

## MULTISYMPLECTIC AND POLYSYMPLECTIC STRUCTURES ON FIBER BUNDLES

MICHAEL FORGER

*Departamento de Matemática Aplicada,  
Instituto de Matemática e Estatística, Universidade de São Paulo,  
Caixa Postal 66281, BR-05314-970 São Paulo, S.P., Brazil  
forger@ime.usp.br*

LEANDRO G. GOMES

*Instituto de Matemática e Computação, Universidade Federal de Itajubá,  
Caixa Postal 50, BR-37500-903 Itajubá, M.G., Brazil  
lggomes@unifei.edu.br*

Received 12 February 2012

Revised 10 September 2013

Accepted 10 September 2013

Published 14 October 2013

We propose new definitions of the concepts of a multisymplectic structure and of a polysymplectic structure which extend previous ones so as to cover the cases that are of interest in mathematical physics: they are tailored to apply to fiber bundles, rather than just manifolds, and at the same time they are sufficiently specific to allow us to prove Darboux theorems for the existence of canonical local coordinates. A key role is played by the notion of “symbol” of a multisymplectic form, which is a polysymplectic form representing its leading order contribution, thus clarifying the relation between these two closely related but not identical concepts.

*Keywords:* Multisymplectic geometry; Classical field theory; Hamiltonian formalism.

Mathematics Subject Classification 2010: 53D99, 53C15, 53B99, 53Zxx, 15A75

### 1. Introduction

Multisymplectic geometry is increasingly recognized as providing the appropriate mathematical framework for classical field theory from the hamiltonian point of view — just as symplectic geometry does for classical mechanics. Unfortunately, the development of this new area of differential geometry has for a long time been hampered by the lack of a fully satisfactory definition of the concept of a multisymplectic structure, which should be mathematically simple as well as in harmony with the needs of applications to physics; the same goes for the closely related notion of a polysymplectic structure.

The main goal of this paper is to provide such a definition and establish a general relation between the two types of structure.

To set the stage, let us consider a simple analogy. The symplectic forms encountered in classical mechanics can locally all be written in the form

$$\omega = dq^i \wedge dp_i, \tag{1.1}$$

where  $q^1, \dots, q^N, p_1, \dots, p_N$  are a particular kind of local coordinates on phase space known as canonical coordinates or Darboux coordinates. Introducing time  $t$  and energy  $E$  as additional variables (which is essential, e.g., for incorporating non-autonomous systems into the symplectic framework of hamiltonian mechanics), this equation is replaced by

$$\omega = dq^i \wedge dp_i + dE \wedge dt, \tag{1.2}$$

where  $t, q^1, \dots, q^N, p_1, \dots, p_N, E$  can be viewed as canonical coordinates on an extended phase space. Similarly, the multisymplectic forms encountered in classical field theory over an  $n$ -dimensional space-time manifold  $M$  can locally all be written in the form

$$\omega = dq^i \wedge dp_i^\mu \wedge d^n x_\mu - dp \wedge d^n x, \tag{1.3}$$

where  $x^\mu, q^i, p_i^\mu, p$  ( $1 \leq \mu \leq n, 1 \leq i \leq N$ ) can again be viewed as canonical coordinates on some extended multiphase space. Here, the  $x^\mu$  are (local) coordinates for  $M$ , while  $p$  is still a single energy variable (except for a sign),  $d^n x$  is the (local) volume form induced by the  $x^\mu$  and  $d^n x_\mu$  is the (local)  $(n-1)$ -form<sup>a</sup> obtained by contracting  $d^n x$  with  $\partial_\mu \equiv \partial/\partial x^\mu$ :

$$d^n x_\mu = i_{\partial_\mu} d^n x.$$

The idea of introducing “multimomentum variables” labeled by an additional space-time index  $\mu$  ( $n$  multimomentum variables  $p_i^\mu$  for each position variable  $q^i$ ) goes back to the work of de Donder [1] and Weyl [2] in the 1930’s (and perhaps even further) and has been recognized ever since as being an essential and unavoidable ingredient in any approach to a generally covariant hamiltonian formulation of classical field theory. Understanding the proper geometric setting for this kind of structure, however, has baffled both mathematicians and physicists for decades, as witnessed by the large number of different proposals for an appropriate global framework that can be found in the literature.

The by now standard example of a globally defined multisymplectic structure starts out from an arbitrary fiber bundle  $E$  over  $M$  called the configuration bundle (since its sections are the basic fields of the field theory under consideration) and whose typical fiber is an  $N$ -dimensional manifold  $Q$  representing the configuration space, as in mechanics. Following [3, 4], for example, consider the vector bundle  $\bigwedge_1^n T^*E$  of  $n$ -forms over  $E$  that are  $(n-1)$ -horizontal (i.e., that vanish whenever

<sup>a</sup>Some authors prefer to write  $d^{n-1}x_\mu$ , but  $d^n x_\mu$  is shorter.

contracted with more than 1 vertical vector field), with projection onto  $E$  denoted by  $\pi_1^n$ : its total space carries a naturally defined  $n$ -form  $\theta$  which we shall refer to as the multicanonical form, given by

$$\theta_\alpha(v_1, \dots, v_n) = \alpha(T_\alpha \pi_1^n \cdot v_1, \dots, T_\alpha \pi_1^n \cdot v_n) \quad (1.4)$$

for  $\alpha \in \bigwedge_1^n T^*E$  and  $v_1, \dots, v_n \in T_\alpha(\bigwedge_1^n T^*E)$ , and which gives rise to a closed  $(n+1)$ -form  $\omega = -d\theta$ . This is the “standard” multisymplectic form encountered in the (first order) covariant hamiltonian formalism of classical field theory. However, the construction can be easily extended to the more general situation of the vector bundle  $\bigwedge_{r-1}^k T^*E$  of  $k$ -forms over  $E$  that are  $(k+1-r)$ -horizontal (i.e., that vanish whenever contracted with more than  $r-1$  vertical vector fields), with projection onto  $E$  denoted by  $\pi_{r-1}^k$ , where  $1 \leq r \leq k$  and  $k+1-r \leq n$ : its total space carries a naturally defined  $k$ -form  $\theta$  which we shall again refer to as the multicanonical form, given by

$$\theta_\alpha(v_1, \dots, v_k) = \alpha(T_\alpha \pi_{r-1}^k \cdot v_1, \dots, T_\alpha \pi_{r-1}^k \cdot v_k) \quad (1.5)$$

for  $\alpha \in \bigwedge_{r-1}^k T^*E$  and  $v_1, \dots, v_k \in T_\alpha(\bigwedge_{r-1}^k T^*E)$ , and which gives rise to a closed  $(k+1)$ -form  $\omega = -d\theta$ . In order to keep the terminology simple, we shall maintain the term “multisymplectic form” even in this case.

The usual local coordinate expressions can be obtained by starting out from local coordinates  $(x^\mu, q^i)$  for  $E$  composed of local coordinates  $x^\mu$  for  $M$  and local coordinates  $q^i$  for  $Q$  together with a local trivialization of  $E$  over  $M$ : these give rise to canonical local coordinates  $(x^\mu, q^i, p_i^\mu, p)$  for  $\bigwedge_1^n T^*E$  in which

$$\theta = p_i^\mu dq^i \wedge d^n x_\mu + p d^n x, \quad (1.6)$$

so

$$\omega = dq^i \wedge dp_i^\mu \wedge d^n x_\mu - dp \wedge d^n x, \quad (1.7)$$

and more generally, to canonical local coordinates  $(x^\mu, q^i, p_{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}})$ , with  $0 \leq s \leq r-1$ , for  $\bigwedge_{r-1}^k T^*E$  in which

$$\theta = \sum_{s=0}^{r-1} \frac{1}{s!} \frac{1}{(k-s)!} p_{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} dq^{i_1} \wedge \dots \wedge dq^{i_s} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{k-s}}, \quad (1.8)$$

so

$$\omega = - \sum_{s=0}^{r-1} \frac{1}{s!} \frac{1}{(k-s)!} dp_{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_s} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{k-s}}. \quad (1.9)$$

In a more general context, when  $\bigwedge_{r-1}^k T^*E$  is replaced by a manifold  $P$  which is only supposed to be the total space of a fiber bundle over a base manifold  $M$ , Darboux’s theorem guarantees the existence of canonical local coordinates in which  $\omega$  is given by the expression in Eq. (1.9), under appropriate conditions on the form  $\omega$ . The central question is to figure out what precisely are these conditions.

A naive first guess would be to simply require the form  $\omega$  to be closed and non-degenerate. However, unlike in the symplectic case, these conditions alone are far too weak to guarantee the validity of a Darboux theorem, even at the purely algebraic level. For certain purposes, they may be sufficient to derive results that are of interest (for an example, see [5–7]), but this version of the definition of a multisymplectic structure – even though often adopted in the literature, mostly for lack of a better one – is clearly inadequate. What is needed is an additional algebraic condition.

An indication of what should be this additional algebraic condition can be found in [8], but the Darboux theorem proved there covers a special situation which is disjoint from the case of interest for the applications to physics because the structure of the underlying manifold as the total space of a fiber bundle over space-time and the corresponding horizontality conditions are completely ignored. More specifically, [8] deals with a multisymplectic form  $\omega$  on a manifold, viewed as the total space of a fiber bundle whose base manifold  $M$  is reduced to a point, so that the pertinent horizontality condition becomes empty; this corresponds to the choice  $n = 0$ ,  $r = k + 1$  and implies that the corresponding expression (1.9) for  $\omega$  in canonical local coordinates takes the form

$$\omega = -\frac{1}{k!} dp_{i_1 \dots i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k}. \quad (1.10)$$

In some sense, this does generalize the concept of a symplectic form to forms of higher degree, but in a direction that is far away from the concept of a multisymplectic form as encountered in classical field theory, where, as we have seen above,  $n = k$ ,  $r = 2$ . This discrepancy with the traditional use of the term “multisymplectic” since the mid 1970’s [11–14], which was recognized and clearly stated only much later [9] (see also [10], for example), has created a great deal of confusion in the literature. Clearly, the incorporation of both cases in the more general scheme treated here serves to overcome this unpleasant situation.

The general idea of our approach is to combine the methods developed in [8] with a new and more profound understanding of the link between multisymplectic and polysymplectic structures.

Polysymplectic structures in the hamiltonian approach to classical field theory seem to have been introduced in [15] and have been further investigated in [16] (where they were called “ $k$ -symplectic structures” — a terminology that we shall not follow in order not to increase the already existing confusion). Roughly speaking, polysymplectic forms are vector-valued analogues of symplectic forms. More generally, the polysymplectic forms to be introduced in this paper are vector-valued analogues of the forms studied in [8].

The standard example of a globally defined polysymplectic structure is the one on the bundle  $T^*E \otimes \hat{T}$  of  $\hat{T}$ -valued 1-forms over a manifold  $E$ , with projection onto  $E$  denoted by  $\pi^1$ , where  $\hat{T}$  is a fixed finite-dimensional auxiliary vector space: its total space carries a naturally defined  $\hat{T}$ -valued 1-form  $\theta$  which we shall refer to

as the polycanonical form, given by

$$\theta_\alpha(v) = \alpha(T_\alpha \pi^1 \cdot v) \tag{1.11}$$

for  $\alpha \in T^*E \otimes \hat{T}$  and  $v \in T_\alpha(T^*E \otimes \hat{T})$ , and which gives rise to a closed  $\hat{T}$ -valued 2-form  $\omega = -d\theta$ . This is the “standard” polysymplectic form introduced, for instance, in [15]. Again, the construction can be easily extended to the more general situation of the vector bundle  $\bigwedge^k T^*E \otimes \hat{T}$  of  $\hat{T}$ -valued  $k$ -forms over  $E$ , with projection onto  $E$  denoted by  $\pi^k$ : its total space carries a naturally defined  $\hat{T}$ -valued  $k$ -form  $\theta$  which we shall again refer to as the polycanonical form, given by

$$\theta_\alpha(v_1, \dots, v_k) = \alpha(T_\alpha \pi^k \cdot v_1, \dots, T_\alpha \pi^k \cdot v_k) \tag{1.12}$$

for  $\alpha \in \bigwedge^k T^*E \otimes \hat{T}$  and  $v_1, \dots, v_k \in T_\alpha(\bigwedge^k T^*E \otimes \hat{T})$ , and which gives rise to a closed  $\hat{T}$ -valued  $(k + 1)$ -form  $\omega = -d\theta$ . In order to keep the terminology simple, we shall maintain the term “polysymplectic form” even in this case.

In terms of the usual local coordinates  $(q^i, p_i)$  for  $T^*E$  and an arbitrary basis  $\{\hat{e}_a \mid 1 \leq a \leq \hat{n}\}$  of  $\hat{T}$ , we have

$$\hat{\theta} = p_i^a dq^i \otimes \hat{e}_a, \tag{1.13}$$

so

$$\hat{\omega} = dq^i \wedge dp_i^a \otimes \hat{e}_a, \tag{1.14}$$

in the special case  $k = 2$ , and

$$\hat{\theta} = \frac{1}{k!} p_{i_1 \dots i_k}^a dq^{i_1} \wedge \dots \wedge dq^{i_k} \otimes \hat{e}_a, \tag{1.15}$$

so

$$\hat{\omega} = -\frac{1}{k!} dp_{i_1 \dots i_k}^a \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k} \otimes \hat{e}_a, \tag{1.16}$$

in the general case of arbitrary  $k$ . In a more general context, when  $\bigwedge^k T^*E \otimes \hat{T}$  is replaced by a manifold  $P$ , Darboux’s theorem guarantees the existence of canonical local coordinates in which  $\hat{\omega}$  is given by the expression in Eq. (1.16), under appropriate conditions on the form  $\hat{\omega}$ . Again, the central question is to figure out what precisely are these conditions.

It should be pointed out that for the special case  $k = 2$ , this problem has been solved in [16], but the fundamental role of what we call the polylagrangian subbundle is not fully appreciated there. As it turns out, this object and its basic properties are the key to the entire subject, allowing to generalize the proof of Darboux’s theorem not only from the special case  $k = 2$  to that of arbitrary  $k$ , but also from polysymplectic structures on manifolds to polysymplectic structures on the total spaces of fiber bundles — a concept that conveys a precise mathematical meaning to the idea of a “smooth family of polysymplectic structures” (each fiber is a polysymplectic manifold in such a way that the entire structure depends smoothly

on the points of the base manifold). This extension is particularly useful in that it finally allows to formulate in precise mathematical terms what is the relation between multisymplectic and polysymplectic structures: the latter appear as the “leading order term” of the former, through a simple formal construction that we introduce below and propose to call the “symbol” because it strongly resembles the construction of the symbol of a differential operator.

