If $\alpha>5 \ln (1 / \underline{\delta})$ and $m_{k}=\left[e^{\alpha k}\right]$, then for almost all realizations of the chains $\left(X_{n}^{(k)}\right)_{n \in \mathrm{Z}}, k=1,2 \ldots$, we have

$$
\frac{\sqrt{R_{m_{k}}^{(k) *}-1}}{\sigma^{(k) *}}\left(\mu^{(k) *}-\hat{\mu}^{(k)}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),
$$

as $k$ tends to $+\infty$.

## 3. Preliminary results

We first introduce some shorthand notation. We define

$$
Z_{i}^{(k)}=\sum_{n=R_{i-1}^{(k)}}^{R_{i}^{(k)}-1}\left(f\left(X_{n}^{(k)}\right)-\hat{\mu}^{(k)}\right),
$$

and its bootstrap version is given by

$$
Z_{i}^{(k) *}=\sum_{n=R_{i-1}^{(k) *}}^{R_{i}^{(k) *}-1}\left(f\left(X_{n}^{(k) *}\right)-\hat{\mu}^{(k)}\right) .
$$

Note that $Z_{i}^{(k) *}=Z_{I_{i}(k)}^{(k)}$.
We use the shorthand $\mathrm{E}^{*}(\cdot)$ to denote $\mathrm{E}\left(\cdot \mid X_{1}^{(k)}, \ldots, X_{R_{m_{k}}^{(k)}}^{(k)}\right)$ and $\operatorname{Var}^{*}(\cdot)$ to denote $\operatorname{Var}\left(\cdot \mid X_{1}^{(k)}, \ldots, X_{R_{m_{k}}^{(k)}}^{(k)}\right)$. We recall that, in both cases, the expectation is taken with respect to the sequence $I_{i}(k), i=1, \ldots, m_{k}$, of i.i.d. random variables uniformly distributed in the set $\left\{1, \ldots, m_{k}\right\}$.

Lemma 3.1. The following equalities hold

$$
\mathrm{E}^{*}\left(Z_{1}^{(k) *}\right)=0,
$$

and

$$
\operatorname{Var}^{*}\left(\sum_{l=1}^{m_{k}} Z_{l}^{(k) *}\right)=\sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{2}
$$

Proof. By definition,

$$
\begin{equation*}
\mathrm{E}^{*}\left(Z_{1}^{(k) *}\right)=\sum_{n=1}^{m_{k}} Z_{n}^{(k)} \mathrm{P}\left(I_{1}^{(k)}=n\right)=\frac{1}{m_{k}} \sum_{n=1}^{m_{k}} Z_{n}^{(k)}=0 \tag{3.1}
\end{equation*}
$$

The second equality follows by a similar computation.
It is convenient to introduce a new family of random variables $\tilde{Z}_{i}^{(k)}$, where $i=1, \ldots, m_{k}$, defined as follows

$$
\begin{equation*}
\tilde{Z}_{i}^{(k)}=\sum_{n=R_{i-1}^{(k)}}^{R_{i}^{(k)}-1}\left(f\left(X_{n}^{(k)}\right)-\mu^{(k)}\right) \tag{3.2}
\end{equation*}
$$

These random variables are not only identically distributed (as it was already the case for $\left(Z_{l}^{(k)}\right)$ ), but also they are independent and have zero mean. Moreover the following relation holds

$$
\begin{equation*}
Z_{l}^{(k)}=\tilde{Z}_{l}^{(k)}+\left(\mu^{(k)}-\hat{\mu}^{(k)}\right)\left(R_{l}^{(k)}-R_{l-1}^{(k)}\right) \tag{3.3}
\end{equation*}
$$

We define $D_{l}^{(k)}=R_{l}^{(k)}-R_{l-1}^{(k)}$ (recall that $R_{0}^{(k)}=1$ ). Similarly, we define $D_{l}^{(k) *}=R_{l}^{(k) *}-R_{l-1}^{(k) *}$.
Lemma 3.2. There is a positive constant $C$ independent of $k$ such that

$$
\left|Z_{1}^{(k)}\right| \leq C D_{1}^{(k)}, \quad \text { and } \quad\left|\tilde{Z}_{1}^{(k)}\right| \leq C D_{1}^{(k)}
$$

Proof. This result follows immediately from the fact that the observable $f$ has finite range.

Lemma 3.3. There is a constant $C>0$ such that, for any $k \geq 1$, the following inequality holds

$$
\mathrm{E}\left(\left(\hat{\mu}^{(k)}-\mu^{(k)}\right)^{2}\right) \leq C \frac{\mathrm{E}\left(\left(D_{1}^{(k)}\right)^{4}\right)}{m_{k}}
$$

Proof. By definition we have

$$
\hat{\mu}^{(k)}-\mu^{(k)}=\frac{\sum_{l=1}^{m_{k}} \tilde{Z}_{l}^{(k)}}{\sum_{l=1}^{m_{k}} D_{l}^{(k)}}
$$

and therefore, using the Markov property and the stationarity of the chain, we have
$\mathrm{E}\left(\left(\hat{\mu}^{(k)}-\mu^{(k)}\right)^{2}\right)=m_{k} \mathrm{E}\left(\frac{\left(\tilde{Z}_{1}^{(k)}\right)^{2}}{\left(\sum_{l=1}^{m_{k}} D_{l}^{(k)}\right)^{2}}\right)+m_{k}\left(m_{k}-1\right) \mathrm{E}\left(\frac{\tilde{Z}_{1}^{(k)} \tilde{Z}_{2}^{(k)}}{\left(\sum_{l=1}^{m_{k}} D_{l}^{(k)}\right)^{2}}\right)$.

