# A POISSON BRACKET ON MULTISYMPLECTIC PHASE SPACE[*]

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A new Poisson bracket for Hamiltonian forms on the full multisymplectic phase space is defined. At least for forms of degree $n - 1$, where $n$ is the dimension of space-time, Jacobi's identity is fulfilled.

**Keywords:** geometric field theory, multisymplectic geometry, Poisson bracket.

## 1.  Introduction

The main idea of the multisymplectic formulation of classical field theory defined by a Lagrangian density $\mathcal{L}$ consists in treating space and time derivatives of fields on an equal footing. The advantage of this approach, as compared to the common canonical formulation, is twofold:

- Lorentz covariance is manifest and automatic,
- phase space is finite-dimensional.

Also twofold are the disadvantages:

- the introduction of several "conjugate momenta" $\pi_i^\mu = \partial\mathcal{L}/\partial\,\partial_\mu\varphi^i$ associated to every field component $\varphi^i$ destroys the usual duality between fields and momenta,
- quantization is unclear in the multisymplectic formalism.

A first step towards multisymplectic quantization is the formulation of multisymplectic Poisson brackets. Pioneer work in this direction has been done by Kijowski [1] (see also [2]). We [3] were motivated by recent innovative work of Kanatchikov [4] that will be briefly sketched below. For a general comprehensive presentation of the multisymplectic formalism, including references to the early literature on the beginnings of the subject, which date back to the second decade of the 20$^{\text{th}}$ century, see [5].

In terms of the covariant De Donder Weyl Hamiltonian,

$$\mathcal{H} \;=\; \pi_i^{\mu} \, \partial_{\mu} \varphi^i - \mathcal{L} \,, \tag{1}$$

the equations of motion can be brought into the form

$$\frac{\partial \mathcal{H}}{\partial \pi_i^{\mu}} \;=\; \partial_{\mu} \varphi^i \,, \tag{2}$$

$$\frac{\partial \mathcal{H}}{\partial \varphi^i} \;=\; -\partial_{\mu} \pi_i^{\mu} \,. \tag{3}$$

The geometry of the multisymplectic phase space $P$ can briefly be described as follows. Let the field $\varphi$ be a section of a fibre bundle $E \xrightarrow{\pi} M$ over an $n$-dimensional space-time manifold $M$ with fibre dimension $N$. Let $(x^{\mu})_{\mu=1,\ldots,n}$ be local coordinates on $M$ and $(q^i)_{i=1,\ldots,N}$ be local coordinates on the standard fibre. The (first) jet bundle $J(E)$ of $E$ is an affine bundle of fibre dimension $nN$ over $E$ and a bundle without special structure and fibre dimension $nN + N$ over $M$. Local coordinates for $J(E)$ can be written as

$$(x^{\mu}, q^i, q_{\mu}^i) \,, \tag{4}$$

where it is understood that the coordinates of the (first) jet of a section $\varphi$ of $E$ at the point $x$ are given by

$$x^{\mu} \,, \qquad q^i \;=\; \varphi^i(x) \,, \qquad q_{\mu}^i \;=\; \partial_{\mu} \varphi^i(x) \,. \tag{5}$$

The multisymplectic phase space $P$ is given by the total space of the (first) cojet bundle $J^*(E)$ of fibrewise affine mappings

$$J(E) \;\longrightarrow\; \pi_E^* \left( \Lambda^n \, T^* M \right) \tag{6}$$

into $n$-forms over $M$. $J^*(E)$ is a vector bundle of fibre dimension $nN + 1$ over $E$. Representing such a fibrewise affine mapping in the form $q_{\mu}^i \;\longmapsto\; \left( p_i^{\mu} q_{\mu}^i + p \right) d^n x$, local coordinates for $J^*(E)$ can be written as

$$(x^{\mu}, q^i, p_i^{\mu}, p) \,. \tag{7}$$

The dimension of the multisymplectic phase space is

$$\dim P = (N + 1)(n + 1) \,. \tag{8}$$

Of fundamental importance are the canonical $n$-form $\theta$ and the multisymplectic $(n + 1)$-form $\omega$ on $J^*(E)$: these can be defined intrinsically and have the coordinate expression