## 2. Polysymplectic Forms on Vector Spaces

We begin by briefly recalling a few basic notions involving vector-valued alternating multilinear forms. Given finite-dimensional real vector spaces<sup>b</sup>  $V$  and  $\hat{T}$ , we consider  $\hat{T}$ -valued  $(k + 1)$ -forms  $\hat{\omega}$  on  $V$ ,

$$\hat{\omega} \in \bigwedge^{k+1} V^* \otimes \hat{T}. \tag{2.1}$$

The **contraction** of such a form  $\hat{\omega}$  is the linear map  $\hat{\omega}^b : V \rightarrow \bigwedge^k V^* \otimes \hat{T}$  given by

$$\hat{\omega}^b(v) = i_v \hat{\omega}, \tag{2.2}$$

and the **kernel** of  $\hat{\omega}$  is defined to be the kernel of  $\hat{\omega}^b$ :  $\ker \hat{\omega} = \ker \hat{\omega}^b$ . If this kernel is  $\{0\}$ , we say that  $\hat{\omega}$  is **non-degenerate**. Next, given a linear form  $\hat{t}^* \in \hat{T}^*$  on  $\hat{T}$ , the **projection** of  $\hat{\omega}$  along  $\hat{t}^*$  is the ordinary  $(k + 1)$ -form on  $V$  given by<sup>c</sup>

$$\omega_{\hat{t}^*} = \langle \hat{t}^*, \hat{\omega} \rangle. \tag{2.3}$$

Note that  $\omega_{\hat{t}^*}$  depends linearly on  $\hat{t}^*$ , so if we choose a basis  $\{\hat{e}_1, \dots, \hat{e}_{\hat{n}}\}$  of  $\hat{T}$ , with dual basis  $\{\hat{e}^1, \dots, \hat{e}^{\hat{n}}\}$  of  $\hat{T}^*$ , we have

$$\hat{\omega} = \omega^a \otimes \hat{e}_a \quad \text{with} \quad \omega^a = \omega_{\hat{e}^a} \quad (1 \leq a \leq \hat{n}). \tag{2.4}$$

Then it is clear that

$$\ker \hat{\omega} = \bigcap_{\hat{t}^* \in \hat{T}^*} \ker \omega_{\hat{t}^*} = \bigcap_{a=1}^{\hat{n}} \ker \omega^a. \tag{2.5}$$

Now suppose that  $L$  is a subspace of  $V$  and  $\ell$  is an integer satisfying  $1 \leq \ell \leq k$ ; then extending the definition given in [9] from ordinary to vector-valued forms, we define the  **$\ell$ -orthogonal complement** of  $L$  (with respect to  $\hat{\omega}$ ) to be the subspace  $L^{\hat{\omega}, \ell}$  of  $V$  given by<sup>d</sup>

$$L^{\hat{\omega}, \ell} = \{v \in V \mid i_v i_{v_1} \dots i_{v_\ell} \hat{\omega} = 0 \text{ for all } v_1, \dots, v_\ell \in L\}. \tag{2.6}$$

<sup>b</sup>In order to simplify the presentation, we assume all vector spaces involved to be real and finite-dimensional: the extension to vector spaces over an arbitrary field of characteristic 0 is straightforward, and generalization to the infinite-dimensional setting, which requires imposing appropriate continuity conditions from functional analysis, will be left to a possible future investigation.

<sup>c</sup>Throughout this paper the symbol  $\langle \cdot, \cdot \rangle$  will stand for the natural bilinear pairing between a vector space and its dual.

<sup>d</sup>We discard the trivial case  $\ell = 0$  since extrapolating the definition to this case would lead to the conclusion that  $L^{\hat{\omega}, 0}$  is simply the kernel of  $\hat{\omega}$ , independently of the subspace  $L$  of  $V$ .

Note that these orthogonal complements form an increasing sequence under inclusion:

$$L^{\hat{\omega},1} \subset \dots \subset L^{\hat{\omega},k}. \tag{2.7}$$

We say that  $L$  is  $\ell$ -**isotropic** (with respect to  $\hat{\omega}$ ) if  $L \subset L^{\hat{\omega},\ell}$  and is **maximal  $\ell$ -isotropic** or, more briefly,  $\ell$ -**lagrangian** (with respect to  $\hat{\omega}$ ) if it is maximal in the partially ordered set formed by the  $\ell$ -isotropic subspaces of  $V$ ; it is a simple exercise to check that, as usual, this is the case if and only if  $L = L^{\hat{\omega},\ell}$ . If  $\ell = 1$ , we omit the prefix 1 and may conclude that a subspace  $L$  of  $V$  will be isotropic if and only if<sup>e</sup>

$$\hat{\omega}^b(L) \subset \hat{\omega}^b(V) \cap (\bigwedge^k L^\perp) \otimes \hat{T} \tag{2.8}$$

and will be maximal isotropic if and only if  $\ker \hat{\omega} \subset L$  and

$$\hat{\omega}^b(L) = \hat{\omega}^b(V) \cap (\bigwedge^k L^\perp) \otimes \hat{T}. \tag{2.9}$$

(For explicit proofs of these elementary statements, the reader may consult [17].) At first sight, the intersection with the subspace  $\hat{\omega}^b(V)$  on the right-hand side of these relations may seem strange, in particular since in the inclusion stated in Eq. (2.8) it is really superfluous, but that is by no means the case for the equality stated in Eq. (2.9). Rather, omitting this intersection leads to a strengthened form of Eq. (2.9) which turns out to provide the key to the theory of polysymplectic forms:

**Definition 2.1.** Let  $V$  and  $\hat{T}$  be finite-dimensional vector spaces ( $\dim \hat{T} \equiv \hat{n}$ ), and let  $\hat{\omega}$  be a non-vanishing  $\hat{T}$ -valued  $(k + 1)$ -form on  $V$ . We say that  $\hat{\omega}$  is a **polypresymplectic form of rank  $N$**  if  $V$  admits a subspace  $L$  of codimension  $N$  which is **polylagrangian**, i.e., such that

$$\hat{\omega}^b(L) = (\bigwedge^k L^\perp) \otimes \hat{T}. \tag{2.10}$$

If  $\hat{\omega}$  is non-degenerate, we call it a **polysymplectic form**. The “standard” case of main interest is when  $\hat{\omega}$  is a 2-form, i.e.,  $k = 1$ .

As a first property of polypresymplectic forms, we note that a polylagrangian subspace, when it exists, contains the kernel of  $\hat{\omega}$  and hence really is a special type of maximal isotropic subspace. But more than that is true.

**Proposition 2.1.** Let  $V$  and  $\hat{T}$  be finite-dimensional vector spaces ( $\dim \hat{T} = \hat{n}$ ), and let  $\hat{\omega}$  be a  $\hat{T}$ -valued polypresymplectic  $(k + 1)$ -form on  $V$  of rank  $N$ , with polylagrangian subspace  $L$ . Then  $N \geq k$ , and  $L$  contains the kernel of  $\hat{\omega}$  as well as the kernel of each of the projected forms  $\omega_{\hat{t}^*}$  ( $\hat{t}^* \in \hat{T}^* \setminus \{0\}$ ):

$$\ker \hat{\omega} \subset \ker \omega_{\hat{t}^*} \subset L \quad \text{for all } \hat{t}^* \in \hat{T}^* \setminus \{0\}. \tag{2.11}$$

<sup>e</sup>Throughout this paper, the symbol  $\cdot^\perp$  will denote the annihilator of a subspace, i.e., given a subspace  $L$  of a vector space  $V$ ,  $L^\perp$  is the subspace of its dual space  $V^*$  consisting of the linear forms on  $V$  that vanish on  $L$ .

**Proof.** First we observe that if  $N < k$ , we have  $\bigwedge^k L^\perp = \{0\}$ , so both sides of Eq. (2.10) vanish, i.e.,  $L$  is contained in  $\ker \hat{\omega}$  and hence  $\ker \hat{\omega}$  has codimension  $< k$  in  $V$ , implying  $\hat{\omega} \equiv 0$ , since the  $(k+1)$ -form on the quotient space  $V/\ker \hat{\omega}$  induced by  $\hat{\omega}$  vanishes identically. (More generally, this argument shows that a nonvanishing vector-valued  $(k+1)$ -form does not permit isotropic subspaces of codimension  $< k$ .) Thus supposing that  $\dim L^\perp = N \geq k$ , we can for any vector  $v \in V \setminus L$  find a linearly independent set of 1-forms  $v_1^*, \dots, v_k^* \in L^\perp$  such that  $\langle v_1^*, v \rangle = 1$  and  $\langle v_i^*, v \rangle = 0$  for  $i > 1$ . Given  $\hat{t}^* \in \hat{T}^*$ , take  $\hat{t} \in \hat{T}$  such that  $\langle \hat{t}^*, \hat{t} \rangle = 1$ . According to the definition of a polylagrangian subspace, there is a vector  $u \in L$  such that

$$i_u \hat{\omega} = v_1^* \wedge \dots \wedge v_k^* \otimes \hat{t} \implies i_v i_u \omega_{\hat{t}^*} = v_2^* \wedge \dots \wedge v_k^* \neq 0,$$

and so  $v \notin \ker \omega_{\hat{t}^*}$ . Hence it follows that  $\ker \hat{\omega} \subset \ker \omega_{\hat{t}^*} \subset L$ .  $\square$

On the other hand, considering the case of main interest, which is that of 2-forms, it must be emphasized that, as shown by the counterexamples presented in Appendix A, by far not every vector-valued 2-form is poly(pre)symplectic, which means that in contrast to lagrangian subspaces, a polylagrangian subspace need not exist, and even if it does exist, not every lagrangian subspace is polylagrangian. However, there is a simple dimension criterion that allows to decide whether a given isotropic subspace is polylagrangian:

**Proposition 2.2.** *Let  $V$  and  $\hat{T}$  be finite-dimensional vector spaces ( $\dim \hat{T} \equiv \hat{n}$ ), and let  $\hat{\omega}$  be a non-vanishing  $\hat{T}$ -valued  $(k+1)$ -form on  $V$ . Given any subspace  $L$  of  $V$ , with  $\dim(V/L) = N$ , such that  $N \geq k$ , the following statements are equivalent:*

- $L$  is a polylagrangian subspace and  $\hat{\omega}$  is a polypresymplectic form of rank  $N$ .
- $L$  contains  $\ker \hat{\omega}$ , is isotropic and has dimension

$$\dim L = \dim \ker \hat{\omega} + \hat{n} \binom{N}{k}. \tag{2.12}$$

**Proof.** Taking into account that, for any isotropic subspace  $L$  of  $V$  containing the kernel of  $\hat{\omega}$ , the contraction map  $\hat{\omega}^\flat$  induces an injective linear map of  $L/\ker \hat{\omega}$  into  $(\bigwedge^k L^\perp) \otimes \hat{T}$ , the result follows from an elementary dimension count.  $\square$

It may be worthwhile to point out that Eq. (2.12) and, in particular, the ensuing dimension formula for  $V$ ,

$$\dim V = \dim \ker \hat{\omega} + \hat{n} \binom{N}{k} + N, \tag{2.13}$$

should perhaps best be read as a restriction on the dimension of a vector space in which a polysymplectic form  $\hat{\omega}$  of given degree  $k+1$  which combines a given number  $\hat{n}$  of scalar  $(k+1)$ -forms into a single vector-valued form can exist. In particular, in the standard symplectic case where we have  $k = 1$ ,  $\hat{n} = 1$  and a trivial kernel, Eq. (2.13) reduces to the familiar statement that symplectic vector spaces must have



even dimension,  $2N$ , and their lagrangian subspaces must have dimension equal to one half of that,  $N$ .

More generally, the case of ordinary or scalar polysymplectic forms ( $\dim \hat{T} = 1$ ) has been studied in the literature [8–10] under the label “multisymplectic forms”, but there, the concept of polylagrangian subspace appears only implicitly, namely through the dimension criterion formulated in Proposition 2.2 above, which is employed as a definition, so that it remains unclear how to extend this purely numerical recipe to other situations, in particular when  $\hat{\omega}$  is taken to be a vector-valued form ( $\dim \hat{T} > 1$ ). The main statement here is

**Proposition 2.3.** *Let  $V$  be a finite-dimensional vector space and let  $\hat{\omega}$  be a scalar polypresymplectic  $(k + 1)$ -form on  $V$  of rank  $N$ , with polylagrangian subspace  $L$ . Then any isotropic subspace  $\tilde{L}$  containing the kernel of  $\hat{\omega}$  and such that*

$$\dim \tilde{L} > \dim \ker \hat{\omega} + \binom{N-1}{k} + 1 \tag{2.14}$$

*is contained in  $L$ . In particular, if  $N > k > 1$ ,  $L$  is unique.*

**Remark 2.1.** Note that the uniqueness statement for  $L$  is of course false for symplectic forms ( $k = 1$ ) and also for volume forms ( $N = k$ ): in both cases, isotropic subspaces  $\tilde{L}$  satisfying the dimension condition (2.14) do not exist, and there is no restriction whatsoever on the relative position of lagrangian subspaces (which for a symplectic form on a  $(2N)$ -dimensional space are  $N$ -dimensional and for a volume form are one-dimensional).

**Proof.** Obviously, passing from  $V$  to the quotient space  $V/\ker \hat{\omega}$  if necessary, and taking into account the previous remark, we may assume without loss of generality that  $\hat{\omega}$  is non-degenerate and also that  $N > k > 1$ . Following [8, 10], we begin by showing that any isotropic subspace of  $V$  of dimension greater than 1 must intersect  $L$  non-trivially. Indeed, if  $v_1$  and  $v_2$  are linearly independent vectors in  $V$  such that the two-dimensional subspace  $\text{span}(v_1, v_2)$  generated by  $v_1$  and  $v_2$  satisfies  $\text{span}(v_1, v_2) \cap L = \{0\}$ , we can find a basis  $\{e_1, \dots, e_N\}$  of a subspace of  $V$  complementary to  $L$  such that  $e_1 = v_1$  and  $e_2 = v_2$ ; then denoting the corresponding dual basis of  $L^\perp$  by  $\{e^1, \dots, e^N\}$ , we use the fact that  $L$  is polylagrangian to conclude that there exists a vector  $u$  in  $L$  such that  $i_u \hat{\omega} = e^1 \wedge \dots \wedge e^k$ , so  $\hat{\omega}(u, e_1, \dots, e_k) = 1$  and hence  $i_{v_1} i_{v_2} \hat{\omega}$  cannot vanish, i.e.,  $\text{span}(v_1, v_2)$  cannot be isotropic. Using this result, we conclude that if  $\tilde{L}$  is any isotropic subspace of  $V$ , then the codimension of  $\tilde{L} \cap L$  in  $\tilde{L}$  is at most 1, so

$$\dim(\tilde{L} + L) - \dim L = \dim \tilde{L} - \dim(\tilde{L} \cap L) \leq 1.$$

Suppose now that  $\tilde{L}$  is an isotropic subspace of  $V$  such that

$$\dim(\tilde{L} + L) - \dim L = \dim \tilde{L} - \dim(\tilde{L} \cap L) = 1.$$

Then  $\tilde{L} + L$  has codimension  $N - 1$  in  $V$  and hence

$$\dim \wedge^k (\tilde{L} + L)^\perp = \binom{N-1}{k},$$

whereas, by hypothesis,

$$\dim(\tilde{L} \cap L) > \binom{N-1}{k},$$

which contradicts the fact that the linear isomorphism  $\hat{\omega}^\flat|_L : L \rightarrow \wedge^k L^\perp$  maps the subspace  $\tilde{L} \cap L$  of  $L$  injectively into the subspace  $\wedge^k (\tilde{L} + L)^\perp$  of  $\wedge^k L^\perp$ .  $\square$

What is remarkable is that the above uniqueness statement remains valid for vector-valued polypresymplectic forms ( $\dim \hat{T} > 1$ ) and that in this case it becomes true even when  $k = 1$  or  $N = k$ : this is a consequence of the following explicit construction of  $L$ .

**Theorem 2.1.** *Let  $V$  and  $\hat{T}$  be finite-dimensional vector spaces such that*

$$\dim \hat{T} \equiv \hat{n} \geq 2, \tag{2.15}$$

*and let  $\hat{\omega}$  be a  $\hat{T}$ -valued polypresymplectic  $(k+1)$ -form of rank  $N$  on  $V$ , with polylagrangian subspace  $L$ . Then  $L$  is given by*

$$L = \sum_{t^* \in \hat{T}^* \setminus \{0\}} \ker \omega_{t^*} \tag{2.16}$$

*and, in particular, is unique. In terms of a basis  $\{\hat{e}_1, \dots, \hat{e}_{\hat{n}}\}$  of  $\hat{T}$  with dual basis  $\{\hat{e}^1, \dots, \hat{e}^{\hat{n}}\}$  of  $\hat{T}^*$ ,*

$$L = \ker \hat{\omega} \oplus K_1 \oplus \dots \oplus K_{\hat{n}}, \tag{2.17}$$

*and for  $1 \leq a \leq \hat{n}$ ,*

$$L = \ker \omega^a \oplus K_a, \tag{2.18}$$

*where for  $1 \leq a \leq \hat{n}$ ,  $K_a$  is a subspace of  $V$  chosen so that*

$$\bigcap_{\substack{b=1 \\ b \neq a}}^{\hat{n}} \ker \omega^b = \ker \hat{\omega} \oplus K_a. \tag{2.19}$$

*The dimensions of these various subspaces are given by*

$$\dim \ker \omega^a = \dim \ker \hat{\omega} + (\hat{n} - 1) \binom{N}{k}, \quad \dim K_a = \binom{N}{k}. \tag{2.20}$$

The idea of the proof is that when  $\hat{\omega}$  is decomposed into a bunch of scalar forms, then even when  $\hat{\omega}$  is non-degenerate, the kernels of these individual forms will be quite big, so we look at their intersections, which are still rather big, so we go down all the way to look at the spaces  $K_a$  obtained as the intersections of all these

kernels except  $\ker \omega^a$ : the essential point is then that the condition that  $L$  should be polylagrangian forces  $L$  to be the direct sum of these subspaces. The details, suitably generalized to include the case where  $\hat{\omega}$  is allowed to be degenerate, follow.