Since $\sum_{l=1}^{m_{k}} D_{l}^{(k)}>m_{k}$, and using Lemma 3.2, we conclude that the first term in the right-hand side of expression (3.4) is bounded above by

$$
\begin{equation*}
C \frac{\mathrm{E}\left(\left(D_{1}^{(k)}\right)^{2}\right)}{m_{k}} \tag{3.5}
\end{equation*}
$$

where $C>0$ is a constant independent of $k$.
To obtain an upper bound for the second term on the right-hand side of expression (3.4), we first observe that for $m_{k} \geq 4$ we have

$$
\begin{aligned}
\mathrm{E}\left(\frac{\tilde{Z}_{1}^{(k)} \tilde{Z}_{2}^{(k)}}{\left(\sum_{l=1}^{m_{k}} D_{l}^{(k)}\right)^{2}}\right)= & \mathrm{E}\left(\frac{\tilde{Z}_{1}^{(k)} \tilde{Z}_{2}^{(k)}}{\left(\sum_{l=3}^{m_{k}} D_{l}^{(k)}\right)^{2}}\right) \\
& -\mathrm{E}\left(\frac{\tilde{Z}_{1}^{(k)} \tilde{Z}_{2}^{(k)}\left(D_{1}^{(k)}+D_{2}^{(k)}\right)^{2}}{\left(\sum_{l=3}^{m_{k}} D_{l}^{(k)}\right)^{2}\left(\sum_{l=1}^{m_{k}} D_{l}^{(k)}\right)^{2}}\right) \\
& -2 \mathrm{E}\left(\frac{\tilde{Z}_{1}^{(k)} \tilde{Z}_{2}^{(k)}\left(D_{1}^{(k)}+D_{2}^{(k)}\right)}{\left(\sum_{l=3}^{m_{k}} D_{l}^{(k)}\right)^{2}\left(\sum_{l=1}^{m_{k}} D_{l}^{(k)}\right)^{2}}\right)
\end{aligned}
$$

The independence of $\tilde{Z}_{1}^{(k)}, \tilde{Z}_{2}^{(k)}$ and $\sum_{l=3}^{m_{k}} D_{l}^{(k)}$ imply that

$$
\mathrm{E}\left(\frac{\tilde{Z}_{1}^{(k)} \tilde{Z}_{2}^{(k)}}{\left(\sum_{l=3}^{m_{k}} D_{l}^{(k)}\right)^{2}}\right)=0
$$

Using again Lemma 3.2, Hölder's inequality and $D_{l}^{(k)} \geq 1$, we deduce that the sum of the absolute values of the two remaining terms is bounded above by

$$
\begin{equation*}
C \frac{\mathrm{E}\left(\left(D_{1}^{(k)}\right)^{3}\right)}{m_{k}^{3}}+\frac{\mathrm{E}\left(\left(D_{1}^{(k)}\right)^{4}\right)}{m_{k}^{3}} \tag{3.6}
\end{equation*}
$$

where $C$ is a positive constant independent of $k$. Since $D_{1}^{(k)} \geq 1$, inequalities (3.5) and (3.6) conclude the proof.

Lemma 3.4. For any integer $k$ and any positive real number $t$ the following inequality holds

$$
\mathrm{P}\left(D_{1}^{(k)}>t\right) \leq\left(1-\underline{\delta}^{k}\right)^{[t / k]}
$$

Proof. We observe that

$$
\mathrm{P}\left(D_{1}^{(k)}>t\right) \leq \mathrm{P}\left(\bigcap_{j=1}^{[t / k]}\left\{X_{j k+1,(j+1) k}^{(k)} \neq X_{1, k}^{(k)}\right\}\right)
$$

Now we rewrite the right-hand side of the above inequality, by conditioning on the values of the initial $k$ symbols

$$
\sum_{a_{1, k}} \mathrm{P}\left(X_{1, k}^{(k)}=a_{1, k}\right) \mathrm{P}\left(\bigcap_{j=1}^{[t / k]}\left\{X_{j k+1,(j+1) k}^{(k)} \neq a_{1, k}\right\} \mid X_{1, k}^{(k)}=a_{1, k}\right)
$$

The second factor in the above sum can be rewritten as

$$
\begin{aligned}
& {\left[1-\mathrm{P}\left(X_{[t / k] k+1,([t / k]+1) k}^{(k)}=a_{1, k} \mid \bigcap_{j=1}^{[t / k]-1}\left\{X_{j k+1,(j+1) k}^{(k)} \neq a_{1, k}\right\}\right.\right.} \\
& \left.\left.\quad \cap\left\{X_{1, k}^{(k)}=a_{1, k}\right\}\right)\right] \times \mathrm{P}\left(\bigcap_{j=1}^{[t / k]-1}\left\{X_{j k+1,(j+1) k}^{(k)} \neq a_{1, k}\right\} \mid X_{1, k}^{(k)}=a_{1, k}\right)
\end{aligned}
$$

Using (2.11) this last expression can be bounded above by

$$
\left(1-\underline{\delta}^{k}\right) \mathrm{P}\left(\bigcap_{j=1}^{[t / k]-1}\left\{X_{j k+1,(j+1) k}^{(k)} \neq a_{1, k}\right\} \mid X_{1, k}^{(k)}=a_{1, k}\right)
$$

The lemma now follows by recursion.
Lemma 3.5. There exists a positive constant $C$ such that for any positive integer $r$ and any positive integer $k$, the following inequality holds

$$
\mathrm{E}\left(\left(D_{1}^{(k)}\right)^{r}\right) \leq r!k^{r}\left(\frac{1}{\delta}\right)^{k r}
$$

Proof. The result follows immediately from Lemma 3.4.