$$\theta \;=\; p_i^\mu \, dq^i \wedge d^n x_\mu \;-\; p \, d^n x \,, \tag{9}$$

$$\omega \;=\; -\, d\theta \;=\; dq^i \wedge dp_i^\mu \wedge d^n x_\mu \;+\; dp \wedge d^n x \;=\; \omega^V \;+\; dp \wedge d^n x \,, \tag{10}$$

where $d^n x_\mu = i_{\partial_\mu} d^n x$ arises by contraction of the volume element $d^n x$ with $\partial_\mu$ [5]; $\omega^V$ is an abbreviation for the "vertical part" of $\omega$ (which, in contrast to $\omega$ itself, has no intrinsic meaning).

$P$ is a field theoretic generalization of the doubly extended phase space of ordinary mechanics, whereas the submanifold

$$P_{\mathcal{H}} \;=\; \{ z \in P \mid p = \mathcal{H}(z) \} \,, \tag{11}$$

carrying forms $\theta_{\mathcal{H}}$, $\omega_{\mathcal{H}}$ and $\omega_{\mathcal{H}}^V$ obtained by restriction from the forms $\theta$, $\omega$ and $\omega^V$ on $P$, respectively, generalizes the extended phase space of mechanics: this is the space used in Kanatchikov's approach [4]. First, the direct generalization of the Hamiltonian vector fields of classical mechanics associated with given Hamiltonians are $n$-multivector fields $X_{\mathcal{H}}$ such that

$$i_{X_{\mathcal{H}}} \omega_{\mathcal{H}}^V \;=\; d^V \mathcal{H} \,, \tag{12}$$

where

$$d^V \;:=\; dq^i \wedge \frac{\partial}{\partial q^i} \;+\; dp_i^\mu \wedge \frac{\partial}{\partial p_i^\mu} \,. \tag{13}$$

Moreover, Kanatchikov defines *Hamiltonian forms* of degree $p$ and *Hamiltonian multivector fields* of degree $n - p$ to be $p$-forms $F$ and $(n - p)$-multivector fields $X$ that can be related through the formula

$$i_X \omega_{\mathcal{H}}^V \;=\; d^V F \,, \tag{14}$$

with the additional restriction that the form $F$ should be *horizontal*. Of course, when $p > 0$, not every $p$-form is Hamiltonian because Eq. (14) imposes a strong integrability condition on $F$. Finally, in analogy with classical mechanics, Kanatchikov defines a generalized Poisson bracket between Hamiltonian forms of arbitrary degree by

$$\left\{ \overset{p}{F_1}, \overset{q}{F_1} \right\} \;=\; (-1)^{n-p} \, i_{X_{F_1}} i_{X_{F_2}} \omega_{\mathcal{H}}^V \,. \tag{15}$$

This bracket is well defined because $F$ determines $X_F$ up to an element in the kernel of $\omega_{\mathcal{H}}^V$. Moreover, it can be checked that Jacobi's identity is fulfilled.

The approach of Kanatchikov provides an important step forward in multisymplectic dynamics, but it suffers from two evident shortcomings.

- The restriction to $P_{\mathcal{H}}$ and horizontal/vertical splitting do a great deal of violence to the multisymplectic structure and introduce nongeneric features like $d^V$ (which may be the price for formulating dynamics in a Poisson Hamiltonian framework).

- The assumption of horizontality of Hamiltonian forms is too restrictive, as can be seen by considering the multimomentum map [5] which provides $(n-1)$-forms associated to generators of symmetries (Noether currents). Horizontality excludes symmetries associated with nontrivial transformations of space-time and an adequate treatment of the energy-momentum tensor.

As far as the first point is concerned, the situation has been alleviated by recent work of Paufler [6] who has shown that a vertical exterior derivative $d^V$ can always be defined and that the bracket (15) does not depend on the ambiguities inherent in its definition.