**Proof.** Fix a basis  $\{\hat{e}_1, \dots, \hat{e}_{\hat{n}}\}$  of  $\hat{T}$  with dual basis  $\{\hat{e}^1, \dots, \hat{e}^{\hat{n}}\}$  of  $\hat{T}^*$  and, recalling Eq. (2.5), choose subspaces  $K_a$  of  $V$  ( $a = 1, \dots, \hat{n}$ ) as indicated in Eq. (2.19). Then the subspaces  $\ker \hat{\omega}$  and  $K_1, \dots, K_{\hat{n}}$  of  $V$  have trivial intersection, so their sum is direct and defines a subspace of  $V$  which we shall, for the moment, denote by  $L'$ . According to the previous proposition,  $L' \subset L$ . To show that  $L' = L$ , it is therefore sufficient to prove that  $\hat{\omega}^b(L) \subset \hat{\omega}^b(L')$ , since both  $L$  and  $L'$  contain  $\ker \hat{\omega}$ . Using the definition of a polylagrangian subspace, we conclude that we must establish the inclusion

$$(\bigwedge^k L^\perp) \otimes \hat{T} \subset \hat{\omega}^b(L').$$

But the equality in Eq. (2.10) guarantees that for any  $\alpha \in \bigwedge^k L^\perp$  and for  $1 \leq a \leq \hat{n}$ , there is a vector  $v_a \in L$  such that

$$i_{v_a} \hat{\omega} = \alpha \otimes \hat{e}_a.$$

Then

$$i_{v_a} \hat{\omega}^b = \alpha \langle \hat{e}^b, \hat{e}_a \rangle = \delta_a^b \alpha,$$

so we see that

$$v_a \in \bigcap_{\substack{b=1 \\ b \neq a}}^{\hat{n}} \ker \hat{\omega}^b.$$

Decomposing  $v_a$  according to Eq. (2.19), we find a vector  $u_a \in K_a$  such that

$$i_{u_a} \hat{\omega} = \alpha \otimes \hat{e}_a,$$

so  $\alpha \otimes \hat{e}_a \in \hat{\omega}^b(K_a) \subset \hat{\omega}^b(L')$ . Conversely, if  $u_a \in K_a \subset L$ , then  $i_{u_a} \hat{\omega}^b = 0$  for  $b \neq a$  and hence  $i_{u_a} \hat{\omega}$  is of the form  $\alpha \otimes \hat{e}_a$  for some  $\alpha \in \bigwedge^k L^\perp$ . Thus we conclude that  $\hat{\omega}^b$  maps  $K_a$  isomorphically onto  $(\bigwedge^k L^\perp) \otimes \text{span}(\hat{e}_a)$ , where  $\text{span}(\hat{e}_a)$  denotes the one-dimensional subspace of  $\hat{T}$  generated by  $\hat{e}_a$ , which proves the second formula in Eq. (2.20). Finally, we observe that combining Eqs (2.5) and (2.19) gives

$$\ker \omega^a \cap K_a = \{0\},$$

so that Eq. (2.18) follows from Eq. (2.17), whereas the first formula in Eq. (2.20) is now a direct consequence of Eqs (2.12) and (2.18).  $\square$

Another fundamental property of polypresymplectic forms is that the polylagrangian subspace has a particular type of direct complement.

**Theorem 2.2.** *Let  $V$  and  $\hat{T}$  be finite-dimensional vector spaces and let  $\hat{\omega}$  be a  $\hat{T}$ -valued polypresymplectic  $(k + 1)$ -form of rank  $N$  on  $V$ , with polylagrangian subspace  $L$ . Then there exists a  $k$ -isotropic subspace  $E$  of  $V$  complementary to  $L$ , i.e., such that*

$$V = E \oplus L. \quad (2.21)$$

**Proof.** Let  $E_0$  be a  $k$ -isotropic subspace of  $V$  of dimension  $N'$  with  $E_0 \cap L = \{0\}$ . (For instance, as long as  $N' \leq k$ ,  $E_0$  can be any subspace of  $V$  with  $E_0 \cap L = \{0\}$ .) If  $N' = N$ , we are done. Otherwise, choose a basis  $\{e_1, \dots, e_{N'}\}$  of a subspace of  $V$  complementary to  $L$  such that the first  $N'$  vectors constitute a basis of  $E_0$ , and denote the corresponding dual basis of  $L^\perp$  by  $\{e^1, \dots, e^{N'}\}$ . We shall prove that there exists a vector  $u \in V \setminus (E_0 \oplus L)$  such that the subspace  $E_1$  of  $V$  spanned by  $u$  and  $E_0$  is  $k$ -isotropic and satisfies  $E_1 \cap L = \{0\}$ ; then since  $\dim E_1 = N' + 1$ , the statement of the theorem follows by induction. To this end, consider an arbitrary basis  $\{\hat{e}_a \mid 1 \leq a \leq \hat{n}\}$  of  $\hat{T}$  with dual basis  $\{\hat{e}^a \mid 1 \leq a \leq \hat{n}\}$  of  $\hat{T}^*$  and, choosing any subspace  $L'$  of  $L$  complementary to  $\ker \hat{\omega}$ , use the fact that  $L$  is polylagrangian to conclude that there exists a unique basis  $\{e_a^{i_1 \dots i_k} \mid 1 \leq a \leq \hat{n}, 1 \leq i_1 < \dots < i_k \leq N'\}$  of  $L'$  such that

$$\hat{\omega}^b(e_a^{i_1 \dots i_k}) = e^{i_1} \wedge \dots \wedge e^{i_k} \otimes \hat{e}_a.$$

Thus, for  $1 \leq i_1 < \dots < i_k \leq N'$  and  $1 \leq j_1 < \dots < j_k \leq N'$ , we have

$$\omega^b(e_a^{i_1 \dots i_k}, e_{j_1}, \dots, e_{j_k}) = \delta_a^b \delta_{j_1}^{i_1} \dots \delta_{j_k}^{i_k}.$$

Therefore, the vector

$$u = e_{N'+1} - \frac{1}{k!} \omega^a(e_{N'+1}, e_{i_1}, \dots, e_{i_k}) e_a^{i_1 \dots i_k}$$

does not belong to the subspace  $E_0 \oplus L$  and, for  $1 \leq j_1 < \dots < j_k \leq N'$ , satisfies

$$\begin{aligned} & \omega^b(u, e_{j_1}, \dots, e_{j_k}) \\ &= \omega^b(e_{N'+1}, e_{j_1}, \dots, e_{j_k}) - \frac{1}{k!} \omega^a(e_{N'+1}, e_{i_1}, \dots, e_{i_k}) \omega^b(e_a^{i_1 \dots i_k}, e_{j_1}, \dots, e_{j_k}) \\ &= 0, \end{aligned}$$

which implies that since the subspace  $E_0$  spanned by  $e_1, \dots, e_{N'}$  is  $k$ -isotropic, the subspace  $E_1$  spanned by  $e_1, \dots, e_{N'}$  and  $u$  is so as well.  $\square$

**Example 2.1 (The Canonical Form).** Let  $E$  and  $\hat{T}$  be vector spaces of dimension  $N$  and  $\hat{n}$ , respectively. Set

$$V_0 = E \oplus \left( (\wedge^k E^*) \otimes \hat{T} \right). \quad (2.22)$$

The **canonical polysymplectic form** of rank  $N$  is the non-degenerate  $\hat{T}$ -valued  $(k + 1)$ -form  $\hat{\omega}_0$  on  $V_0$  defined by

$$\hat{\omega}_0((u_0, \alpha_0 \otimes \hat{t}_0), \dots, (u_k, \alpha_k \otimes \hat{t}_k)) = \sum_{i=0}^k (-1)^i \alpha_i(u_0, \dots, \hat{u}_i, \dots, u_k) \hat{t}_i. \quad (2.23)$$

Again, the “standard” case of main interest is when  $\hat{\omega}_0$  is a 2-form, i.e.,  $k = 1$ . (For scalar forms, where  $\hat{T} = \mathbb{R}$ , this construction can be found, e.g., in [8–10].)

To justify this terminology, note that it is a straightforward exercise to show that  $\hat{\omega}_0$  is non-degenerate and that, considering  $E$  and

$$L = (\bigwedge^k E^*) \otimes \hat{T} \tag{2.24}$$

as subspaces of  $V_0$ , we have the direct decomposition  $V_0 = E \oplus L$  where

$$L \text{ is polylagrangian and } E \text{ is } k\text{-isotropic.} \tag{2.25}$$

In terms of bases, let  $\{\hat{e}_a \mid 1 \leq a \leq \hat{n}\}$  be any basis of  $\hat{T}$  with dual basis  $\{\hat{e}^a \mid 1 \leq a \leq \hat{n}\}$  of  $\hat{T}^*$  and let  $\{e_i \mid 1 \leq i \leq N\}$  be any basis of  $E$  with dual basis  $\{e^i \mid 1 \leq i \leq N\}$  of  $E^*$ . For  $1 \leq a \leq \hat{n}$  and  $1 \leq i_1 < \dots < i_k \leq N$ , define

$$e_a^{i_1 \dots i_k} = e^{i_1} \wedge \dots \wedge e^{i_k} \otimes \hat{e}_a, \quad e_{i_1 \dots i_k}^a = e_{i_1} \wedge \dots \wedge e_{i_k} \otimes \hat{e}^a.$$

This provides a basis  $\{e_i, e_a^{i_1 \dots i_k} \mid 1 \leq a \leq \hat{n}, 1 \leq i \leq N, 1 \leq i_1 < \dots < i_k \leq N\}$  of  $V_0$  with dual basis  $\{e^i, e_{i_1 \dots i_k}^a \mid 1 \leq a \leq \hat{n}, 1 \leq i \leq N, 1 \leq i_1 < \dots < i_k \leq N\}$  of  $V_0^*$ , both of which we shall refer to as a **canonical basis** or **Darboux basis**, such that

$$\hat{\omega}_0 = \frac{1}{k!} (e_{i_1 \dots i_k}^a \wedge e^{i_1} \wedge \dots \wedge e^{i_k}) \otimes \hat{e}_a. \tag{2.26}$$

Now it is easy to derive the algebraic Darboux theorem for general polypresymplectic forms: let  $\{\hat{e}_a \mid 1 \leq a \leq \hat{n}\}$  be an arbitrary basis of  $\hat{T}$ , with dual basis  $\{\hat{e}^a \mid 1 \leq a \leq \hat{n}\}$  of  $\hat{T}^*$ , and let  $\{e_i \mid 1 \leq i \leq N\}$  be an arbitrary basis of a  $k$ -isotropic subspace  $E$  complementary to  $L$  in  $V$ , with dual basis  $\{e^i \mid 1 \leq i \leq N\}$  of  $L^\perp \cong E^*$ . Choosing an arbitrary subspace  $L'$  of  $L$  complementary to  $\ker \hat{\omega}$  and taking into account the identity (2.10), we define a basis  $\{e_a^{i_1 \dots i_k} \mid 1 \leq a \leq \hat{n}, 1 \leq i_1 < \dots < i_k \leq N\}$  of  $L'$  by

$$\hat{\omega}^\flat(e_a^{i_1 \dots i_k}) = e^{i_1} \wedge \dots \wedge e^{i_k} \otimes \hat{e}_a.$$

It is easy to see that the union of this basis with that of  $E$  gives a canonical basis of  $V$  (or more precisely, of  $E \oplus L'$ , which is a subspace of  $V$  complementary to  $\ker \hat{\omega}$ ). Thus we have proved<sup>f</sup>

**Theorem 2.3 (Darboux Theorem for Polypresymplectic Vector Spaces).**  
*Every polypresymplectic vector space admits a canonical basis.*

<sup>f</sup>Clearly, the inductive construction of a  $k$ -isotropic subspace  $E$  complementary to the polylagrangian subspace  $L$ , as presented in the proof of Theorem 2.2, provides an explicit iteration procedure for building canonical bases in a way similar to the well known Gram-Schmidt orthogonalization process.

### 3. Multisymplectic Forms on Vector Spaces

In this section we deal with ordinary alternating multilinear forms which are partially horizontal with respect to a given vertical subspace. To explain what this means, assume that we are given a finite-dimensional real vector space  $W$  together with a fixed subspace  $V$  and a projection  $\pi$  from  $W$  to another finite-dimensional real vector space  $T$  which has  $V$  as its kernel, so that  $T \cong W/V$ , i.e., we have a short exact sequence of vector spaces<sup>b</sup>

$$0 \longrightarrow V \longrightarrow W \xrightarrow{\pi} T \longrightarrow 0. \quad (3.1)$$

Motivated by standard jargon of fiber bundle theory, we shall refer to  $W$  as the **total space**,  $V$  as the **vertical space** and  $T$  as the **base space**. Then an  $r$ -form  $\alpha \in \bigwedge^r W^*$  on  $W$  is said to be  $(r - s)$ -**horizontal** (with respect to  $\pi$ ), where  $0 \leq s \leq r$ , if its contraction with more than  $s$  vertical vectors vanishes, i.e., if

$$i_{v_1} \dots i_{v_{s+1}} \alpha = 0 \quad \text{for } v_1, \dots, v_{s+1} \in V. \quad (3.2)$$

The vector space of  $(r - s)$ -horizontal  $r$ -forms on  $W$  will be denoted by  $\bigwedge_s^r W^*$ . Note that as  $s$  is varied (with  $r$  fixed), these spaces form an increasing sequence<sup>g</sup> under inclusion:

$$\bigwedge^r T^* \cong \bigwedge_0^r W^* \subset \dots \subset \bigwedge_s^r W^* \subset \dots \subset \bigwedge_r^r W^* = \bigwedge^r W^*. \quad (3.3)$$

At the two extremes, we have  $\bigwedge_r^r W^* = \bigwedge^r W^*$  since the condition of 0-horizontality is void, whereas the space  $\bigwedge_0^r W^*$  of fully horizontal  $r$ -forms on  $W$ , which are precisely the horizontal  $r$ -forms as defined in [18, 19], is canonically isomorphic to the space  $\bigwedge^r T^*$  of all  $r$ -forms on  $T$ :  $\bigwedge_0^r W^* \cong \bigwedge^r T^*$ . This canonical isomorphism is simply given by pull-back with the projection  $\pi$ , i.e.,  $\alpha_W = \pi^* \alpha_T$  or

$$\alpha_W(w_1, \dots, w_r) = \alpha_T(\pi(w_1), \dots, \pi(w_r)) \quad \text{for } w_1, \dots, w_r \in W. \quad (3.4)$$

Its inverse is given by  $\alpha_T = \mathfrak{s}^* \alpha_W$  or

$$\alpha_T(t_1, \dots, t_r) = \alpha_W(\mathfrak{s}(t_1), \dots, \mathfrak{s}(t_r)) \quad \text{for } t_1, \dots, t_r \in T, \quad (3.5)$$

where  $\mathfrak{s}$  is any splitting of the exact sequence (3.1), i.e., any linear mapping from  $T$  to  $W$  such that  $\pi \circ \mathfrak{s} = \text{id}_T$ .<sup>h</sup>

Extending this construction to partially horizontal forms leads us naturally to the concept of symbol, which will provide the link between polysymplectic and multisymplectic structures.

<sup>g</sup>The first few terms of this sequence may be trivial, since  $\bigwedge_s^r W^* = \{0\}$  if  $s < r - \dim T$ .