## 4. Proof of Theorem 2.2

We can now start the proof of Theorem 2.2. We first observe that

$$
\begin{equation*}
\frac{\sqrt{R_{m_{k}}^{(k) *}-1}}{\sigma^{(k)_{*}}}\left(\mu^{(k) *}-\hat{\mu}^{(k)}\right)=\frac{\sum_{i=1}^{m_{k}} Z_{i}^{(k) *}}{\sqrt{\operatorname{Var}^{*}\left(\sum_{l=1}^{m_{k}} Z_{l}^{(k) *}\right)}} . \tag{4.1}
\end{equation*}
$$

We want to prove that the right-hand side of (4.1) converges in distribution to a standard normal distribution, when $k \rightarrow+\infty$. By the Lindeberg - Feller Central Limit Theorem for double arrays (see, for instance, [3]), this will follow once we show that for any $\varepsilon>0$

$$
\begin{equation*}
\lim _{k \rightarrow+\infty} \frac{\mathrm{E}^{*}\left(\left(Z_{1}^{(k) *}\right)^{2} \mathbf{1}\left\{\left(Z_{1}^{(k) *}\right)^{2}>\varepsilon m_{k} \operatorname{Var}^{*}\left(Z_{1}^{(k) *}\right)\right\}\right)}{\operatorname{Var}^{*}\left(Z_{1}^{(k) *}\right)}=0 \tag{4.2}
\end{equation*}
$$

Using Lemma 3.1 we can rewrite (4.2) as

$$
\begin{equation*}
\lim _{k \rightarrow+\infty} \frac{\sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{2} \mathbf{1}\left\{\left(Z_{l}^{(k)}\right)^{2}>\varepsilon \sum_{j=1}^{m_{k}}\left(Z_{j}^{(k)}\right)^{2}\right\}}{\sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{2}}=0 . \tag{4.3}
\end{equation*}
$$

Since

$$
\begin{equation*}
\mathbf{1}\left\{\left(Z_{l}^{(k)}\right)^{2}>\varepsilon \sum_{j=1}^{m_{k}}\left(Z_{j}^{(k)}\right)^{2}\right\} \leq \frac{\left(Z_{l}^{(k)}\right)^{2}}{\varepsilon \sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{2}} \tag{4.4}
\end{equation*}
$$

the fraction at the left-hand side of expression (4.3) is bounded above by

$$
\begin{equation*}
\frac{\sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{4}}{\varepsilon\left(\sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{2}\right)^{2}} \tag{4.5}
\end{equation*}
$$

To prove that expression (4.5) vanishes as $k$ diverges, we will obtain a sequence of almost sure upper bounds for its numerator and a sequence of almost sure lower bounds for its denominator.

Lemma 4.1. For any $\alpha>0$ and for any $v>1+4 \ln (1 / \underline{\delta}) / \alpha$, if $m_{k}=\left[e^{\alpha k}\right]$, then for almost all samples the upper bound

$$
\sum_{i=1}^{m_{k}}\left(Z_{i}^{(k)}\right)^{4} \leq m_{k}^{v}
$$

holds, for all $k$ large enough.
Proof. Markov's inequality and Lemmas 3.2 and 3.5 imply that

$$
\begin{equation*}
\mathrm{P}\left(\sum_{i=1}^{m_{k}}\left(Z_{i}^{(k)}\right)^{4}>m_{k}^{v}\right) \leq \frac{\mathrm{E}\left(\left(Z_{1}^{(k)}\right)^{4}\right)}{m_{k}^{v-1}} \leq \frac{C k^{4}}{m_{k}^{v-1} \underline{\delta}^{4 k}}, \tag{4.6}
\end{equation*}
$$

where $C>0$ does not depend on $k$. Since by hypothesis $\alpha(v-1)>4 \ln (1 / \underline{\delta})$, we conclude that the right-hand side of expression (4.6) is summable. This together with the Borel-Cantelli lemma concludes the proof of the lemma.

The next step is to find a lower bound for the denominator.
Lemma 4.2. For any $\alpha>4 \ln (1 / \underline{\delta})$, and for any summable sequence of nonnegative real numbers $\eta_{k}, k=1,2, \ldots$, if $m_{k}=\left[e^{\alpha k}\right]$, then, for almost all samples, the lower bound

$$
\sum_{i=1}^{m_{k}}\left(\tilde{Z}_{i}^{(k)}\right)^{2} \geq m_{k} \eta_{k} \mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{2}\right)
$$

holds, for all $k$ large enough.

Proof. To simplify the notation, let us call

$$
W^{(k)}=\sum_{i=1}^{m_{k}}\left(\tilde{Z}_{i}^{(k)}\right)^{2}
$$

By definition we have

$$
\begin{equation*}
\mathrm{E}\left(W^{(k)}\right)=m_{k} \mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{2}\right) \tag{4.7}
\end{equation*}
$$

Using the fact that the random variables

$$
\left(\tilde{Z}_{i}^{(k)}\right)^{2}-\mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{2}\right)
$$

are independent, identically distributed and have zero mean we get

$$
\begin{equation*}
\mathrm{E}\left(\left(W^{(k)}\right)^{2}\right)=m_{k}\left(m_{k}-1\right)\left(\mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{2}\right)\right)^{2}+m_{k} \mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{4}\right) \tag{4.8}
\end{equation*}
$$

Using the inequality of Paley-Zygmund, for $0<\eta<1$, together with the identities (4.7) and (4.8) we obtain the inequality

$$
\mathrm{P}\left(W^{(k)} \geq \eta \mathrm{E}\left(W^{(k)}\right)\right) \geq \frac{(1-\eta)^{2} m_{k}^{2}\left(\mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{2}\right)\right)^{2}}{m_{k}\left(m_{k}-1\right)\left(\mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{2}\right)\right)^{2}+m_{k} \mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{4}\right)}
$$

The right-hand side of the above expression can be rewritten as

$$
\begin{equation*}
(1-\eta)^{2}\left(1-\frac{1}{m_{k}}+\frac{\mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{4}\right)}{m_{k}\left(\mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{2}\right)\right)^{2}}\right)^{-1} \tag{4.9}
\end{equation*}
$$

Therefore Lemma 3.2 and hypothesis (2.12) imply that

$$
\begin{equation*}
\mathrm{P}\left(W^{(k)} \geq \eta \mathrm{E}\left(W^{(k)}\right)\right) \geq(1-\eta)^{2}\left(1-\frac{1}{m_{k}}+\frac{C \mathrm{E}\left(\left(D_{1}^{(k)}\right)^{4}\right)}{m_{k}}\right)^{-1} \tag{4.10}
\end{equation*}
$$

where $C>0$ does not depend on $k$. From this it follows immediately that

$$
\begin{equation*}
\mathrm{P}\left(W^{(k)}<\eta \mathrm{E}\left(W^{(k)}\right)\right) \leq \frac{-1 / m_{k}+C \mathrm{E}\left(\left(D_{1}^{(k)}\right)^{4}\right) / m_{k}+2 \eta-\eta^{2}}{1-1 / m_{k}+C \mathrm{E}\left(\left(D_{1}^{(k)}\right)^{4}\right) / m_{k}} \tag{4.11}
\end{equation*}
$$