Our proposal [3] is to completely avoid all these problems by working directly on the full multisymplectic phase space $P = J^*(E)$. Hamiltonian forms $f$ and Hamiltonian multivector fields $X_f$ are defined on $P$, without any horizontality restriction on $f$, and are related by means of the full multisymplectic form $\omega$, according to

$$i_{X_f}\,\omega \;=\; df. \tag{16}$$

## 2.  Hamiltonian multivector fields and forms

The Lie derivative of differential forms along vector fields can be generalized to a Lie derivative of differential forms along multivector fields, defined as the graded commutator between the exterior derivative $d$ and the respective contraction operator: for a $p$-multivector field $X$ on $P$,

$$L_X\,\alpha \;=\; [\,d,i_X]\,\alpha \;=\; \Big(d\,i_X - (-1)^p\,i_X\,d\Big)\alpha\,. \tag{17}$$

On the other hand, we have the Schouten bracket between multivector fields, which is the (unique) extension of the Lie bracket between vector fields by graded derivations (provided one uses an appropriately shifted degree). These operations satisfy the following relations:

$$[\,d,L_X] \;=\; L_X - (-1)^p L_X\,d \;=\; 0\,, \tag{18}$$

$$i_{[X,Y]}\alpha \;=\; (-1)^{(p-1)(q-1)}\Big(L_X\,i_Y - (-1)^{(p-1)q}i_Y\,L_X\Big)\alpha\,, \tag{19}$$

$$L_{[X,Y]}\alpha \;=\; (-1)^{(p-1)q}\Big(L_X L_Y - (-1)^{(p-1)(q-1)}L_Y L_X\Big)\alpha\,. \tag{20}$$

A multivector field $X$ on $P$ is called *locally Hamiltonian* or *multisymplectic* if

$$L_X\,\omega \;=\; 0\,. \tag{21}$$

A direct consequence of Eq. (17) is

LEMMA.  *Every Hamiltonian multivector field is multisymplectic.*

From Eq. (19) we readily infer another lemma.

LEMMA.  *The Schouten bracket $[X,Y]$ of two multisymplectic multivector fields $X$ and $Y$ is Hamiltonian.*

*Proof*:

$$i_{[X,Y]}\,\omega \;=\; \pm\,[\,i_Y, L_X\,]\,\omega \;=\; \pm\,L_X\,i_Y\,\omega$$
$$=\; \pm\,i_X\,d\,i_Y\,\omega \pm d\,i_X\,d\,i_Y\,\omega$$
$$=\; \pm\,d\,i_X\,i_Y\,\omega\,.$$

In what follows, we shall consider mainly vector fields, rather than the more general multivector fields, on $M$, $E$, $J(E)$ and $J^*(E)$. In particular, Hamiltonian vector fields $X$ on $J^*(E)$ will play a prominent role. Their associated Hamiltonian forms $f$ are of degree $n-1$, and the Poisson bracket of two Hamiltonian $(n-1)$-forms will again be a Hamiltonian $p$-form. (More generally, the Poisson bracket of a Hamiltonian $(n-1)$-form with a Hamiltonian $p$-form will again be a Hamiltonian $p$-form.)

We begin with vector fields $X_M$ on $M$ and vector fields $X_E$ on $E$: they generate diffeomorphisms of $M$ and of $E$, respectively. $E$ being not just any manifold but the total space of a fibre bundle $E \xrightarrow{\ \pi\ } M$, there are two special classes of vector fields on $E$, namely *projectable vector fields* that generate bundle automorphisms of $E$ (covering diffeomorphisms of $M$) and *vertical vector fields* that generate strict bundle automorphisms of $E$ (covering the identity on $M$). By definition, a vector field $X_E$ on $E$ is projectable (or more precisely, $M$-projectable) iff there exists a vector field $X_M$ on $M$ such that

$$T\pi\,X_E(e) \;=\; X_M(\pi(e)) \tag{22}$$

for all $e \in E$, and is vertical if this formula holds with $X_M = 0$. In local coordinates $(x^\mu)$ on $M$ and $(x^\mu, q^i)$ on $E$, writing

$$X_M \;=\; X^\mu\frac{\partial}{\partial x^\mu}\,, \tag{23}$$

$$X_E \;=\; X^\mu\frac{\partial}{\partial x^\mu} \;+\; X^i\frac{\partial}{\partial q^i}\,, \tag{24}$$

we see that $X_E$ is projectable iff the $X^\mu$ are independent of the fibre coordinates $q^i$ and that $X_E$ is vertical iff the $X^\mu$ vanish. Now the jet bundles $J(E)$ and the cojet bundle $J^*(E)$ are bundles over $E$ for which we have the following.