<sup>h</sup>It is a straightforward exercise to verify that the expression on the right-hand side of Eq. (3.5) does not depend on the choice of the splitting  $\mathfrak{s}$  when  $\alpha_W$  is horizontal.

**Definition 3.1.** Let  $W$ ,  $V$  and  $T$  be finite-dimensional vector spaces related by the short exact sequence (3.1). The **symbol** of an  $(r-s)$ -horizontal  $r$ -form  $\alpha$  on  $W$ ,  $\alpha \in \bigwedge_s^r W^*$ , is the  $\bigwedge^{r-s} T^*$ -valued  $s$ -form  $\hat{\alpha}$  on  $V$ ,  $\hat{\alpha} \in \bigwedge^s V^* \otimes \bigwedge^{r-s} T^*$ , given by

$$\hat{\alpha}(v_1, \dots, v_s) = i_{v_1} \dots i_{v_s} \alpha \quad \text{for } v_1, \dots, v_s \in V. \quad (3.6)$$

Explicitly, we may use Eq. (3.5) to arrive at the following formula for  $\hat{\alpha}$  in terms of  $\alpha$ :

$$\begin{aligned} \hat{\alpha}(v_1, \dots, v_s)(t_1, \dots, t_{r-s}) &= \alpha(v_1, \dots, v_s, \mathfrak{s}(t_1), \dots, \mathfrak{s}(t_{r-s})) \\ &\text{for } v_1, \dots, v_s \in V, t_1, \dots, t_{r-s} \in T, \end{aligned} \quad (3.7)$$

where, once again,  $\mathfrak{s}$  is any splitting of the exact sequence (3.1).<sup>i</sup> It follows that passage to the symbol can be regarded as a projection, from the space  $\bigwedge_s^r W^*$  of  $(r-s)$ -horizontal  $r$ -forms on  $W$  to the space  $\bigwedge^s V^* \otimes \bigwedge^{r-s} T^*$  of  $s$ -forms on  $V$  with values in the space  $\bigwedge^{r-s} T^*$  of  $(r-s)$ -forms on  $T$ , whose kernel is the subspace  $\bigwedge_{s-1}^r W^*$  of  $(r-s+1)$ -horizontal  $r$ -forms on  $W$ .

If we introduce a basis  $\{e_1^V, \dots, e_m^V, e_1^T, \dots, e_n^T\}$  of  $W$  such that the first  $m$  vectors span  $V$  while the last  $n$  vectors span a subspace complementary to  $V$  and hence isomorphic to  $T$ , then in terms of the dual basis  $\{e_V^1, \dots, e_V^m, e_T^1, \dots, e_T^n\}$  of  $W^*$ , an arbitrary form  $\alpha \in \bigwedge_s^r W^*$  is represented as<sup>j</sup>

$$\alpha = \sum_{t=0}^s \frac{1}{t!} \frac{1}{(r-t)!} \alpha_{i_1 \dots i_t; \mu_1 \dots \mu_{r-t}} e_V^{i_1} \wedge \dots \wedge e_V^{i_t} \wedge e_T^{\mu_1} \wedge \dots \wedge e_T^{\mu_{r-t}}, \quad (3.8)$$

while its symbol  $\hat{\alpha} \in \bigwedge^s V^* \otimes \bigwedge^{r-s} T^*$  is represented as

$$\hat{\alpha} = \frac{1}{s!} \frac{1}{(r-s)!} \alpha_{i_1 \dots i_s; \mu_1 \dots \mu_{r-s}} e_V^{i_1} \wedge \dots \wedge e_V^{i_s} \otimes e_T^{\mu_1} \wedge \dots \wedge e_T^{\mu_{r-s}}, \quad (3.9)$$

which also shows that

$$\dim \bigwedge_s^r W^* = \sum_{t=0}^s \binom{\dim V}{t} \binom{\dim T}{r-t}, \quad (3.10)$$

where it is to be understood that  $\binom{k}{l} = 0$  if  $l > k$ .

As in the polysymplectic case, our definition of a multisymplectic form will be based on the existence of a special type of maximal isotropic subspace, the only restriction being that we consider only isotropic subspaces of the vertical space  $V$ ; correspondingly, the concept of maximality should in this context be understood to mean maximality in the partially ordered set formed by the isotropic subspaces

<sup>i</sup>Again, it is a straightforward exercise to verify that the expression on the right-hand side of Eq. (3.7) does not depend on the choice of the splitting  $\mathfrak{s}$  when  $\alpha$  is  $(r-s)$ -horizontal.

<sup>j</sup>The expansion in Eq. (3.8) explains why forms in  $\bigwedge_s^r W^*$  are called  $(r-s)$ -horizontal: they are represented as linear combinations of exterior products of 1-forms such that, in each term of the sum, at least  $r-s$  of them are horizontal.

of  $V$  (not  $W$ ). More precisely, suppose we are given a fixed  $(k + 1 - r)$ -horizontal  $(k + 1)$ -form  $\omega$  on  $W$ ,

$$\omega \in \bigwedge_r^{k+1} W^*, \tag{3.11}$$

where  $1 \leq r \leq k + 1$ ,<sup>k</sup> and denote by  $\omega^b$  the contraction of  $\omega$ , as defined in the previous section, which is a linear map  $\omega^b : W \rightarrow \bigwedge^k W^*$ , as well as its restriction to the vertical subspace  $V$ , which is a linear map  $\omega^b : V \rightarrow \bigwedge_{r-1}^k W^*$ . Then defining, for any subspace  $L$  of  $V$ ,

$$\bigwedge_{r-1}^k L^\perp = \bigwedge^k L^\perp \cap \bigwedge_{r-1}^k W^*, \tag{3.12}$$

we conclude as in the previous section that a subspace  $L$  of  $V$  will be isotropic (with respect to  $\omega$ ) if and only if

$$\omega^b(L) \subset \omega^b(V) \cap \bigwedge_{r-1}^k L^\perp, \tag{3.13}$$

and will be maximal isotropic (with respect to  $\omega$ ) if and only if  $V \cap \ker \omega \subset L$  and

$$\omega^b(L) = \omega^b(V) \cap \bigwedge_{r-1}^k L^\perp. \tag{3.14}$$

As in the previous section, the intersection with the subspace  $\omega^b(V)$  on the right-hand side of these relations is superfluous in Eq. (3.13) but not in Eq. (3.14), and omitting it here leads to a strengthened form of Eq. (3.14) which turns out to provide the key to the theory of multisymplectic forms:

**Definition 3.2.** Let  $W$ ,  $V$  and  $T$  be finite-dimensional vector spaces related by the short exact sequence (3.1), with  $\dim T = n$ , and let  $\omega$  be a non-vanishing  $(k + 1 - r)$ -horizontal  $(k + 1)$ -form on  $W$ , where  $1 \leq r \leq k + 1$ . We say that  $\omega$  is a **multipresymplectic form of rank  $N$**  and **horizontality degree  $k + 1 - r$**  if  $V$  admits a subspace  $L$  of codimension  $N$  which is **multilagrangian**, i.e., such that

$$\omega^b(L) = \bigwedge_{r-1}^k L^\perp. \tag{3.15}$$

If  $\omega$  is non-degenerate, we call it a **multisymplectic form**. The “standard” case of main interest is when  $\omega$  is an  $(n - 1)$ -horizontal  $(n + 1)$ -form, i.e.,  $k = n$  and  $r = 2$ .

The first two propositions on multipresymplectic forms are entirely analogous to the corresponding ones for polypresymplectic forms. To begin with, we note that a multilagrangian subspace, when it exists, contains the kernel of  $\omega$  (implying, in particular, that  $\ker \omega \subset V$ ) and hence really is a special type of maximal isotropic subspace.

<sup>k</sup>The extreme case of fully horizontal forms ( $r = 0$ ) will be excluded right from the start since it can be reduced to the other extreme case where the horizontality condition is void ( $r = k + 1$ ), substituting the total space  $W$  by the quotient space  $T$ . Additional restrictions that serve to exclude other trivial or uninteresting cases will be imposed as we go along.



**Proposition 3.1.** *Let  $W, V$  and  $T$  be finite-dimensional vector spaces related by the short exact sequence (3.1), with  $\dim T = n$ , and let  $\omega$  be a multipresymplectic  $(k+1)$ -form on  $W$  of rank  $N$  and horizontality degree  $k+1-r$ , where  $1 \leq r \leq k+1$ , and with multilagrangian subspace  $L$ . Then  $N+n \geq k$  and  $k+1-r \leq n$ , and  $L$  contains the kernel of  $\omega$ :*

$$\ker \omega \subset L. \tag{3.16}$$

**Proof.** First we observe that if  $N+n < k$ , we have  $\bigwedge^k L^\perp = \{0\}$ , while if  $k+1-r > n$ , we have  $\bigwedge_{r-1}^k W^* = \{0\}$ , so in either case, both sides of Eq. (3.15) vanish, i.e.,  $L$  is contained in  $\ker \omega$  and hence  $\ker \omega$  has codimension  $< k$  in  $W$ , implying  $\omega \equiv 0$ , since the  $(k+1)$ -form on the quotient space  $W/\ker \omega$  induced by  $\omega$  vanishes identically. Thus supposing that  $\dim L^\perp = N+n \geq k$  and using that  $\dim V^\perp = n$  and  $V^\perp \subset L^\perp$ , we conclude that we can, for any vector  $w \in W \setminus L$ , find a linearly independent set of 1-forms  $w_1^*, \dots, w_k^* \in L^\perp$ , with  $w_r^*, \dots, w_k^* \in V^\perp$ , such that  $\langle w_1^*, w \rangle = 1$  and  $\langle w_i^*, w \rangle = 0$  for  $i > 1$ . According to the definition of a multilagrangian subspace, there is a vector  $u \in L$  such that

$$i_u \omega = w_1^* \wedge \dots \wedge w_k^* \implies i_w i_u \omega = w_2^* \wedge \dots \wedge w_k^* \neq 0,$$

and so  $w \notin \ker \omega$ . Hence it follows that  $\ker \omega \subset L$ . □

The second gives a simple dimension criterion that allows to decide whether a given isotropic subspace of  $V$  is multilagrangian:

**Proposition 3.2.** *Let  $W, V$  and  $T$  be finite-dimensional vector spaces related by the short exact sequence (3.1), with  $\dim T = n$ , and let  $\omega$  be a non-vanishing  $(k+1-r)$ -horizontal  $(k+1)$ -form on  $W$ , where  $1 \leq r \leq k+1$  and  $k+1-r \leq n$ . Given any subspace  $L$  of  $V$ , with  $\dim(V/L) = N$ , such that  $N+n \geq k$ , the following statements are equivalent:*

- $L$  is a multilagrangian subspace and  $\omega$  is a multipresymplectic form of rank  $N$ .
- $L$  contains  $\ker \omega$ , is isotropic and has dimension

$$\dim L = \dim \ker \omega + \sum_{s=0}^{r-1} \binom{N}{s} \binom{n}{k-s}, \tag{3.17}$$

where it is to be understood that  $\binom{N}{s} = 0$  if  $s > N$ .

**Proof.** Taking into account that, for any isotropic subspace  $L$  of  $V$  containing the kernel of  $\omega$ , the contraction map  $\omega^b$  induces an injective linear map of  $L/\ker \omega$  into  $\bigwedge_{r-1}^k L^\perp$ , we obtain

$$\dim L - \dim \ker \omega = \dim \omega^b(L) = \dim \bigwedge_{r-1}^k L^\perp.$$

To compute this dimension, introduce a basis  $\{e_1^L, \dots, e_l^L, e_1^{L'}, \dots, e_N^{L'}, e_1^T, \dots, e_n^T\}$  of  $W$  such that the first  $l$  vectors form a basis of  $L$ , the following  $N$  vectors form

a basis of a subspace  $L'$  complementary to  $L$  in  $V$  and the last  $n$  vectors form a basis of a subspace  $H$  complementary to  $V$  in  $W$ , which is isomorphic to  $T$ . Then in terms of the dual basis  $\{e_L^1, \dots, e_L^l, e_{L'}^1, \dots, e_{L'}^N, e_T^1, \dots, e_T^n\}$  of  $W^*$ , we conclude that  $\{e_{L'}^{i_1} \wedge \dots \wedge e_{L'}^{i_s} \wedge e_T^{\mu_1} \wedge \dots \wedge e_T^{\mu_{k-s}} \mid 0 \leq s \leq r-1, 1 \leq i_1 < \dots < i_s \leq N, 1 \leq \mu_1 < \dots < \mu_{k-s} \leq n\}$  is a basis of  $\bigwedge_{r-1}^k L^\perp$ .  $\square$

Again, it may be worthwhile to point out that Eq. (3.17) and, in particular, the ensuing dimension formulas for  $V$  and  $W$ ,

$$\dim V = \dim \ker \omega + \sum_{s=0}^{r-1} \binom{N}{s} \binom{n}{k-s} + N, \quad (3.18)$$

and

$$\dim W = \dim \ker \omega + \sum_{s=0}^{r-1} \binom{N}{s} \binom{n}{k-s} + N + n, \quad (3.19)$$

should perhaps best be read as a restriction on the dimension of a vector space in which a multisymplectic form  $\omega$  of given degree  $k+1$  and horizontality degree  $k+1-r$  can exist. In particular, in the standard multisymplectic case encountered in field theory where we have  $k=n, r=2$  and a trivial kernel, Eq. (3.19) reduces to the statement that the dimension of  $W$  must be a multiple of  $n+1$  (more precisely, equal to  $(N+1)(n+1)$ ).

The relation between multipresymplectic and polypresymplectic forms is established through the symbol:

**Theorem 3.1.** *Let  $W, V$  and  $T$  be finite-dimensional vector spaces related by the short exact sequence (3.1), with  $\dim T = n$ , let  $\omega$  be a non-vanishing  $(k+1-r)$ -horizontal  $(k+1)$ -form on  $W$ , where  $1 \leq r \leq k+1$  and  $k+1-r \leq n$ , and let  $\hat{\omega}$  be its symbol, which is a  $\bigwedge^{k+1-r} T^*$ -valued  $r$ -form on  $V$ . Suppose that  $\omega$  is multipresymplectic, with multilagrangian subspace  $L$ . Then  $\hat{\omega}$  is polypresymplectic, with polylagrangian subspace  $L$ , and*

$$\ker \omega \subset \ker \hat{\omega}. \quad (3.20)$$

Here, the “standard” cases correspond to each other: if  $\omega$  is an  $(n-1)$ -horizontal  $(n+1)$ -form on  $W$ , then  $\hat{\omega}$  is a  $\bigwedge^{n-1} T^*$ -valued 2-form on  $V$  and, in this case,

$$\dim \ker \hat{\omega} - \dim \ker \omega \leq 1. \quad (3.21)$$

**Proof.** Fixing an arbitrary horizontal subspace  $H$  of  $W$  and using the direct decompositions  $W = V \oplus H$  and  $W^* = H^\perp \oplus V^\perp$ , with  $H^\perp \cong V^*$  and  $V^\perp \cong H^*$ , we note that in order to show that  $L$  is polylagrangian with respect to  $\hat{\omega}$ , we must establish the equality

$$\hat{\omega}^\flat(L) \cong \bigwedge^{r-1} (L^\perp \cap H^\perp) \otimes \bigwedge^{k+1-r} T^*.$$

To do so, we use the isomorphism  $\bigwedge^{k+1-r} T^* \cong \bigwedge_0^{k+1-r} W^* = \bigwedge^{k+1-r} V^\perp$  with the inclusion  $V^\perp \subset L^\perp$  and note that the space  $\bigwedge^{r-1}(L^\perp \cap H^\perp) \otimes \bigwedge^{k+1-r} V^\perp$  is generated by elements which can be written in the form

$$\hat{\alpha} = (w_1^* \wedge \dots \wedge w_{r-1}^*) \otimes (w_r^* \wedge \dots \wedge w_k^*) \quad \text{where} \quad \alpha = w_1^* \wedge \dots \wedge w_k^*$$

with  $w_1^*, \dots, w_{r-1}^* \in L^\perp \cap H^\perp$  and  $w_r^*, \dots, w_k^* \in V^\perp \subset L^\perp$ . Since  $L$  is multi-lagrangian with respect to  $\omega$ , there is a vector  $u \in L$  such that  $\alpha = i_u \omega$  and so  $\hat{\alpha} = i_u \hat{\omega}$ , showing that  $L$  is polylagrangian with respect to  $\hat{\omega}$ . The inclusion (3.20) is obvious from the definition (3.6) or (3.7), and the estimate (3.21) follows by observing that the linear map  $\omega^\flat : V \rightarrow \bigwedge_{r-1}^k W^*$  induces an injective linear map from  $V/\ker \omega$  to  $\bigwedge_{r-1}^k W^*$ , which takes  $\ker \hat{\omega}/\ker \omega$  into  $\bigwedge_{r-2}^k W^*$ , and that in the “standard” case ( $k = n, r = 2$ ), this space is isomorphic to  $\bigwedge^n T^*$ , which is one-dimensional.  $\square$

In particular, we note that even when  $\omega$  is multisymplectic,  $\hat{\omega}$  is not necessarily polysymplectic, i.e., non-degeneracy of  $\omega$  does *not* imply non-degeneracy of  $\hat{\omega}$ . All we can say in general is that a multisymplectic form  $\omega$  on  $W$  induces a polysymplectic form on the quotient space  $\hat{W} = W/\ker \hat{\omega}$  (which is usually again denoted by  $\hat{\omega}$ , by abuse of notation). This happens even in the “standard” case ( $k = n, r = 2$ ) because there, both possible situations (namely,  $\hat{\omega}$  either is non-degenerate or else has a one-dimensional kernel) do arise in practice. That is the main reason why, in the present paper, we have refrained from imposing non-degeneracy right from the outset, even though this is forcing us to carry the somewhat annoying additional syllable “pre” all along the way.