Lemma 3.5 and the choice of $\alpha$ imply that the quantity

$$
\begin{equation*}
\left|\frac{1}{m_{k}}-\frac{C \mathrm{E}\left(\left(D_{1}^{(k)}\right)^{4}\right)}{m_{k}}\right| \leq \frac{1}{2} \tag{4.12}
\end{equation*}
$$

for $k$ large enough. Therefore inequality (4.11) implies that

$$
\begin{equation*}
\mathrm{P}\left(W^{(k)}<\eta \mathrm{E}\left(W^{(k)}\right)\right) \leq 2 \frac{C \mathrm{E}\left(\left(D_{1}^{(k)}\right)^{4}\right)}{m_{k}}+4 \eta \tag{4.13}
\end{equation*}
$$

for $k$ large enough. Using again Lemma 3.5 it follows from (4.13) that

$$
\begin{equation*}
\sum_{k=1}^{+\infty} \mathrm{P}\left(W^{(k)}<\eta_{k} \mathrm{E}\left(W^{(k)}\right)\right)<+\infty \tag{4.14}
\end{equation*}
$$

for any summable sequence of non negative real numbers $\eta_{k}, k=1,2, \ldots$ As a consequence, the lemma of Borel-Cantelli implies that

$$
\begin{equation*}
\sum_{i=1}^{m_{k}}\left(\tilde{Z}_{i}^{(k)}\right)^{2} \geq \eta_{k} m_{k} \mathrm{E}\left(\left(\tilde{Z}_{1}^{(k)}\right)^{2}\right) \tag{4.15}
\end{equation*}
$$

almost surely for $k$ large enough.
Lemma 4.3. For any $\alpha>4 \ln (1 / \underline{\delta})$, if $m_{k}=\left[e^{\alpha k}\right]$, then, for almost all samples, the following limit holds

$$
\lim _{k \rightarrow+\infty} \frac{\sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{4}}{\left(\sum_{l=1}^{m_{k}}\left(\tilde{Z}_{l}^{(k)}\right)^{2}\right)^{2}}=0
$$

Proof. The result follows at once from Lemmas 4.1 and 4.2 and the BorelCantelli lemma by taking $1+4 \ln (1 / \underline{\delta}) / \alpha<v<2$ and, for instance, $\eta_{k}=1 / k^{2}$.

The expression in the statement of the above lemma is similar to (4.5) with $Z_{l}^{(k)}$ replaced by $\tilde{Z}_{l}^{(k)}$ in the denominator. Therefore to conclude the proof of Theorem 2.2 we need the following lemma.

Lemma 4.4. For any $\alpha>5 \ln (1 / \underline{\delta})$, if $m_{k}=\left[e^{\alpha k}\right]$, then, for almost all samples, the following limit holds

$$
\lim _{k \rightarrow+\infty} \frac{\sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{2}}{\sum_{l=1}^{m_{k}}\left(\tilde{Z}_{l}^{(k)}\right)^{2}}=1
$$

Proof. An elementary computation shows that for any real numbers $a$ and $b$, and for any $\varepsilon>0$ one has

$$
(1-\varepsilon) a^{2}+\left(1-\varepsilon^{-1}\right) b^{2} \leq(a+b)^{2} \leq(1+\varepsilon) a^{2}+\left(1+\varepsilon^{-1}\right) b^{2} .
$$

We apply this inequality for each $l=1, \ldots, m_{k}$ with $a=\tilde{Z}_{l}^{(k)}$, and $b=\left(\hat{\mu}^{(k)}-\right.$ $\left.\mu^{(k)}\right) D_{l}^{(k)}$. Summing up over $l$ and using identity (3.3) we obtain the inequalities

$$
\begin{equation*}
1-\varepsilon+\left(1-\varepsilon^{-1}\right) \zeta^{(k)} \leq \frac{\sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{2}}{\sum_{l=1}^{m_{k}}\left(\tilde{Z}_{l}^{(k)}\right)^{2}} \leq 1+\varepsilon+\left(1+\varepsilon^{-1}\right) \zeta^{(k)} \tag{4.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\zeta^{(k)}=\left(\hat{\mu}^{(k)}-\mu^{(k)}\right)^{2} \frac{\sum_{l=1}^{m_{k}}\left(D_{l}^{(k)}\right)^{2}}{\sum_{l=1}^{m_{k}}\left(\tilde{Z}_{l}^{(k)}\right)^{2}} \tag{4.17}
\end{equation*}
$$

To conclude the proof it remains to show that $\zeta^{(k)}$ converges to zero almost surely as $k$ diverges.

Using Lemma 3.3, Markov's inequality and the Borel-Cantelli lemma, it follows immediately that for any summable sequence of positive numbers $\rho_{k}$, $k \geq 1$, and for almost all samples, the following inequality holds

$$
\begin{equation*}
\left(\hat{\mu}^{(k)}-\mu^{(k)}\right)^{2} \leq \frac{C}{\rho_{k}}\left[\frac{\mathrm{E}\left(\left(D_{1}^{(k)}\right)^{3}\right)}{m_{k}}+\frac{\mathrm{E}\left(\left(D_{1}^{(k)}\right)^{4}\right)}{m_{k}^{2}}\right] \tag{4.18}
\end{equation*}
$$

for all $k$ large enough, where $C$ is a positive constant independent of $k$. We also observe that for the same sequence $\rho_{k}$ the inequality

$$
\begin{equation*}
\sum_{l=1}^{m_{k}}\left(D_{l}^{(k)}\right)^{2} \leq \frac{m_{k}}{\rho_{k}} \mathrm{E}\left(\left(D_{1}^{(k)}\right)^{2}\right) \tag{4.19}
\end{equation*}
$$