THEOREM. *Bundle automorphisms $\Phi_E$ of $E$ over $M$ can be lifted to bundle automorphisms $\Phi_{J(E)}$ of $J(E)$ and $\Phi_{J^*(E)}$ of $J^*(E)$ over $E$. Similarly, $M$-projectable vector fields $X_E$ on $E$ can be lifted to $E$-projectable vector fields $X_{J(E)}$ on $J(E)$ and $X_{J^*(E)}$ on $J^*(E)$.*

*Proof*: These statements can all be inferred from the following formula, which describes how a bundle automorphism $\Phi_E$ of $E$ over $M$ is lifted to a bundle automorphism $\Phi_{J(E)}$ of $J(E)$ over $E$, namely simply by taking the derivative. Indeed, we may think of a point $u_e \in J_e(E)$ as the jet or derivative $T_m\varphi$ of a local section $\varphi$ of $E$ at $m$ satisfying $e = \varphi(m)$, so in particular, $u_e$ is a linear map from $T_mM$ to $T_eE$. Correspondingly, we may set

$$\Phi_{J(E)}\,u_e \;=\; T_e\Phi_E \circ u_e \circ (T_m\Phi_M)^{-1}\,. \tag{25}$$

In local coordinates, the lifting of projectable vector fields is given by

$$X_{J(E)} \;=\; X^\mu \frac{\partial}{\partial x^\mu} \;+\; X^i \frac{\partial}{\partial q^i} \;-\; \left( \frac{\partial X^i}{\partial q^j} q^j_\mu \;-\; \frac{\partial X^\nu}{\partial x^\mu} q^i_\nu \;+\; \frac{\partial X^i}{\partial x^\mu} \right) \frac{\partial}{\partial q^i_\mu}, \qquad (26)$$

and

$$\begin{aligned} X_{J^*(E)} \;=\;& X^\mu \frac{\partial}{\partial x^\mu} \;+\; X^i \frac{\partial}{\partial q^i} \;-\; \left( \frac{\partial X^j}{\partial q^i} p^\mu_j \;-\; \frac{\partial X^\mu}{\partial x^\nu} p^\nu_i \;+\; \frac{\partial X^\nu}{\partial x^\nu} p^\mu_i \right) \frac{\partial}{\partial p^\mu_i} \\ &-\; \left( \frac{\partial X^i}{\partial x^\mu} p^\mu_i \;+\; \frac{\partial X^\nu}{\partial x^\nu} p \right) \frac{\partial}{\partial p}. \end{aligned} \qquad (27)$$

Just as in ordinary mechanics on cotangent bundles, one uses this lift to define the *multimomentum map* [5] which to each projectable vector field $X_E$ on $E$ associates the $(n-1)$-form $J(X_E)$ on $J^*(E)$ defined by contraction with the canonical $n$-form $\theta$:

$$J(X_E) \;=\; i_{X_{J^*(E)}} \theta. \qquad (28)$$

Now invariance of $\theta$ under bundle automorphisms of $J^*(E)$ that arise from bundle automorphisms of $E$ by lifting implies that

$$L_{X_{J^*(E)}} \omega \;=\; 0, \qquad (29)$$

so that

$$i_{X_{J^*(E)}} \omega \;=\; d J(X_E), \qquad (30)$$

which means that $J(X_E)$ is a Hamiltonian $(n-1)$-form. In coordinates one finds [5]

$$J(X_E) \;=\; \left( p^\mu_i X^i + p X^\mu \right) d^n x_\mu \;-\; \tfrac{1}{2} \left( p^\mu_i X^\nu - p^\nu_i X^\mu \right) dq^i \wedge d^n x_{\mu\nu}. \qquad (31)$$

The first term on the right-hand side of this equation, the only one present in Kanatchikov's approach, corresponds to internal symmetry transformations, whereas the remaining terms describe transformations (diffeomorphisms) that act nontrivially on space-time; it is from this part of the multimomentum map that one extracts the energy momentum tensor of field theory [7].