Before going on to explore general consequences of the relation between multipresymplectic and polypresymplectic forms that we have just established, let us pause for a moment to comment on the situations encountered when the parameter  $r$  is assumed to take one of its extreme values. On the one hand, when  $r = k + 1$ , the horizontality condition becomes void, the choice of  $V$  and  $T$  becomes irrelevant, the space  $\bigwedge^{k+1-r} T^*$  is one-dimensional and the form  $\omega$  coincides with its symbol  $\hat{\omega}$ . This is, once again, the particular case that has been studied before in the literature [8–10] under the label “multisymplectic forms”. Thus it becomes clear that the multisymplectic forms in the sense of [8–10] appear at the intersection between polysymplectic and multisymplectic forms in the more general sense employed here: they are polysymplectic without being vector-valued as well as multisymplectic without satisfying any non-trivial horizontality conditions. On the other hand, note that we have already excluded the case  $r = 0$ , where the  $(k + 1)$ -form  $\omega$  is fully horizontal (i.e.,  $(k + 1)$ -horizontal), since this situation can be reduced to the previous one if we substitute  $W$  by  $T$ , but even the next case  $r = 1$ , where the  $(k + 1)$ -form  $\omega$  is  $k$ -horizontal, is essentially trivial, since every  $k$ -horizontal multipresymplectic  $(k + 1)$ -form has rank 0 and multilagrangean subspace  $V$ . (In fact, the condition that  $\omega$  should be  $k$ -horizontal is equivalent to the condition that  $V$  should be isotropic, and in this case,  $V$  does satisfy the remaining criteria of Proposition 3.2.)

Another fundamental property of multipresymplectic forms that can be derived from the corresponding property for polypresymplectic forms is that the multi-lagrangian subspace has a particular type of direct complement.

**Theorem 3.2.** *Let  $W$ ,  $V$  and  $T$  be finite-dimensional vector spaces related by the short exact sequence (3.1), with  $\dim T = n$ , and let  $\omega$  be a multipresymplectic  $(k+1)$ -form on  $W$  of rank  $N$  and horizontality degree  $k+1-r$ , where  $1 \leq r \leq k+1$  and  $k+1-r \leq n$ , and with multilagrangian subspace  $L$ . Then there exists a  $k$ -isotropic subspace  $F$  of  $W$  such that the intersection  $E = V \cap F$  is an  $(r-1)$ -isotropic subspace of  $V$  and*

$$W = F \oplus L, \quad V = E \oplus L. \quad (3.22)$$

**Proof.** First we construct an  $(r-1)$ -isotropic subspace  $E$  of  $V$  of dimension  $N$  which is complementary to  $L$  in  $V$ . (If  $r = 1$ , there is nothing to prove since in this case the vertical subspace  $V$  is isotropic and so we have  $L = V$ ,  $E = \{0\}$  and  $N = 0$ . If  $r > 1$ , we apply Theorem 2.2 to the symbol  $\hat{\omega}$  of  $\omega$  to conclude that there is a subspace  $E$  of  $V$  of dimension  $N$  which is complementary to  $L$  in  $V$  and is  $(r-1)$ -isotropic with respect to  $\hat{\omega}$ . Now taking into account that the whole vertical subspace  $V$  is  $r$ -isotropic with respect to  $\omega$ , it follows that  $E$  is  $(r-1)$ -isotropic with respect to  $\omega$  as well.) Now let  $F_0$  be a subspace of  $W$  of dimension  $N+n'$  which is  $k$ -isotropic with respect to  $\omega$  and such that  $F_0 \cap V = E$ . (For instance, if  $n' = 0$ ,  $F_0 = E$ .) If  $n' = n$ , we are done. Otherwise, choose a basis  $\{e_1^E, \dots, e_N^E, e_1, \dots, e_n\}$  of a subspace of  $W$  complementary to  $L$  such that the first  $N$  vectors constitute a basis of  $E$  and the first  $N+n'$  vectors constitute a basis of  $F_0$ , and denote the corresponding dual basis of  $L^\perp$  by  $\{e_E^1, \dots, e_E^N, e^1, \dots, e^n\}$ . We will prove that there exists a vector  $u \in W \setminus (F_0 \oplus L)$  such that the subspace  $F_1$  spanned by  $u$  and  $F_0$  is  $k$ -isotropic and satisfies  $F_1 \cap V = E$ ; then since  $\dim F_1 = N+n'+1$ , the statement of the theorem follows by induction. To this end, choose any subspace  $L'$  of  $L$  complementary to  $\ker \omega$  and use the fact that  $L$  is multilagrangian to conclude that there exists a unique basis  $\{e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} \mid 0 \leq s \leq r-1, 1 \leq i_1 < \dots < i_s \leq N, 1 \leq \mu_1 < \dots < \mu_{k-s} \leq n\}$  of  $L'$  such that

$$\omega_V^\flat(e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}}) = e_E^{i_1} \wedge \dots \wedge e_E^{i_s} \wedge e^{\mu_1} \wedge \dots \wedge e^{\mu_{k-s}}.$$

Thus, for  $0 \leq s, t \leq r-1, 1 \leq i_1 < \dots < i_s \leq N, 1 \leq j_1 < \dots < j_t \leq N, 1 \leq \mu_1 < \dots < \mu_{k-s} \leq n, 1 \leq \nu_1 < \dots < \nu_{k-t} \leq n$ , we have that

$$\omega(e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}}, e_{j_1}^E, \dots, e_{j_t}^E, e_{\nu_1}, \dots, e_{\nu_{k-t}})$$

is equal to  $\delta_{j_1}^{i_1} \dots \delta_{j_s}^{i_s} \delta_{\nu_1}^{\mu_1} \dots \delta_{\nu_{k-s}}^{\mu_{k-s}}$  if  $s = t$  and equal to 0 otherwise. Therefore, the vector

$$u = e_{n'+1} - \sum_{s=0}^{r-1} \frac{1}{s!} \frac{1}{(k-s)!} \omega(e_{n'+1}, e_{i_1}^E, \dots, e_{i_s}^E, e_{\mu_1}, \dots, e_{\mu_{k-s}}) e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}}$$

does not belong to the subspace  $F_0 \oplus L$  and, for  $0 \leq t \leq r-1$ ,  $1 \leq j_1 < \dots < j_t \leq N$  and  $1 \leq \nu_1 < \dots < \nu_{k-t} \leq n$ , satisfies

$$\begin{aligned} & \omega(u, e_{j_1}^E, \dots, e_{j_t}^E, e_{\nu_1}, \dots, e_{\nu_{k-t}}) \\ &= \omega(e_{n'+1}, e_{j_1}^E, \dots, e_{j_t}^E, e_{\nu_1}, \dots, e_{\nu_{k-t}}) \\ & \quad - \sum_{s=0}^{r-1} \frac{1}{s!} \frac{1}{(k-s)!} \omega(e_{n'+1}, e_{i_1}^E, \dots, e_{i_s}^E, e_{\mu_1}, \dots, e_{\mu_{k-s}}) \\ & \quad \quad \times \omega(e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}}, e_{j_1}^E, \dots, e_{j_t}^E, e_{\nu_1}, \dots, e_{\nu_{k-t}}) \\ &= 0, \end{aligned}$$

which implies that since the subspace  $F_0$  spanned by  $e_1^E, \dots, e_N^E, e_1, \dots, e_{n'}$  is  $k$ -isotropic, the subspace  $F_1$  spanned by  $e_1^E, \dots, e_N^E, e_1, \dots, e_{n'}$  and  $u$  is so as well.  $\square$

**Example 3.1 (The Canonical Form).** Let  $F$  be a vector space of dimension  $N+n$  and  $E$  be a fixed  $N$ -dimensional subspace of  $F$ . Denoting the  $n$ -dimensional quotient space  $F/E$  by  $T$  and the canonical projection of  $F$  onto  $T$  by  $\rho$ , we obtain the following exact sequence of vector spaces:

$$0 \longrightarrow E \longrightarrow F \xrightarrow{\rho} T \longrightarrow 0. \quad (3.23)$$

Set

$$W_0 = F \oplus \bigwedge_{r-1}^k F^*, \quad V_0 = E \oplus \bigwedge_{r-1}^k F^*, \quad \pi_0 = \rho \circ \text{pr}_1, \quad (3.24)$$

where  $\text{pr}_1 : W_0 \rightarrow F$  is the canonical projection, which leads us to the following exact sequence of vector spaces:

$$0 \longrightarrow V_0 \longrightarrow W_0 \xrightarrow{\pi_0} T \longrightarrow 0. \quad (3.25)$$

The **canonical multipresymplectic form** (if  $r = 1$ ) or **canonical multisymplectic form** (if  $r > 1$ ) of rank  $N$  and horizontality degree  $k+1-r$  is the  $(k+1-r)$ -horizontal  $(k+1)$ -form  $\omega_0$  on  $W_0$  defined by

$$\omega_0((u_0, \omega_0), \dots, (u_k, \omega_k)) = \sum_{i=0}^k (-1)^i \omega_i(u_0, \dots, \widehat{u}_i, \dots, u_k). \quad (3.26)$$

Again, the “standard” case of main interest is when  $\omega_0$  is an  $(n-1)$ -horizontal  $(n+1)$ -form, i.e.,  $k = n$  and  $r = 2$ .

It is a straightforward exercise to show that  $\omega_0$  is  $(k+1-r)$ -horizontal and is degenerate when  $r = 1$ , with

$$\ker \omega_0 = E \quad \text{if } r = 1, \quad (3.27)$$

but is non-degenerate when  $r > 1$ . In what follows, we shall assume that  $N > 0$ , since when  $E = \{0\}$ , we are back to the polysymplectic case on  $T \oplus \bigwedge^k T^*$ , with  $\hat{T} = \mathbb{R}$ , which has already been studied in [8–10]. For the same reason, we shall also

assume that  $r > 1$ , since for  $r = 1$  we have  $\bigwedge_0^k F^* \cong \bigwedge^k T^*$ , so that after passing to the quotient by the kernel of  $\omega_0$ , we are once again back to the polysymplectic case on  $T \oplus \bigwedge^k T^*$ , with  $\hat{T} = \mathbb{R}$ . Then considering  $F$  and

$$L = \bigwedge_{r-1}^k F^* \quad (3.28)$$

as subspaces of  $W_0$ , we have the direct decompositions  $W_0 = F \oplus L$  and  $V_0 = E \oplus L$  where

$$L \text{ is multilagrangian, } F \text{ is } k\text{-isotropic, } E \text{ is } (r-1)\text{-isotropic.} \quad (3.29)$$

In terms of bases, let  $\{e_i, e_\mu \mid 1 \leq i \leq N, 1 \leq \mu \leq n\}$  be a basis of  $F$  with dual basis  $\{e^i, e^\mu \mid 1 \leq i \leq N, 1 \leq \mu \leq n\}$  of  $F^*$  such that  $\{e_i \mid 1 \leq i \leq N\}$  is a basis of  $E$  and  $\{e_\mu \mid 1 \leq \mu \leq n\}$  is a basis of a subspace  $H$  of  $F$  complementary to  $E$ , isomorphic to  $T$ . For  $0 \leq s \leq r$ ,  $1 \leq i_1 < \dots < i_s \leq N$  and  $1 \leq \mu_1 < \dots < \mu_{k-s} \leq N$ , define:

$$\begin{aligned} e_{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} &= e_{i_1} \wedge \dots \wedge e_{i_s} \wedge e_{\mu_1} \wedge \dots \wedge e_{\mu_{k-s}} \\ e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} &= e^{i_1} \wedge \dots \wedge e^{i_s} \wedge e^{\mu_1} \wedge \dots \wedge e^{\mu_{k-s}} \end{aligned}$$

This provides a basis

$$\left\{ e_i, e_\mu, e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} \mid 0 \leq s \leq r-1, \begin{array}{l} 1 \leq i \leq N, 1 \leq i_1 < \dots < i_s \leq N \\ 1 \leq \mu \leq n, 1 \leq \mu_1 < \dots < \mu_{k-s} \leq n \end{array} \right\}$$

of  $W_0$  with dual basis

$$\left\{ e^i, e^\mu, e_{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} \mid 0 \leq s \leq r-1, \begin{array}{l} 1 \leq i \leq N, 1 \leq i_1 < \dots < i_s \leq N \\ 1 \leq \mu \leq n, 1 \leq \mu_1 < \dots < \mu_{k-s} \leq n \end{array} \right\}$$

of  $W_0^*$ , both of which we shall refer to as a **canonical basis** or **Darboux basis**, such that

$$\omega_0 = \sum_{s=0}^{r-1} \frac{1}{s!} \frac{1}{(k-s)!} e_{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} \wedge e^{i_1} \wedge \dots \wedge e^{i_s} \wedge e^{\mu_1} \wedge \dots \wedge e^{\mu_{k-s}}, \quad (3.30)$$

and for the symbol

$$\begin{aligned} \hat{\omega}_0 &= \frac{1}{(r-1)!} \frac{1}{(k+1-r)!} e_{i_1 \dots i_{r-1}; \mu_1 \dots \mu_{k+1-r}} \wedge e^{i_1} \wedge \dots \wedge e^{i_{r-1}} \\ &\quad \otimes e^{\mu_1} \wedge \dots \wedge e^{\mu_{k+1-r}}. \end{aligned} \quad (3.31)$$

Now it is easy to derive the algebraic Darboux theorem for general multipre-symplectic forms: let  $\{e_i, e_\mu \mid 1 \leq i \leq N, 1 \leq \mu \leq n\}$  be a basis of a  $k$ -isotropic subspace  $F$  complementary to  $L$  in  $W$ , with dual basis  $\{e^i, e^\mu \mid 1 \leq i \leq N, 1 \leq \mu \leq n\}$  of  $L^\perp \cong F^*$ , such that  $\{e_i \mid 1 \leq i \leq N\}$  is a basis of the  $(r-1)$ -isotropic subspace  $E = V \cap F$  which is complementary to  $L$  in  $V$ . Choosing any subspace  $L'$  of  $L$  complementary to  $\ker \omega$  and taking into account the identity (3.15), we define a basis

$$\left\{ e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} \mid 0 \leq s \leq r-1, \begin{array}{l} 1 \leq i_1 < \dots < i_s \leq N \\ 1 \leq \mu_1 < \dots < \mu_{k-s} \leq n \end{array} \right\}$$

of  $L'$  by

$$\omega_V^b(e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}}) = e^{i_1} \wedge \dots \wedge e^{i_s} \wedge e^{\mu_1} \wedge \dots \wedge e^{\mu_{k-s}}.$$

It is easy to see that the union of this basis with that of  $F$  gives a canonical basis of  $W$  (or more precisely, of  $F \oplus L'$ , which is a subspace of  $W$  complementary to  $\ker \omega$ ). Thus we have proved<sup>1</sup>

**Theorem 3.3 (Darboux Theorem for Multipresymplectic Vector Spaces).**  
*Every multipresymplectic vector space admits a canonical basis.*

#### 4. Cartan Calculus and the Symbol

In order to extend the structures studied in the previous two sections and, in particular, the concept of symbol that interrelates them, from a purely algebraic setting to the realm of differential geometry, we shall need a variant of Cartan's calculus, which in its standard formulation deals with differential forms on manifolds, to handle vertical differential forms on total spaces of fiber bundles.