holds almost surely for all $k$ large enough.
Combining Lemma 4.2 and hypothesis (2.12), we conclude that for any summable sequence $\eta_{k}, k \geq 1$, and for almost all sample, the following inequality holds

$$
\begin{equation*}
\sum_{l=1}^{m_{k}}\left(\tilde{Z}_{l}^{(k)}\right)^{2} \geq C m_{k} \eta_{k} \tag{4.20}
\end{equation*}
$$

for all $k$ large enough, where $C$ is a strictly positive constant independent of $k$.
Using inequalities (4.18), (4.19), (4.20), and using Lemma 3.5 we deduce that for almost all samples, the following inequality holds

$$
\zeta^{(k)} \leq C \frac{\exp \{-k(\alpha-5 \ln (1 / \underline{\delta}))\}}{\rho_{k}^{2} \eta_{k}}
$$

for all $k$ large enough, where $C$ is a positive constant independent of $k$. Since by hypothesis, $\alpha>5 \ln (1 / \underline{\delta})$, it is enough to take for instance $\rho_{k}=\eta_{k}=1 / k^{2}$ to conclude $\zeta^{(k)}$ converges to zero almost surely. Recalling that inequality (4.16) holds for any fixed $\varepsilon>0$, the lemma follows.

Combining Lemmas 4.3 and 4.4, it follows that almost surely

$$
\begin{equation*}
\lim _{k \rightarrow+\infty} \frac{\sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{4}}{\varepsilon\left(\sum_{l=1}^{m_{k}}\left(Z_{l}^{(k)}\right)^{2}\right)^{2}}=0 \tag{4.21}
\end{equation*}
$$

This implies (4.2) and concludes the proof of Theorem 2.2.

## 5. Proof of Theorem 2.1

The basic idea of the proof is to approximate the chain of infinite order by a sequence of Markov chains of increasing order satisfying the hypotheses of Theorem 2.2. We will use for this purpose the canonical Markov approximation $\left(X_{n}^{[k]}\right)_{n \in \mathrm{Z}}$ of the chain $\left(X_{n}\right)_{n \in \mathrm{Z}}$ which is the Markov chain of order $k$ whose transition probabilities are defined by

$$
\begin{equation*}
P^{[k]}\left(b \mid a_{1}, \ldots, a_{k}\right):=\mathrm{P}\left(X_{k+1}=b \mid X_{j}=a_{j}, 1 \leq j \leq k\right) \tag{5.1}
\end{equation*}
$$

for all integer $k \geq 1$ and $a_{1}, \ldots, a_{k}, b \in A$.
From now on we only consider stationary chains. The sequence of stationary canonical Markov approximations can be constructed together with the stationary chain of infinite order on the same probability space $(\Omega, \mathcal{F}, \mathrm{P})$. In particular they can be constructed together using the well-known maximal coupling (see, for instance, Appendix A. 1 in [2]). For details of this construction in the present context we refer the reader to [10].

Before starting the proof of Theorem 2.1 we will recall a few results from the literature which will be used in the sequel. The following theorem was proven by Fernández and Galves in [10].

Theorem 5.1. Let $\left(X_{n}\right)_{n \in \mathrm{Z}}$ be a chain of infinite order on the finite alphabet $A$ and satisfying the conditions

$$
\sum_{a \in A} \inf _{\left(\ldots, x_{-2}, x_{-1}\right) \in \mathcal{A}_{a}} p\left(a \mid x_{-1}, x_{-2}, \ldots\right)>0 \quad \text { and } \quad \sum_{l \geq 1} \beta_{l}<+\infty
$$

Then the construction of the chains using the maximal coupling satisfies the following inequality

$$
\begin{equation*}
\mathrm{P}\left\{X_{0}^{[k]} \neq X_{0}\right\} \leq \beta_{k} \tag{5.2}
\end{equation*}
$$

The following theorem is a particular case of the main theorem by Bressaud, Fernández and Galves [5]. For convenience of the reader we will reformulate the result in the framework in which it will be used in the proofs below.

Theorem 5.2. If hypotheses $\mathbf{H}_{\mathbf{1}}$ and $\mathbf{H}_{\mathbf{2}}$ are satisfied, then the chain $\left(X_{n}\right)_{n \in \mathrm{Z}}$ is exponentially $\varphi$-mixing.

For a definition of $\varphi$-mixing chains we refer the reader to [3]. To make the connection between the present hypotheses and the assumptions of [5] we note that hypotheses $\mathbf{H}_{\mathbf{1}}$ and $\mathbf{H}_{\mathbf{2}}$ imply that the sequence of log-continuity rates $\left(\gamma_{l}\right)$ defined by

$$
\gamma_{l}=\max _{a \in A} \sup _{\substack{\left(\ldots, x_{-2}, x_{-1}\right) \in \mathcal{A}_{a} \\ x_{i}=y_{i}, i=-l, \ldots,-1}}\left|\frac{p\left(a \mid x_{-1}, x_{-2}, \ldots\right)}{p\left(a \mid y_{-1}, y_{-2}, \ldots\right)}-1\right|
$$

is exponentially decreasing and therefore satisfies the hypotheses of this paper.
We can now start the proof of Theorem 2.1. First of all we will use the result by Fernández and Galves, mentioned above, to obtain an upper bound for the probability of discrepancies in the first $r$ symbols for the coupled realizations of the chain $\left(X_{n}\right)_{n \in Z}$ and its canonical Markov approximation of order $k,\left(X_{n}^{[k]}\right)_{n \in \mathrm{Z}}$. More precisely let us define

$$
\Delta_{r}^{[k]}:=\left\{X_{t}^{[k]}=X_{t}, t=1, \ldots, r\right\}
$$

which is the set of coincidence up to time $r$ of the chains $\left(X_{n}^{[k]}\right)_{n \in \mathrm{Z}}$ and $\left(X_{n}\right)_{n \in \mathrm{Z}}$.
Lemma 5.1. Let $\left(X_{n}\right)_{n \in \mathrm{Z}}$ be a chain of infinite order satisfying conditions $\mathbf{H}_{\mathbf{1}}$ and $\mathbf{H}_{\mathbf{2}}$ with $\beta_{l}$ summable. Then there exists a positive constant $C$ such that

$$
\mathrm{P}\left\{\left(\Delta_{r}^{[k]}\right)^{c}\right\} \leq C r \beta_{k}
$$

We will now check that the hypotheses of Theorem 2.2 are satisfied by the sequence of canonical Markov approximations $\left(X_{n}^{[k]}\right)_{n \in \mathrm{Z}}, k \geq 1$.