## 3.   Hamiltonian forms of degree $n-1$ and their Poisson bracket

In the previous section, we saw that the multimomentum map, which encompasses the energy-momentum tensor as well as the Noether currents associated with any kind of continuous symmetry in field theory, produces Hamiltonian $(n-1)$-forms. The structure of all Hamiltonian $(n-1)$-forms is completely described by the following.

THEOREM [3]. *Hamiltonian $(n-1)$-forms on $J^*(E)$ are the sum of three contributions:*

1. *the Noether current $J(X_E)$ associated to a projectable vector field $X_E$ on $E$,*

2. *the pull-back of a horizontal $(n-1)$-form on $E$ to $J^*(E)$,*

3. *any closed $(n-1)$-form on $J^*(E)$.*

In local coordinates, this decomposition (which is of course not unique) can be written explicitly as follows. Let

$$X_E = X^\mu \frac{\partial}{\partial x^\mu} + X^i \frac{\partial}{\partial q^i} \tag{32}$$

be a projectable vector field on $E$ and

$$f_0 = f_0^\mu \, d^n x_\mu \tag{33}$$

be an $(n-1)$-form on $J^*(E)$ obtained from a horizontal $(n-1)$-form on $E$ (with the same local coordinate expression) by pull-back: this means that the coefficient functions $X^\mu$ depend only on the variables $x^\nu$ while the coefficient functions $X^i$ and $f_0^\mu$ depend only on the variables $x^\nu$ and $q^j$. Define

$$f = J(X_E) + f_0 \,; \tag{34}$$

so

$$f = \left(p_i^\mu X^i + p X^\mu + f_0^\mu\right) d^n x_\mu - \frac{1}{2}\left(p_i^\mu X^\nu - p_i^\nu X^\mu\right) dq^i \wedge d^n x_{\mu\nu} \,. \tag{35}$$

Then $f$ is a Hamiltonian $(n-1)$-form, and the corresponding Hamiltonian vector field $X_f$ reads

$$X_f = X^\mu \frac{\partial}{\partial x^\mu} + X^i \frac{\partial}{\partial q^i} - \left(\frac{\partial X^j}{\partial q^i} p_j^\mu - \frac{\partial X^\mu}{\partial x^\nu} p_i^\nu + \frac{\partial X^\nu}{\partial x^\nu} p_i^\mu + \frac{\partial X^\mu}{\partial q^i} p + \frac{\partial f_0^\mu}{\partial q^i}\right) \frac{\partial}{\partial p_i^\mu}$$
$$- \left(\frac{\partial X^i}{\partial x^\mu} p_i^\mu + \frac{\partial X^\nu}{\partial x^\nu} p + \frac{\partial f_0^\mu}{\partial x^\mu}\right) \frac{\partial}{\partial p} \,. \tag{36}$$

We shall also write these expressions for $f$ and for $X_f$ in the form

$$f = f^\mu \, d^n x_\mu + \tfrac{1}{2} f_i^{\mu\nu} \, dq^i \wedge d^n x_{\mu\nu} \,, \tag{37}$$

and

$$X_f = \frac{\partial f^\mu}{\partial p} \frac{\partial}{\partial x^\mu} + \frac{1}{n} \frac{\partial f^\mu}{\partial p_i^\mu} \frac{\partial}{\partial q^i} - \left(\frac{\partial f^\mu}{\partial q^i} - \frac{\partial f_i^{\mu\nu}}{\partial x^\nu}\right) \frac{\partial}{\partial p_i^\mu} - \frac{\partial f^\mu}{\partial x^\mu} \frac{\partial}{\partial p} \,. \tag{38}$$

The theorem claims that up to a closed $(n-1)$-form, $f$ is the most general Hamiltonian $(n-1)$-form $f$ on $J^*(E)$. Note the integrability constraints that express themselves through the dependence of the coefficient functions on the variables $p_i^\mu$ and $p$, which is affine (linear plus constant).