Let  $P$  be a fiber bundle over a base manifold  $M$ , with projection  $\pi : P \rightarrow M$ . Then the vector bundle  $\bigwedge_s^r T^*P$  over  $P$  whose fiber at any point  $p$  in  $P$  is the space  $\bigwedge_s^r T_p^*P$  of  $(r-s)$ -horizontal  $r$ -forms on the tangent space  $T_pP$  to  $P$  at  $p$  is called the **bundle of  $(r-s)$ -horizontal  $r$ -forms** on  $P$ , and its sections are called  **$(r-s)$ -horizontal differential  $r$ -forms** or simply  **$(r-s)$ -horizontal  $r$ -forms** on  $P$ ; the space of such forms will be denoted by  $\Omega_s^r(P)$ . Similarly, assuming in addition that  $\hat{T}$  is a vector bundle over the same base manifold  $M$ , with projection  $\hat{\tau} : \hat{T} \rightarrow M$ , and denoting the pull-back of  $\hat{T}$  to  $P$  by  $\pi^*\hat{T}$  and the vertical bundle of  $P$  by  $VP$  (both are vector bundles over  $P$ ), the vector bundle  $\bigwedge^r V^*P \otimes \pi^*\hat{T}$  over  $P$  is called the **bundle of vertical  $r$ -forms** on  $P$ , and its sections are called **vertical differential  $r$ -forms** or simply **vertical  $r$ -forms** on  $P$ , with **values** or **coefficients** in  $\pi^*\hat{T}$  or, by abuse of language, in  $\hat{T}$ : the space of such forms will be denoted by  $\Omega_V^r(P; \pi^*\hat{T})$ . Finally, the sections of the vertical bundle  $VP$  itself are called **vertical vector fields** or simply **vertical fields** on  $P$ : the space of such fields will be denoted by  $\mathfrak{X}_V(P)$ . Obviously,  $\Omega_s^r(P)$ ,  $\Omega_V^r(P; \pi^*\hat{T})$  and  $\mathfrak{X}_V(P)$  are (locally finite) modules over the algebra  $\mathfrak{F}(P)$  of functions on  $P$ .

It should be noted that speaking of vertical forms involves a certain abuse of language because these "forms" are really equivalence classes of differential  $r$ -forms on  $P$ :  $\Omega_V^r(P; \pi^*\hat{T})$  is not a subspace of the space  $\Omega^r(P; \pi^*\hat{T})$  of all differential  $r$ -forms on  $P$  but rather its quotient space

$$\Omega_V^r(P; \pi^*\hat{T}) = \Omega^r(P; \pi^*\hat{T}) / \Omega_{r-1}^r(P; \pi^*\hat{T})$$

by the subspace  $\Omega_{r-1}^r(P; \pi^*\hat{T})$  of all 1-horizontal differential  $r$ -forms on  $P$ .

<sup>1</sup>Once again, the inductive construction of a  $k$ -isotropic subspace  $F$  complementary to the multi-lagrangian subspace  $L$ , as presented in the proof of Theorem 3.2, provides an explicit iteration procedure for building canonical bases in a way similar to the well known Gram-Schmidt orthogonalization process.

An interesting aspect of this construction is that it is possible to develop a variant of the usual Cartan calculus for differential forms on the manifold  $P$  in which vector fields on  $P$  are replaced by vertical fields  $X$  on  $P$  and differential forms on  $P$  (taking values in some fixed vector space) are replaced by vertical differential forms  $\alpha$  on  $P$  (taking values in some fixed vector bundle over the base manifold), in such a way that all operations of this calculus such as exterior multiplication, contraction, Lie derivative and exterior derivative continue to be well defined and to satisfy the standard rules. (See [18, Vol. 1, Problem 8, p. 313] for the special case where the vector bundle of coefficients is the trivial line bundle  $M \times \mathbb{R}$ .) Here, we shall only need the **vertical exterior derivative**

$$\begin{aligned} d_V : \Omega_V^r(P; \pi^*\hat{T}) &\longrightarrow \Omega_V^{r+1}(P; \pi^*\hat{T}) \\ \alpha &\longmapsto d_V\alpha \end{aligned} \quad (4.1)$$

which is defined by exactly the same formula as in the standard case, namely

$$\begin{aligned} d_V\alpha(X_0, \dots, X_r) &= \sum_{i=0}^r (-1)^i X_i \cdot \left( \alpha(X_0, \dots, \hat{X}_i, \dots, X_r) \right) \\ &+ \sum_{0 \leq i < j \leq r} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r), \end{aligned} \quad (4.2)$$

where  $X_0, X_1, \dots, X_r \in \mathfrak{X}_V(P)$ : this makes sense since  $VP$  is an involutive distribution on  $P$ , provided we correctly define the **vertical directional derivative**

$$\begin{aligned} \mathfrak{X}_V(P) \times \Gamma(\pi^*\hat{T}) &\longrightarrow \Gamma(\pi^*\hat{T}) \\ (X, \varphi) &\longmapsto X \cdot \varphi \end{aligned} \quad (4.3)$$

as an  $\mathbb{R}$ -bilinear operator which is  $\mathfrak{F}(P)$ -linear in the first entry and satisfies a Leibniz rule in the second entry,

$$X \cdot (f\varphi) = (X \cdot f)\varphi + f(X \cdot \varphi). \quad (4.4)$$

Explicitly, for  $X \in \mathfrak{X}_V(P)$  and  $\varphi \in \Gamma(\pi^*\hat{T})$ ,  $X \cdot \varphi \in \Gamma(\pi^*\hat{T})$  is defined as the standard directional derivative of vector valued functions along the fibers, that is, for any point  $m$  in  $M$ ,  $(X \cdot \varphi)|_{P_m} \in C^\infty(P_m, \hat{T}_m)$  is given in terms of  $X|_{P_m} \in \mathfrak{X}(P_m)$  and  $\varphi|_{P_m} \in C^\infty(P_m, \hat{T}_m)$  by

$$(X \cdot \varphi)|_{P_m} = X|_{P_m} \cdot \varphi|_{P_m}. \quad (4.5)$$

Since the Lie bracket is natural under restriction to submanifolds, we have

$$X \cdot (Y \cdot \varphi) - Y \cdot (X \cdot \varphi) = [X, Y] \cdot \varphi \quad \text{for } X, Y \in \mathfrak{X}_V(P), \varphi \in \Gamma(\pi^*\hat{T}),$$

which implies that  $d_V^2 = 0$ . On the other hand, sections  $\varphi$  of  $\pi^*\hat{T}$  obtained from sections  $\hat{t}$  of  $\hat{T}$  by composing with  $\pi$  are constant along the fibers and hence their vertical directional derivative vanishes:

$$X \cdot (\hat{t} \circ \pi) = 0 \quad \text{for } X \in \mathfrak{X}_V(P), \hat{t} \in \Gamma(\hat{T}). \quad (4.6)$$



In the same way, substituting  $\hat{T}$  by  $\hat{T}^*$ , we get

$$X \cdot \langle \hat{t}^* \circ \pi, \varphi \rangle = 0 \quad \text{for } X \in \mathfrak{X}_V(P), \hat{t}^* \in \Gamma(\hat{T}^*), \quad (4.7)$$

which means that

$$X \cdot \langle \hat{t}^* \circ \pi, \varphi \rangle = \langle \hat{t}^* \circ \pi, X \cdot \varphi \rangle \quad \text{for } X \in \mathfrak{X}_V(P), \varphi \in \Gamma(\pi^* \hat{T}). \quad (4.8)$$

More generally, given a  $\pi^* \hat{T}$ -valued vertical  $r$ -form  $\hat{\alpha}$  on  $P$  and a section  $\hat{t}^*$  of the dual vector bundle  $\hat{T}^*$  of  $\hat{T}$ , we define the **projection** of  $\hat{\alpha}$  along  $\hat{t}^*$  to be the ordinary vertical  $r$ -form  $\hat{\alpha}_{\hat{t}^*}$  on  $P$  given by<sup>m</sup>

$$\hat{\alpha}_{\hat{t}^*}(p) = \langle \hat{t}^*(\pi(p)), \hat{\alpha}(p) \rangle \quad \text{for } p \in P, \quad (4.9)$$

and obtain

$$d_V \hat{\alpha}_{\hat{t}^*} = (d_V \hat{\alpha})_{\hat{t}^*}. \quad (4.10)$$

Hence, if  $\hat{\alpha}$  is closed,  $\hat{\alpha}_{\hat{t}^*}$  will be closed as well, and a standard argument shows that, in this case, the **kernel** of  $\hat{\alpha}$  and of  $\hat{\alpha}_{\hat{t}^*}$ , given by

$$\ker_p \hat{\alpha} = \ker \hat{\alpha}_p = \{ u_p \in V_p P \mid i_{u_p} \hat{\alpha}_p = 0 \} \quad \text{for } p \in P, \quad (4.11)$$

and

$$\ker_p \hat{\alpha}_{\hat{t}^*} = \ker (\hat{\alpha}_{\hat{t}^*})_p = \{ u_p \in V_p P \mid i_{u_p} (\hat{\alpha}_{\hat{t}^*})_p = 0 \} \quad \text{for } p \in P, \quad (4.12)$$

respectively, define involutive distributions on  $P$ , provided they have constant dimension.

With these tools at our disposal, we can now formulate the construction of the symbol in the differential geometric setting. To deal directly with the case of interest, suppose that  $\omega$  is a  $(k+1-r)$ -horizontal  $(k+1)$ -form on  $P$ , where  $1 \leq r \leq k+1$  and  $k+1-r \leq n$ , that is,

$$\omega \in \Omega_r^{k+1}(P) = \Gamma(\bigwedge_r^{k+1} T^* P). \quad (4.13)$$

Explicitly, the horizontality condition means that contraction of  $\omega$  with more than  $r$  vertical fields on  $P$  gives zero. Then the **symbol**  $\hat{\omega}$  of  $\omega$ , whose value at every point  $p$  of  $P$  is defined to be the symbol  $\hat{\omega}_p$  of  $\omega_p$ , is a vertical  $r$ -form on  $P$  taking values in the vector bundle  $\pi^*(\bigwedge^{k+1-r} T^* M)$ , that is,

$$\hat{\omega} \in \Omega_V^r(P, \pi^*(\bigwedge^{k+1-r} T^* M)) = \Gamma(\bigwedge^r V^* P \otimes \pi^*(\bigwedge^{k+1-r} T^* M)). \quad (4.14)$$

Using the canonical isomorphism  $\pi^*(\bigwedge^{k+1-r} T^* M) \cong \bigwedge_0^{k+1-r} T^* P$  of vector bundles over  $P$  as an identification, we have

$$\hat{\omega}(X_1, \dots, X_r) = i_{X_1} \dots i_{X_r} \omega \quad \text{for } X_1, \dots, X_r \in \mathfrak{X}_V(P). \quad (4.15)$$

<sup>m</sup>Here, it is important that  $\hat{t}^*$  be a section of  $\hat{T}^*$  and not of  $\pi^* \hat{T}^*$ .

More explicitly, using the horizontal lift

$$\begin{aligned} \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(P) \\ \xi &\longmapsto \xi^H \end{aligned}$$

of vector fields induced by some fixed connection on  $P$ , we have

$$\begin{aligned} \hat{\omega}(X_1, \dots, X_r) \cdot (\xi_1 \circ \pi, \dots, \xi_{k+1-r} \circ \pi) &= \omega(X_1, \dots, X_r, \xi_1^H, \dots, \xi_{k+1-r}^H) \\ \text{for } X_1, \dots, X_r \in \mathfrak{X}_V(P), \xi_1, \dots, \xi_{k+1-r} \in \mathfrak{X}(M), \end{aligned} \quad (4.16)$$

where it should be noted that although composition of vector fields on  $M$  with the projection  $\pi$  provides only a subspace of the vector space of all sections of the pull-back  $\pi^*TM$  of  $TM$  by  $\pi$ , this formula is sufficient to fix the value of  $\hat{\omega}$  at each point  $p$  of  $P$ .

**Theorem 4.1.** *With the same notations as above, suppose that the form  $\omega$  satisfies*

$$d\omega \in \Omega_r^{k+2}(P) = \Gamma(\wedge_r^{k+2} T^*P),$$

*i.e.,  $d\omega$  is  $(k+2-r)$ -horizontal. Then the form  $\hat{\omega}$  is vertically closed:*

$$d_V \hat{\omega} = 0.$$

In particular, for  $\hat{\omega}$  to be vertically closed, it is sufficient (but not necessary) that  $\omega$  be closed.

**Proof.** Let  $X_0, \dots, X_r$  be vertical fields on  $P$  and  $\xi_1, \dots, \xi_{k+1-r}$  be vector fields on  $M$ , and denote the horizontal lifts of the latter with respect to some fixed connection on  $P$  by  $\xi_1^H, \dots, \xi_{k+1-r}^H$ , respectively. Then we have the relation

$$d\omega(X_0, \dots, X_r, \xi_1^H, \dots, \xi_{k+1-r}^H) = \left( d_V \hat{\omega}(X_0, \dots, X_r) \right) (\xi_1 \circ \pi, \dots, \xi_{k+1-r} \circ \pi),$$

which can be derived from Cartan's formula for  $d\omega$  and its analogue for  $d_V \hat{\omega}$ , Eq. (4.2), using that  $\omega$  is a  $(k+1-r)$ -horizontal  $(k+1)$ -form and the Lie brackets  $[X_i, \xi_j^H]$  are vertical fields, by applying Eq. (4.8) to expressions of the form  $\varphi = \hat{\omega}(X_0, \dots, \hat{X}_i, \dots, X_r)$  and  $\hat{t}^* = \xi_1 \wedge \dots \wedge \xi_{k+1-r}$ .  $\square$

## 5. Polysymplectic and Multisymplectic Fiber Bundles

Now we are ready to transfer the poly- and multisymplectic structures introduced in the first two sections from the algebraic to the differential geometric context. All we need to do is add the appropriate integrability condition, which is the expected one: the differential forms in question should be closed. For the sake of brevity, we begin directly with the notion of a poly(pre)symplectic fiber bundle, which formalizes the idea of a “family of poly(pre)symplectic manifolds smoothly parametrized by the points of a base manifold  $M$ ” and includes that of a poly(pre)symplectic manifold as a special case.

**Definition 5.1.** A **polypresymplectic fiber bundle** is a fiber bundle  $P$  over an  $n$ -dimensional manifold  $M$  equipped with a vertical  $(k+1)$ -form  $\hat{\omega}$  of constant rank on the total space  $P$  taking values in a fixed  $\hat{n}$ -dimensional vector bundle  $\hat{T}$  over the same manifold  $M$ , called the **polypresymplectic form along the fibers** of  $P$ , or simply the **polypresymplectic form**, and said to be of **rank**  $N$ , such that  $\hat{\omega}$  is vertically closed,

$$d_V \hat{\omega} = 0, \tag{5.1}$$

and such that at every point  $p$  of  $P$ ,  $\hat{\omega}_p$  is a polypresymplectic form of rank  $N$  on the vertical space  $V_p P$ . If the polylagrangian subspaces at the different points of  $P$  fit together into a distribution  $L$  on  $P$  (which is contained in the vertical bundle  $VP$  of  $P$ ), we call it the **polylagrangian distribution** of  $\hat{\omega}$ . If  $\hat{\omega}$  is non-degenerate, we say that  $P$  is a **polysymplectic fiber bundle** and  $\hat{\omega}$  is a **polysymplectic form along the fibers** of  $P$ , or simply a **polysymplectic form**. If  $M$  reduces to a point, we speak of a **poly(pre)symplectic manifold**. As before, the “standard” case of main interest is when  $\hat{\omega}$  is a 2-form, i.e.,  $k = 1$ .