Lemma 5.2. Under assumption $\mathbf{H}_{\mathbf{1}}$ we have

$$
\inf \left\{\delta^{[k]}: k \geq 1\right\} \geq \delta
$$

where

$$
\delta^{[k]}=\min _{a \in A} \inf _{\left(x_{-k}, \ldots, x_{-1}\right) \in \mathcal{A}_{a}^{(k)}} p^{[k]}\left(a \mid x_{-1}, \ldots, x_{-k}\right)
$$

Proof. It follows at once from the properties of the conditional probability.
This lemma establishes condition (2.11). The proof that condition (2.12) holds follows from the next three lemmas. Let us define

$$
Z_{i}(k)=\sum_{n=R_{i-1}(k-1)}^{R_{i}(k)-1}\left(f\left(X_{n}\right)-\mu\right) \quad \text { and } \quad Z_{i}^{[k]}=\sum_{n=R_{i-1}^{[k]}}^{R_{i}^{[k]}-1}\left(f\left(X_{n}^{[k]}\right)-\mu^{[k]}\right)
$$

where $R_{1}^{[k]}$ is defined as in expression (2.7) using the chain $\left(X_{n}^{[k]}\right)_{n \in \mathrm{Z}}$ and $\mu^{[k]}=$ $\mathrm{E}\left(f\left(X_{1}^{[k]}\right)\right)$.

Lemma 5.3. Under hypotheses $\mathbf{H}_{\mathbf{1}}, \mathbf{H}_{\mathbf{2}}$ and $\mathbf{H}_{\mathbf{3}}$ the chain $\left(X_{n}\right)_{n \in \mathrm{Z}}$ satisfies the inequality

$$
\liminf _{k \rightarrow+\infty} \mathrm{E}\left(\left(Z_{1}(k)\right)^{2}\right)>0
$$

Proof. Markov's inequality implies that

$$
\mathrm{E}\left(\left(Z_{1}(k)\right)^{2}\right) \geq u^{2} \mathrm{P}\left\{\left(Z_{1}(k)\right)^{2}>u^{2}\right\}
$$

for any real number $u$. Recalling that $R_{1}(k) \geq 1$, we obtain the lower bound

$$
\begin{equation*}
\mathrm{E}\left(\left(Z_{1}(k)\right)^{2}\right) \geq u^{2} \mathrm{P}\left\{\frac{\left|Z_{1}(k)\right|}{\sqrt{R_{1}(k)}}>u\right\} \tag{5.3}
\end{equation*}
$$

By the above mentioned theorem from [5], the process $\left(f\left(X_{n}\right)\right)_{n}$ is exponentially $\varphi$-mixing. Therefore it follows from classical results on the Central Limit Theorem (see for instance Theorems 20.1 and 20.3 from [3])

$$
\frac{Z_{1}(k)}{\sqrt{R_{1}(k)}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^{2}\right)
$$

as $k$ diverges. Hypothesis $\mathbf{H}_{\mathbf{3}}$ ensures that $\sigma>0$. This implies that for any fixed $u$ and any $k$ large enough the lower bound provided by inequality (5.3) is greater than a fixed strictly positive real number. This concludes the proof of the lemma.

We define $D_{l}(k)=R_{l}(k)-R_{l-1}(k)$.
Lemma 5.4. For any integer $k$, any integer $r \leq 4$ and any positive real number $t$ the following inequalities hold

$$
\mathrm{P}\left(D_{1}(k)>t\right) \leq\left(1-\delta^{k}\right)^{[t / k]} \quad \text { and } \quad \mathrm{E}\left(\left(D_{1}(k)\right)^{r}\right) \leq C k^{r}\left(\frac{1}{\delta}\right)^{k r}
$$

where $C$ is a positive constant.
Proof. The proof is exactly the same as the proofs of Lemmas 3.4 and 3.5.
Lemma 5.5. Under the conditions of Theorem 2.1 the sequence of canonical Markov approximations satisfies the inequality

$$
\liminf _{k \rightarrow+\infty} \mathrm{E}\left(\left(Z_{1}^{[k]}\right)^{2}\right)>0
$$

Proof. We will first derive an upper bound for the modulus of the difference

$$
\left|\mathrm{E}\left(\left(Z_{1}(k)\right)^{2}-\left(Z_{1}^{[k]}\right)^{2}\right)\right|=\left|\mathrm{E}\left(\left(Z_{1}(k)-Z_{1}^{[k]}\right)\left(Z_{1}(k)+Z_{1}^{[k]}\right)\right)\right|
$$

The finiteness of the alphabet $A$ implies that

$$
\begin{equation*}
\left|Z_{1}(k)+Z_{1}^{[k]}\right| \leq C\left|R_{1}(k)+R_{1}^{[k]}\right| \tag{5.4}
\end{equation*}
$$

where $C=\max \{|f(a)|: a \in A\}$. We observe also that

$$
\begin{equation*}
\left|Z_{1}(k)-Z_{1}^{[k]}\right| \leq \sum_{n=1}^{R_{1}(k) \wedge R_{1}^{[k]}}\left|Y_{n}-Y_{n}^{[k]}\right|+C\left|R_{1}(k)-R_{1}^{[k]}\right|, \tag{5.5}
\end{equation*}
$$

where $Y_{n}=f\left(X_{n}\right)-\mu$ and $Y_{n}^{[k]}=f\left(X_{n}^{[k]}\right)-\mu^{[k]}$.
In the sequel we will no longer specify the different positive constants appearing in the various estimates. Moreover they will be all denoted by the letter $C$. Combining inequalities (5.4) and (5.5) we obtain

$$
\begin{align*}
\left|\mathrm{E}\left(\left(Z_{1}(k)\right)^{2}-\left(Z_{1}^{[k]}\right)^{2}\right)\right| \leq & C \mathrm{E}\left(\left|R_{1}(k)-R_{1}^{[k]}\right|\left|R_{1}(k)+R_{1}^{[k]}\right|\right) \\
& +C \mathrm{E}\left(\sum_{n=1}^{R_{1}(k) \wedge R_{1}^{[k]}}\left|Y_{n}-Y_{n}^{[k]}\right|\left|R_{1}(k)+R_{1}^{[k]}\right|\right) . \tag{5.6}
\end{align*}
$$