For the definition of a Poisson bracket between Hamiltonian $(n-1)$-forms $f$ and $g$, the first idea would be to set

$$\{f, g\}' = i_{X_g} i_{X_f} \omega \,, \tag{39}$$

since this gives

$$[X_f, X_g] = - X_{\{f,g\}'} . \tag{40}$$

But this bracket satisfies Jacobi's identity only up to an exact term,

$$\{f, \{g, h\}'\}' + \{g, \{h, f\}'\}' + \{h, \{f, g\}'\}' = d\left(i_{X_f} i_{X_g} i_{X_h} \theta\right) . \tag{41}$$

This disease can be cured [3] by adding a correction term, which is a uniquely defined exact $(n-1)$-form in order to guarantee that, as before,

$$[X_f, X_g] = - X_{\{f,g\}} . \tag{42}$$

Explicitly,

$$\{f, g\} = i_{X_g} i_{X_f} \omega + d\left(i_{X_g} f - i_{X_f} g - i_{X_g} i_{X_f} \theta\right) . \tag{43}$$

It can be checked [3] that this new bracket does satisfy Jacobi's identity and hence provides the space of Hamiltonian $(n-1)$-forms on $J^*(E)$ with the structure of a Lie algebra. By an explicit calculation in local coordinates using the above expressions for $f$, $X_f$ and analogous ones for $g$, $X_g$, one finds

$$
\begin{aligned}
\{f, g\} = & \left[\frac{\partial X^\nu}{\partial x^\nu} g^\mu - f^\mu \frac{\partial Y^\nu}{\partial x^\nu} + \frac{\partial f^\mu}{\partial q^i} Y^i - X^i \frac{\partial g^\mu}{\partial q^i}\right] d^n x_\mu \\
& - \left[\left(\frac{\partial X^\nu}{\partial q^i} g^\mu - f^\mu \frac{\partial Y^\nu}{\partial q^i}\right) + p_i^\mu \left(\frac{\partial X^\nu}{\partial x^\rho} Y^\rho - X^\rho \frac{\partial Y^\nu}{\partial x^\rho}\right) \right. \\
& \left. - p\left(\frac{\partial X^\nu}{\partial q^i} Y^\mu - X^\mu \frac{\partial Y^\nu}{\partial q^i}\right)\right] dq^i \wedge d^n x_{\mu\nu} .
\end{aligned}
\tag{44}
$$

The calculation shows that the correction terms in the definition of the new bracket lead to strong cancellations and greatly simplifies the final result.

In order to extend the new Poisson bracket to Hamiltonian forms of arbitrary degree, two problems need to be solved.

- Eq. (43) has to be modified by introducing signs depending on the degrees of the Hamiltonian forms such that a graded version of Jacobi's identity still holds.
- For forms $f$ of degree other than $n-1$, $df$ no longer determines the Hamiltonian multivector field $X_f$ uniquely. This ambiguity has to be fixed in a consistent way.

These questions are presently under investigation.

## REFERENCES

[1] J. Kijowski: *Commun. Math. Phys.* **30**(1973), 99 ; *Bull. Acad. Sci. Polon.* **22** (1974), 1219.
[2] J. Kijowski and W. Tulczyew: *A Symplectic Framework for Field Theories*, LNP **107**, Springer, New York 1979.
[3] M. Forger and H. Römer: *The Poisson Bracket for Differential Forms on Multisymplectic Phase Space*, in preparation.
[4] I. V. Kanatchikov: *Rep. Math. Phys.* **40** (1997), 225; *Rep. Math. Phys.* **41** (1998), 49.
[5] M. J. Gotay, J. Isenberg and J. E. Marsden: *Momentum Maps and Classical Relativistic Fields*, Part I: *Covariant Field Theory*, **physics/9801019**.
[6] C. Paufler: *Rep. Math. Phys.* **47** (2001), 101.
[7] M. J. Gotay and J. E. Marsden: *Contemp. Math.* **132** (1992), 367.