The notion of a multi(pre)symplectic fiber bundle is defined similarly.

**Definition 5.2.** A **multipresymplectic fiber bundle** is a fiber bundle  $P$  over an  $n$ -dimensional manifold  $M$  equipped with a  $(k+1-r)$ -horizontal  $(k+1)$ -form  $\omega$  of constant rank on the total space  $P$ , where  $1 \leq r \leq k+1$  and  $k+1-r \leq n$ , called the **multipresymplectic form** and said to be of **rank**  $N$  and **horizontality degree**  $k+1-r$ , such that  $\omega$  is closed,

$$d\omega = 0, \tag{5.2}$$

and such that at every point  $p$  of  $P$ ,  $\omega_p$  is a multipresymplectic form of rank  $N$  on the tangent space  $T_p P$ . If the multilagrangian subspaces at the different points of  $P$  fit together into a distribution  $L$  on  $P$  (which is contained in the vertical bundle  $VP$  of  $P$ ), we call it the **multilagrangian distribution** of  $\omega$ . If  $\omega$  is non-degenerate, we say that  $P$  is a **multisymplectic fiber bundle** and  $\omega$  is a **multisymplectic form**. If  $M$  reduces to a point, we speak of a **multi(pre)symplectic manifold**. As before, the “standard” case of main interest is when  $\omega$  is an  $(n-1)$ -horizontal  $(n+1)$ -form, i.e.,  $k = n$  and  $r = 2$ .

Combining Theorems 3.1 and 4.1, we obtain

**Theorem 5.1.** *Let  $P$  be a fiber bundle over an  $n$ -dimensional manifold  $M$ , with projection  $\pi : P \rightarrow M$ , let  $\omega$  be a  $(k+1-r)$ -horizontal  $(k+1)$ -form of constant rank on  $P$ , where  $1 \leq r \leq k+1$  and  $k+1-r \leq n$ , and let  $\hat{\omega}$  be its symbol, which is a vertical  $r$ -form on  $P$  taking values in the bundle of  $(k+1-r)$ -forms on  $M$ . Suppose that  $\omega$  is multipresymplectic, with multilagrangian distribution  $L$ . Then  $\hat{\omega}$  is polypresymplectic, with polylagrangian distribution  $L$ , and*

$$\ker \omega \subset \ker \hat{\omega}. \tag{5.3}$$

As before, the “standard” cases correspond to each other: if  $\omega$  is an  $(n-1)$ -horizontal  $(n+1)$ -form, then  $\hat{\omega}$  is a  $\bigwedge^{n-1} T^*M$ -valued 2-form and, in this case,

$$\dim \ker \hat{\omega} - \dim \ker \omega \leq 1. \tag{5.4}$$

In particular, we repeat that even when  $\omega$  is multisymplectic,  $\hat{\omega}$  is not necessarily polysymplectic, i.e., non-degeneracy of  $\omega$  does *not* imply non-degeneracy of  $\hat{\omega}$ . All we can say in general is that a multisymplectic form  $\omega$  on  $P$  induces a polysymplectic form on the quotient bundle  $\hat{P} = P/\ker \hat{\omega}$  (which is usually again denoted by  $\hat{\omega}$ , by abuse of notation). This happens even in the “standard” case ( $k = n$ ,  $r = 2$ ) because there, both possible situations (namely,  $\omega$  either is non-degenerate or else has a one-dimensional kernel) do arise in practice. That is the main reason why, in the present paper, we have refrained from imposing non-degeneracy right from the outset, even though this is forcing us to carry the somewhat annoying additional syllable “pre” all along the way.

**Remark 5.1.** Note that when  $\hat{n} \geq 2$ , in the case of polypresymplectic structures, or when  $1 \leq r \leq k$ , in the case of multipresymplectic structures, the explicit construction of  $L$  in terms of the kernels of the projections of  $\hat{\omega}$  (see Theorem 2.1) implies that the polylagrangian or multilagrangian subspaces at the different points of  $P$  do fit together to form a uniquely determined, smooth distribution  $L$  on  $P$ . (The proof uses the fact that sums and intersections of smooth vector subbundles of a vector bundle are again smooth vector subbundles if they have constant rank and, more generally, that kernels and images of smooth vector bundle homomorphisms of constant rank are smooth vector subbundles; see, e.g., [20, Exercise 1.6F(c), p. 51].) But in the case of scalar polypresymplectic structures ( $\hat{n} = 1$ ) and of multipresymplectic structures for which the horizontality condition is void ( $r = k + 1$ ), such a distribution may fail to exist, even if we assume the base manifold to be trivial (i.e.,  $M$  reduces to a point,  $n = 0$ ) and the form  $\hat{\omega}$  or  $\omega$  to be non-degenerate. Indeed, for symplectic forms ( $k = 1$ ) or volume forms ( $N = k$ ), it is easy to find examples of manifolds whose tangent bundle does not admit any (smooth) lagrangian subbundle: the simplest of them all is just the 2-sphere  $S^2$ , according to the “no hair theorem”. And even in the remaining cases ( $N > k > 1$ ), where according to Proposition 2.3, the polylagrangian or multilagrangian subspaces at the different points of  $P$  are unique, an explicit construction for them does not seem to be available, so we do not know whether it is true that smoothness of  $\hat{\omega}$  or  $\omega$  by itself implies smoothness of  $L$ . Thus when we simply refer to “the polylagrangian distribution  $L$ ” or “the multilagrangian distribution  $L$ ” without further specification, as will often be done in what follows, existence and smoothness of  $L$  is in these cases tacitly assumed.

## 6. Integrability

A further remarkable property of polypresymplectic/multipresymplectic structures is that, generically, the polylagrangian/multilagrangian distribution is

automatically integrable and thus gives rise to a **polylagrangian/multilagrangian foliation**.

**Theorem 6.1 (Integrability Theorem).** *Let  $P$  be a polypresymplectic fiber bundle over an  $n$ -dimensional manifold  $M$  with polypresymplectic  $(k + 1)$ -form  $\hat{\omega}$  of rank  $N$  taking values in a fixed  $\hat{n}$ -dimensional vector bundle  $\hat{T}$  over the same manifold  $M$ . Then if*

$$\hat{n} \geq 3, \tag{6.1}$$

*the polylagrangian distribution  $L$  is integrable. Similarly, let  $P$  be a multipresymplectic fiber bundle over an  $n$ -dimensional manifold  $M$  with multipresymplectic  $(k + 1)$ -form  $\omega$  of rank  $N$  and horizontality degree  $k + 1 - r$ , where  $1 \leq r \leq k + 1$  and  $k + 1 - r \leq n$ . Then if*

$$\binom{n}{k + 1 - r} \geq 3, \tag{6.2}$$

*the multilagrangian distribution  $L$  is integrable.*

**Proof.** Using Theorem 5.1, the second statement is easily reduced to the first. To prove this, suppose that  $X$  and  $Y$  are vector fields on  $P$  which are sections of  $L \subset VP$ . Using the decomposition

$$L = K_0 \oplus K_1 \oplus \dots \oplus K_{\hat{n}},$$

with  $K_0 = \ker \hat{\omega}$ , as in Eq. (2.17), we can decompose  $X$  and  $Y$  according to

$$X = \sum_{a=0}^{\hat{n}} X_a, \quad Y = \sum_{b=0}^{\hat{n}} Y_b,$$

where  $X_a$  and  $Y_b$  are sections of  $K_a$  and  $K_b$ , respectively. Using that  $\hat{n} \geq 3$ , we can for each value of  $a$  and  $b$  find a value  $c \neq 0$  such that  $c \neq a$  and  $c \neq b$ ; then  $K_a \subset \ker \omega^c$  and  $K_b \subset \ker \omega^c$ . Since  $\omega^c$  is vertically closed and has constant rank,  $\ker \omega^c \subset VP$  is involutive. Therefore the vector field  $[X_a, Y_b]$  is a section of  $\ker \omega^c \subset L$ .  $\square$

It must be emphasized that when the above inequalities are not satisfied, the statement of Theorem 6.1 is false, i.e.,  $L$  may fail to be involutive. For the multipresymplectic case, it is useful to spell out explicitly under what circumstances this may happen:

- (1)  $r = k + 1$ ,  $n = 0, 1, 2, 3, \dots$  arbitrary: this is the extreme case discussed before in which the horizontality condition is void.
- (2)  $r = k + 1 - n$ ,  $n = 1, 2, 3, \dots$  arbitrary: this includes the symplectic case, obtained by choosing  $k = n + 1$  and  $r = 2$  (i.e.,  $\omega \in \Omega_2^{n+2}(P)$ ) and supposing in addition that  $\ker \hat{\omega} = \{0\}$  and that  $M$  is orientable, so that  $\hat{\omega} \in \Omega_V^2(P, \pi^*(\wedge^n T^*M))$  represents a “family of symplectic forms smoothly parametrized by the points of a base manifold  $M$  of dimension  $n$ ”. Here, it is not difficult to construct examples of lagrangian distributions which are not involutive.

- (3)  $r = k, n = 2$  (the only possibility to have  $\binom{n}{k+1-r} \leq 2$  with  $1 \leq r \leq k$  and  $k + 1 - r < n$ ): this includes the “standard” multisymplectic case over a two-dimensional base manifold  $M$ , obtained by choosing  $r = k = n = 2$ . An explicit example of this situation is constructed in Example 6.3 below.

A simple example of a “standard” polysymplectic manifold with a two-dimensional non-involutive polylagrangian distribution of rank 1 is the following.

**Example 6.1.** Let  $P$  be  $SU(2)$  and let  $\alpha \in \Omega^1(P, \mathfrak{su}(2))$  be the left invariant Maurer-Cartan form on  $P$ , where  $\mathfrak{su}(2)$  is the Lie algebra of  $SU(2)$ . Consider the left invariant  $\mathfrak{su}(2)^*$ -valued 2-form  $\beta$  on  $P$  obtained by taking the exterior product of component forms whose values are multiplied using the commutator in  $\mathfrak{su}(2)$  and finally passing to the dual using the invariant scalar product  $(\cdot, \cdot)$  on  $\mathfrak{su}(2)$ , i.e.,

$$\langle \beta, X \rangle = \frac{1}{2}([\alpha \wedge \alpha], X) \quad \text{for } X \in \mathfrak{su}(2).$$

Using the isomorphism  $\mathfrak{su}(2) \cong \mathbb{R}^3$  induced by employing the standard orthonormal basis  $\{\sigma_a/2i \mid a \in \{1, 2, 3\}\}$  of  $\mathfrak{su}(2)$ ,<sup>n</sup> and working in components, we see that  $\alpha$  is represented by a triplet of left invariant 1-forms  $\alpha^1, \alpha^2, \alpha^3$  and  $\beta$  by a triplet of left invariant 2-forms  $\beta_1, \beta_2, \beta_3$  on  $P$  such that

$$\beta_a = \frac{1}{2} \epsilon_{abc} \alpha^b \wedge \alpha^c.$$

Moreover, taking  $\hat{T} = (\mathbb{R}^2)^*$ , let  $e$  be a given 2-frame in  $\mathbb{R}^3$ , i.e., a given injective linear map  $e$  from  $\mathbb{R}^2$  to  $\mathbb{R}^3$ . Its transpose will be a surjective linear map (projection)  $\text{pr}$  from  $(\mathbb{R}^3)^*$  to  $(\mathbb{R}^2)^*$ ,<sup>o</sup> and we define

$$\hat{\omega} = \text{pr} \circ \beta. \tag{6.3}$$

In components,  $e$  is represented by a triplet of vectors  $e^1, e^2, e^3$  in  $(\mathbb{R}^2)^*$  (not a basis, of course), and we have

$$\hat{\omega} = \frac{1}{2} \epsilon_{abc} \alpha^a \wedge \alpha^b \otimes e^c. \tag{6.4}$$

<sup>n</sup>Note that under the isomorphism  $\mathfrak{su}(2) \cong \mathbb{R}^3$  given by mapping the basis  $\{\sigma_a/2i \mid a \in \{1, 2, 3\}\}$  of  $\mathfrak{su}(2)$  to the standard basis of  $\mathbb{R}^3$ , where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the usual Pauli matrices, the commutator in  $\mathfrak{su}(2)$  corresponds to the vector product in  $\mathbb{R}^3$  whereas the invariant scalar product in  $\mathfrak{su}(2)$  given by  $(X, Y) = -2 \text{tr}(XY)$  corresponds to the standard scalar product in  $\mathbb{R}^3$ . With respect to this orthonormal basis, the (totally covariant) structure constants for  $\mathfrak{su}(2)$  are given by the components of the  $\epsilon$ -tensor, which are those of the standard volume form on  $\mathbb{R}^3$ .

<sup>o</sup>It may seem overly pedantic not to identify the spaces  $\mathbb{R}^2$  and  $\mathbb{R}^3$  with their respective duals  $(\mathbb{R}^2)^*$  and  $(\mathbb{R}^3)^*$ , but we have refrained from doing so right from the start since this turns out to facilitate the understanding of the generalizations to be discussed in the next two examples. Moreover, maintaining this distinction is not completely irrelevant since  $e$  is *not* assumed to be isometric: the space  $\mathbb{R}^2$  is not even supposed to carry a scalar product.

It follows immediately from the Maurer-Cartan structure equations that each of the 2-forms  $\beta_1, \beta_2, \beta_3$  is closed, and hence so is  $\hat{\omega}$ . Moreover, it is easily verified that the image of  $\mathbb{R}^2$  under  $e$ , regarded as a subspace of  $\mathfrak{su}(2) \cong \mathbb{R}^3$ , generates a two-dimensional left invariant distribution  $L$  on  $P$  which is isotropic under  $\hat{\omega}$  (since the expression  $\epsilon_{abc} e^a(u) e^b(v) e^c(w)$  vanishes identically, for any three vectors  $u, v, w$  in  $\mathbb{R}^2$ ), so  $\hat{\omega}$  is polysymplectic and  $L$  is the corresponding polylagrangian distribution. Of course,  $L$  is not involutive.

Similar but somewhat more complicated constructions lead to examples of polysymplectic and multisymplectic fiber bundles over an arbitrary two-dimensional base manifold  $M$  whose polylagrangian or multilagrangian distribution, again of rank 1, is not involutive. All of them are principal bundles over  $M$ , with structure group  $SU(2)$  in the polysymplectic case and  $U(2)$  in the multisymplectic case, carrying an invariant polysymplectic or multisymplectic form built from data that can be interpreted in terms of concepts from Yang-Mills-Higgs field theory, namely an  $SU(2)$ -connection form  $A$  and a Higgs field  $\phi$  in the adjoint representation of  $SU(2)$  in the polysymplectic case and a  $U(1)$ -connection form  $A$  and a Higgs field  $\phi$  in the truncated adjoint representation of  $U(2)$  in the multisymplectic case.<sup>P</sup> The details follow.