We will estimate separately each term. For the second term we have

$$
\begin{aligned}
& \mathrm{E}\left(\sum_{n=1}^{R_{1}(k) \wedge R_{1}^{[k]}}\left|Y_{n}-Y_{n}^{[k]}\right|\left|R_{1}(k)+R_{1}^{[k]}\right|\right) \\
& \quad=\mathrm{E}\left(\mathbf{1}\left\{\left(\Delta_{k}^{[k]}\right)^{c}\right\} \sum_{n=1}^{R_{1}(k) \wedge R_{1}^{[k]}}\left|Y_{n}-Y_{n}^{[k]}\right|\left|R_{1}(k)+R_{1}^{[k]}\right|\right) \\
& \quad \leq C \mathrm{E}\left(\mathbf{1}\left\{\left(\Delta_{k}^{[k]}\right)^{c}\right\}\left(R_{1}(k)+R_{1}^{[k]}\right)^{2}\right) .
\end{aligned}
$$

Using Schwarz inequality and Lemmas 3.5, 5.1 and 5.4. we obtain the upper bound

$$
\mathrm{E}\left(\mathbf{1}\left\{\left(\Delta_{k}^{[k]}\right)^{c}\right\}\right)^{1 / 2} \mathrm{E}\left(\left(R_{1}(k)+R_{1}^{[k]}\right)^{4}\right)^{1 / 2} \leq C k^{5 / 2} \beta_{k}^{1 / 2} \delta^{-2 k}
$$

We now come to the estimation of the first term in (5.6). Using Schwarz inequality and Lemmas 3.5 and 5.4 we get

$$
\begin{aligned}
& \mathrm{E}\left(\left|R_{1}(k)-R_{1}^{[k]}\right|\left|R_{1}(k)+R_{1}^{[k]}\right|\right) \\
& \quad \leq \mathrm{E}\left(\left(R_{1}(k)-R_{1}^{[k]}\right)^{2}\right)^{1 / 2} \mathrm{E}\left(\left(R_{1}(k)+R_{1}^{[k]}\right)\right)^{1 / 2} \\
& \quad \leq C k \delta^{-k} \mathrm{E}\left(\left(R_{1}(k)-R_{1}^{[k]}\right)^{2}\right)^{1 / 2} .
\end{aligned}
$$

We now have
$\mathrm{E}\left(\left(R_{1}(k)-R_{1}^{[k]}\right)^{2}\right)=\mathrm{E}\left(\mathbf{1}\left\{\Delta_{k}^{[k]}\right\}\left(R_{1}(k)-R_{1}^{[k]}\right)^{2}\right)+\mathrm{E}\left(\mathbf{1}\left\{\left(\Delta_{k}^{[k]}\right)^{c}\right\}\left(R_{1}(k)-R_{1}^{[k]}\right)^{2}\right)$
and the last term is estimated as above. For the first term, we have

$$
\begin{aligned}
& \mathrm{E}\left(1\left\{\Delta_{k}^{[k]}\right\}\left(R_{1}(k)-R_{1}^{[k]}\right)^{2}\right) \\
& \quad \leq \mathrm{E}\left(R_{1}(k)+R_{1}^{[k]}\right)^{2}\left(1-\prod_{j=R_{1}(k) \wedge R_{1}^{[k]}}^{R_{1}(k) \wedge R_{1}^{[k]}+k-1} 1\left\{X_{j}^{[k]}=X_{j}\right\}\right) \\
& \quad \leq \mathrm{E}\left(\left(R_{1}(k)+R_{1}^{[k]}\right)^{4}\right)^{1 / 2} \mathrm{E}\left(1-\prod_{j=R_{1}(k) \wedge R_{1}^{[k]}}^{R_{1}(k) \wedge R_{1}^{[k]}+k-1} 1\left\{X_{j}^{[k]}=X_{j}\right\}\right)^{1 / 2} \\
& \quad \leq C k^{2} \delta^{-2 k} \mathrm{E}\left(1-\prod_{j=R_{1}(k) \wedge R_{1}^{[k]}}^{R_{1}(k) \wedge R_{1}^{[k]}+k-1} 1\left\{X_{j}^{[k]}=X_{j}\right\}\right)^{1 / 2}
\end{aligned}
$$

where we have used again Schwarz inequality and Lemmas 3.5 and 5.4. We now have

$$
\begin{aligned}
& \mathrm{E}\left(1-\prod_{j=R_{1}(k) \wedge R_{1}^{[k]}}^{R_{1}(k) \wedge R_{1}^{[k]}+k-1} \mathbf{1}\left\{X_{j}^{[k]}=X_{j}\right\}\right) \\
& \quad=\sum_{p=1}^{\infty} \mathrm{E}\left(\mathbf{1}\left\{R_{1}(k) \wedge R_{1}^{[k]}=p\right\}\left(1-\prod_{j=p}^{p+k-1} \mathbf{1}\left\{X_{j}^{[k]}=X_{j}\right\}\right)\right) .
\end{aligned}
$$

Using Schwarz inequality, stationarity and Lemmas 3.5, 5.1 and 5.4 this is bounded above by

$$
\begin{aligned}
& \left(\sum_{p=1}^{\infty} p^{2} \mathrm{E}\left(\mathbf{1}\left\{R_{1}(k) \wedge R_{1}^{[k]}=p\right\}\right)\right)^{1 / 2} \mathrm{E}\left(\mathbf{1}\left\{\left(\Delta_{k}^{[k]}\right)^{c}\right\}\right)^{1 / 2} \\
& \quad \leq \mathrm{E}\left(\left(R_{1}(k)+R_{1}^{[k]}\right)^{2}\right)^{1 / 2} \mathrm{E}\left(\mathbf{1}\left\{\left(\Delta_{k}^{[k]}\right)^{c}\right\}\right)^{1 / 2} \\
& \quad \leq C k^{3 / 2} \delta^{-k} \beta_{k}^{1 / 2}
\end{aligned}
$$

Collecting together the above bounds we get

$$
\left|\mathrm{E}\left(\left(Z_{1}(k)\right)^{2}-\left(Z_{1}^{[k]}\right)^{2}\right)\right| \leq C\left(k^{5 / 2} \delta^{-2 k} \beta_{k}^{1 / 2}+k^{19 / 8} \delta^{-9 k / 4} \beta_{k}^{1 / 8}\right)
$$

It follows from this inequality and assumption $c>18 \log \delta^{-1}$ that

$$
\lim _{k \rightarrow \infty}\left|\mathrm{E}\left(\left(Z_{1}(k)\right)^{2}-\left(Z_{1}^{[k]}\right)^{2}\right)\right|=0
$$

This together with Lemma 5.3 concludes the proof of the lemma.

In order to prove Theorem 2.1 we need to construct together the bootstrap samples of $\left(X_{n}\right)_{n \in Z}$ and $\left(X_{n}^{[k]}\right)_{n \in Z}$. We recall that we have already assumed that $\left(X_{n}\right)_{n \in Z}$ and $\left(X_{n}^{[k]}\right)_{n \in Z}$ are constructed together using the maximal coupling. Now, given two coupled realizations of theses chains we will use the same realization of the sequence of random indices to choose the blocks entering in the bootstrap samples of the chains. Formally, for every fixed $k \geq 1$ the bootstrap blocks will be defined as

$$
\xi_{l}^{*}(k)=\xi_{I_{l}(k)}(k) \quad \text { and } \quad \xi_{l}^{[k] *}=\xi_{I_{l}(k)}^{[k]}
$$

where $I_{1}(k), \ldots, I_{m_{k}}(k)$ are the same independent random variables with uniform distribution in the set $\left\{1, \ldots, m_{k}\right\}$.

The next lemma says that the coupled samples of $\left(X_{n}\right)_{n \in \mathrm{Z}}$ and $\left(X_{n}^{[k]}\right)_{n \in \mathrm{Z}}$ coincide up to time $R_{m_{k}}(k)$ with overwhelming probability.

Lemma 5.6. Under the hypotheses of Theorem 2.1 we have

$$
\lim _{k \rightarrow+\infty} \mathrm{P}\left(\left(\Delta_{R_{m_{k}}}(k)\right)^{c}\right)=0
$$

Proof. We observe that for any $r>0$ we have

$$
\begin{equation*}
\mathrm{P}\left(\left(\Delta_{R_{m_{k}}}(k)\right)^{c}\right) \leq \mathrm{P}\left(\left(\Delta_{r}\right)^{c}\right)+\mathrm{P}\left(R_{m_{k}}(k)>r\right) \tag{5.7}
\end{equation*}
$$

By Lemma 5.1 the first term in the right-hand side of (5.7) is bounded above by $C r \beta_{k}$.

It follows from Lemmas 5.4 and 5.2 that the second term of the right-hand side of (5.7) is bounded above by

$$
m_{k} \mathrm{P}\left(D_{1}(k)>r / m_{k}\right) \leq m_{k}\left(1-\delta^{k}\right)^{\left[r /\left(k m_{k}\right)\right]}
$$

We now set $r=\lambda k^{2} m_{k} \delta^{-k}$, where $\lambda$ is a fixed number strictly larger than $\alpha$. With this choice of $r$ the two terms in inequality (5.7) tend to 0 when $k$ diverges. This concludes the proof of the lemma.

We can now conclude the proof of Theorem 2.1. First of all we observe that

$$
\begin{aligned}
\frac{\sqrt{R_{m_{k}}^{*}(k)}}{\sigma_{k}^{*}}\left(\mu_{k}^{*}-\hat{\mu}_{k}\right)= & \frac{\sigma^{[k] *}}{\sigma_{k}^{*}} \sqrt{\frac{R_{m_{k}}^{*}(k)}{R_{m_{k}}^{[k] *}}} \frac{\sqrt{R_{m_{k}}^{[k] *}}}{\sigma^{[k] *}}\left(\mu^{[k] *}-\hat{\mu}^{[k]}\right) \\
& +\frac{\sqrt{R_{m_{k}}^{*}(k)}}{\sigma_{k}^{*}}\left(\hat{\mu}^{[k]}-\hat{\mu}_{k}\right) \\
& +\frac{\sqrt{R_{m_{k}}^{*}(k)}}{\sigma_{k}^{*}}\left(\mu_{k}^{*}-\mu^{[k] *}\right) .
\end{aligned}
$$

Lemma (5.6) ensures that last two terms are equal to zero with probability tending to 1 when $k$ tends to infinity. Theorem 2.2 implies

$$
\sqrt{\frac{R_{m_{k}}^{*}(k)}{R_{m_{k}}^{[k] *}}} \frac{\sqrt{R_{m_{k}}^{[k] *}}}{\sigma^{[k] *}}\left(\mu^{[k] *}-\hat{\mu}^{[k]}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) .
$$

Finally we observe that Lemma (5.6) ensures that

$$
\lim _{k \rightarrow \infty} \mathrm{P}\left(\frac{\sigma^{[k] *}}{\sigma_{k}^{*}} \sqrt{\frac{R_{m_{k}}^{*}(k)}{R_{m_{k}}^{[k] *}}}=1\right)=1
$$

This concludes the proof of Theorem 2.1.

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