**Example 6.2.** Let  $P$  be the total space of a principal bundle over a two-dimensional manifold  $M$  with structure group  $SU(2)$  and bundle projection denoted by  $\pi : P \rightarrow M$ , and let  $A \in \Omega^1(P, \mathfrak{su}(2))$  be a given connection form on  $P$ , where  $\mathfrak{su}(2)$  is the Lie algebra of  $SU(2)$ . Consider the equivariant  $\mathfrak{su}(2)^*$ -valued 2-form  $B$  on  $P$  obtained by taking the exterior product of component forms whose values are multiplied using the commutator in  $\mathfrak{su}(2)$  and finally passing to the dual using the invariant scalar product  $(\cdot, \cdot)$  on  $\mathfrak{su}(2)$ , i.e.,

$$\langle B, X \rangle = \frac{1}{2} ([A \wedge A], X) \quad \text{for } X \in \mathfrak{su}(2).$$

Using the isomorphism  $\mathfrak{su}(2) \cong \mathbb{R}^3$  as before,<sup>n</sup> and working in components, we see that  $A$  is represented by a triplet of 1-forms  $A^1, A^2, A^3$  and  $B$  by a triplet of 2-forms  $B_1, B_2, B_3$  on  $P$  such that

$$B_a = \frac{1}{2} \epsilon_{abc} A^b \wedge A^c.$$

Moreover, taking  $\hat{T} = T^*M$ , let  $e$  be a given 1-form on  $M$  taking values in the adjoint bundle  $P \times_{SU(2)} \mathbb{R}^3$  which is an “immersion” in the sense that, when interpreted as a vector bundle homomorphism from  $TM$  to  $P \times_{SU(2)} \mathbb{R}^3$ , it is fiberwise injective. When pulled back to  $P$ ,  $e$  corresponds to an equivariant horizontal  $\mathbb{R}^3$ -valued 1-form  $\pi^*e$  on  $P$  which, once again, is an “immersion” in the sense that,

<sup>P</sup>By the truncated adjoint representation of a Lie group  $G$ , we mean the restriction of the adjoint representation of  $G$  on its Lie algebra  $\mathfrak{g}$  to its derived algebra  $[\mathfrak{g}, \mathfrak{g}]$ , which is an  $\text{Ad}(G)$ -invariant subspace (in particular, an ideal) of  $\mathfrak{g}$ . Note that the derived algebra of  $\mathfrak{u}(n)$  is  $\mathfrak{su}(n)$ .

when interpreted as an equivariant vector bundle homomorphism from  $\pi^*(TM)$  to  $P \times \mathbb{R}^3$ , it is fiberwise injective. Its transpose will be a fiberwise surjective equivariant vector bundle homomorphism (equivariant projection)  $\text{pr}$  from  $P \times (\mathbb{R}^3)^*$  to  $\pi^*(T^*M)$ , and we define

$$\hat{\omega} = \text{pr} \circ B|_{VP}, \tag{6.5}$$

where  $\cdot|_{VP}$  denotes restriction to the vertical bundle  $VP$  of  $P$ . In components,  $\pi^*e$  is represented by a triplet of sections  $\pi^*e^1, \pi^*e^2, \pi^*e^3$  of  $\pi^*(T^*M)$ , and we have

$$\hat{\omega} = \frac{1}{2} \epsilon_{abc} A^a|_{VP} \wedge A^b|_{VP} \otimes \pi^*e^c. \tag{6.6}$$

It follows immediately from the structure equation

$$d_V A^a|_{VP} = (dA^a)|_{VP} = -\frac{1}{2} \epsilon^a{}_{bc} A^b|_{VP} \wedge A^c|_{VP},$$

together with the formula  $\epsilon_{abc} \epsilon^a{}_{de} = \delta_{bd} \delta_{ce} - \delta_{be} \delta_{cd}$ , that  $\hat{\omega}$  is vertically closed:

$$\begin{aligned} 2 d_V \hat{\omega} &= \epsilon_{abc} d_V A^a|_{VP} \wedge A^b|_{VP} \otimes \pi^*e^c - \epsilon_{abc} A^a|_{VP} \wedge d_V A^b|_{VP} \otimes \pi^*e^c \\ &= -\epsilon_{abc} \epsilon^a{}_{de} A^d|_{VP} \wedge A^e|_{VP} \wedge A^b|_{VP} \otimes \pi^*e^c \\ &\quad + \epsilon_{abc} \epsilon^b{}_{de} A^a|_{VP} \wedge A^d|_{VP} \wedge A^e|_{VP} \otimes \pi^*e^c \\ &= 0. \end{aligned}$$

Moreover, it is easily verified that the image of  $\pi^*(TM)$  under  $\pi^*e$ , when transferred from  $P \times \mathbb{R}^3$  to  $VP$  by means of the canonical isomorphism that exists between the two since  $P$  is a principal bundle, constitutes a two-dimensional invariant distribution  $L$  on  $P$  which is isotropic under  $\hat{\omega}$ , so  $\hat{\omega}$  is polysymplectic and  $L$  is the corresponding polylagrangian distribution. Of course,  $L$  is not involutive.

**Example 6.3.** Let  $P$  be the total space of a principal bundle over a two-dimensional manifold  $M$  with structure group  $U(2)$  and bundle projection denoted by  $\pi : P \rightarrow M$ , and let  $A \in \Omega^1(P, \mathfrak{u}(2))$  be a given connection form on  $P$ , where  $\mathfrak{u}(2)$  is the Lie algebra of  $U(2)$ . Consider the equivariant  $\mathfrak{su}(2)^*$ -valued 2-form  $B$  on  $P$  obtained by taking the exterior product of component forms whose values are multiplied using the commutator in  $\mathfrak{u}(2)$  (which maps into the derived algebra  $\mathfrak{su}(2)$ ) and finally passing to the dual using the invariant scalar product  $(\cdot, \cdot)$  on  $\mathfrak{su}(2)$ , i.e.,

$$\langle B, X \rangle = \frac{1}{2} ([A \wedge A], X) \quad \text{for } X \in \mathfrak{su}(2).$$

Using the isomorphism  $\mathfrak{u}(2) \cong \mathbb{R}^4$  induced by employing the standard orthonormal basis  $\{\sigma_a/2i \mid a \in \{0, 1, 2, 3\}\}$  of  $\mathfrak{u}(2)$ ,<sup>9</sup> and working in components, we see that  $A$

<sup>9</sup>Note that under the isomorphism  $\mathfrak{u}(2) \cong \mathbb{R}^4$  given by mapping the basis  $\{\sigma_a/2i \mid a \in \{0, 1, 2, 3\}\}$  of  $\mathfrak{u}(2)$  to the standard basis of  $\mathbb{R}^4$ , where  $\sigma_0$  is the unit matrix and  $\sigma_1, \sigma_2, \sigma_3$  are the Pauli matrices as before, the commutator in  $\mathfrak{u}(2)$  corresponds to that induced by the (associative) quaternion



is represented by a quartet of 1-forms  $A^0, A^1, A^2, A^3$  and  $B$  by a triplet of 2-forms  $B_1, B_2, B_3$  on  $P$  such that

$$B_a = \frac{1}{2} \epsilon_{abc} A^b \wedge A^c.$$

Moreover, let  $\phi$  be a given section of the truncated adjoint bundle  $P \times_{U(2)} \mathbb{R}^3$  which is an “immersion” in the sense that its covariant derivative  $D_A \phi$  with respect to the given connection, which is a 1-form on  $M$  taking values in  $P \times_{U(2)} \mathbb{R}^3$ , when interpreted as a vector bundle homomorphism from  $TM$  to  $P \times_{U(2)} \mathbb{R}^3$ , is fiberwise injective. When pulled back to  $P$ ,  $\phi$  corresponds to an equivariant  $\mathbb{R}^3$ -valued function  $\pi^* \phi$  on  $P$  and  $D_A \phi$  corresponds to an equivariant horizontal  $\mathbb{R}^3$ -valued 1-form  $\pi^* D_A \phi = D_A \pi^* \phi$  on  $P$  which, once again, is an “immersion” in the sense that, when interpreted as an equivariant vector bundle homomorphism from  $\pi^*(TM)$  to  $P \times \mathbb{R}^3$ , it is fiberwise injective. In components,  $\pi^* \phi$  is represented by a triplet of functions  $\pi^* \phi^1, \pi^* \phi^2, \pi^* \phi^3$  and  $D_A \pi^* \phi$  by a triplet of horizontal 1-forms  $D_A \pi^* \phi^1, D_A \pi^* \phi^2, D_A \pi^* \phi^3$  on  $P$ , where

$$D_A \pi^* \phi^a = d\pi^* \phi^a + \epsilon^a_{bc} A^b \pi^* \phi^c.$$

By standard results [18, 19], the covariant exterior derivative of  $D_A \pi^* \phi$ , given by

$$d_A D_A \pi^* \phi^a = d D_A \pi^* \phi^a + \epsilon^a_{bc} A^b \wedge D_A \pi^* \phi^c,$$

can also be written in the form

$$d_A D_A \pi^* \phi^a = \epsilon^a_{bc} F^b \pi^* \phi^c,$$

where the  $F^a$  are the components of the  $\mathfrak{su}(2)$ -part of the curvature form of the connection form  $A$ , defined by

$$F^0 = dA^0, \quad F^a = dA^a + \frac{1}{2} \epsilon^a_{bc} A^b \wedge A^c,$$

and there is a product rule relating the ordinary exterior derivative to the covariant exterior derivative which in the case of importance here reads

$$\begin{aligned} & d(\epsilon_{abc} \pi^* \phi^a D_A \pi^* \phi^b \wedge D_A \pi^* \phi^c) \\ &= \epsilon_{abc} D_A \pi^* \phi^a \wedge D_A \pi^* \phi^b \wedge D_A \pi^* \phi^c \\ & \quad + \epsilon_{abc} \pi^* \phi^a d_A D_A \pi^* \phi^b \wedge D_A \pi^* \phi^c \\ & \quad - \epsilon_{abc} \pi^* \phi^a D_A \pi^* \phi^b \wedge d_A D_A \pi^* \phi^c. \end{aligned}$$

product in  $\mathbb{R}^4$  whereas the invariant scalar product in  $\mathfrak{u}(2)$  given by  $(X, Y) = -2 \operatorname{tr}(XY)$  corresponds to the standard scalar product in  $\mathbb{R}^4$ . With respect to this orthonormal basis, the (totally covariant) structure constants for  $\mathfrak{u}(2)$  are again given by the components of the  $\epsilon$ -tensor, which are those of the standard volume form on  $\mathbb{R}^3$ : all structure constants for which one of the indices takes the value 0 vanish.

In fact, this expression vanishes identically since it represents an invariant horizontal 3-form on  $P$  which corresponds to a 3-form on  $M$ , but there exists no non-zero 3-form on a two-dimensional manifold. Now define

$$\omega = \frac{1}{2} \epsilon_{abc} A^a \wedge A^b \wedge D_A \pi^* \phi^c - \frac{1}{2} \epsilon_{abc} \pi^* \phi^a A^0 \wedge D_A \pi^* \phi^b \wedge D_A \pi^* \phi^c. \quad (6.7)$$

Then using the above relations and the identity  $\epsilon_{abc} \epsilon^a_{de} = \delta_{bd} \delta_{ce} - \delta_{be} \delta_{cd}$ , we can calculate the exterior derivative of  $\omega$ :

$$\begin{aligned} 4 d\omega &= 2 \epsilon_{abc} dA^a \wedge A^b \wedge D_A \pi^* \phi^c - 2 \epsilon_{abc} A^a \wedge dA^b \wedge D_A \pi^* \phi^c \\ &\quad + 2 \epsilon_{abc} A^a \wedge A^b \wedge d_A D_A \pi^* \phi^c - 2 \epsilon_{abc} \epsilon^c_{de} A^a \wedge A^b \wedge A^d \wedge D_A \pi^* \phi^e \\ &\quad - 2 \epsilon_{abc} \pi^* \phi^a dA^0 \wedge D_A \pi^* \phi^b \wedge D_A \pi^* \phi^c \\ &\quad + 2 A^0 \wedge d(\epsilon_{abc} \pi^* \phi^a D_A \pi^* \phi^b \wedge D_A \pi^* \phi^c) \\ &= 2 \epsilon_{abc} F^a \wedge A^b \wedge D_A \pi^* \phi^c - 2 \epsilon_{abc} A^a \wedge F^b \wedge D_A \pi^* \phi^c \\ &\quad - \epsilon_{abc} \epsilon^a_{de} A^d \wedge A^e \wedge A^b \wedge D_A \pi^* \phi^c + \epsilon_{abc} \epsilon^b_{de} A^a \wedge A^d \wedge A^e \wedge D_A \pi^* \phi^c \\ &\quad + 2 \epsilon_{abc} \epsilon^c_{de} \pi^* \phi^e A^a \wedge A^b \wedge F^d \\ &\quad - 2 \epsilon_{abc} \epsilon^c_{de} A^a \wedge A^b \wedge A^d \wedge D_A \pi^* \phi^e \\ &\quad - 2 \epsilon_{abc} \pi^* \phi^a F^0 \wedge D_A \pi^* \phi^b \wedge D_A \pi^* \phi^c. \end{aligned}$$

The terms in the first and fifth line of the second equation vanish by the same argument of horizontality as before, while the terms in the second and fourth line vanish by symmetry. However, the term in the third line survives, i.e., we have

$$d\omega = \delta_{ac} \delta_{bd} \pi^* \phi^a A^b \wedge A^c \wedge F^d.$$

This means that  $\omega$  is closed if (and also only if) the  $\mathfrak{su}(2)$ -part of the curvature form  $F$  vanishes, whereas there is no restriction on its  $\mathfrak{u}(1)$ -part. Moreover, it is easily verified that the image of  $\pi^*(TM)$  under  $D_A \pi^* \phi$  together with the (one-dimensional) orthogonal complement of  $P \times \mathbb{R}^3$  in  $P \times \mathbb{R}^4$ , when transferred from  $P \times \mathbb{R}^4$  to  $VP$  by means of the canonical isomorphism that exists between the two since  $P$  is a principal bundle, constitutes a three-dimensional invariant distribution  $L$  on  $P$  which is isotropic under  $\omega$ , so provided  $\omega$  is closed,  $\omega$  is multisymplectic and  $L$  is the corresponding multilagrangian distribution. Of course,  $L$  is not involutive.

**Remark 6.1.** We emphasize again that the input data for the construction of the polysymplectic and multisymplectic forms in the last two examples allow for a natural interpretation in terms of Yang-Mills-Higgs field theory. Indeed, in the polysymplectic case,  $A$  is an arbitrary  $SU(2)$ -connection and so we only have to make one assumption that is more restrictive than stated in Example 6.2: namely that the 1-form  $e$  is (covariantly) holonomous, that is, of the form  $e = D_A \phi$  where the Higgs field  $\phi$  is a section of the adjoint bundle  $P \times_{SU(2)} \mathbb{R}^3$ . Moreover, since only the vertical part of  $A$  appears in Eq. (6.6), it is clear that  $\hat{\omega}$  is completely independent of the choice of connection! In the multisymplectic case, we can argue that the relevant input data are (a) a principal  $U(1)$ -bundle  $P_0$  over  $M$  (or equivalently,

a complex line bundle  $L_0$  over  $M$  with fixed hermitean fiber metric) and a given  $U(1)$ -connection form  $A_0$  on  $P_0$  (or equivalently, a given linear connection in  $L_0$  preserving this fiber metric) and (b) a given embedding  $\phi$  of  $M$  into  $\mathbb{R}^3$ . Indeed, the principal  $U(2)$ -bundle  $P$  over  $M$  (corresponding to a rank 2 complex vector bundle  $E_0$  over  $M$  with fixed hermitean fiber metric) and the  $U(2)$ -connection form  $A$  on  $P$  (corresponding to a linear connection in  $E_0$  preserving this fiber metric) that appear in Example 6.3 are then obtained by the process of extension of structure group, using the embedding homomorphism of  $U(1)$  into  $U(2)$  that takes a phase to that phase times the unit  $(2 \times 2)$ -matrix; moreover, this construction guarantees that the  $\mathfrak{su}(2)$ -part of the curvature form  $F$  will vanish. Conversely, if this condition is satisfied, then according to the Ambrose-Singer theorem,  $P$  can be reduced to the holonomy bundle of  $A$ , whose structure group is the holonomy group of  $A$  which has connected one-component  $U(1)$ : this means that except for a possible discrete part, we are back to the previous situation. It should also be emphasized that from a purely topological point of view, reduction from  $U(2)$  to  $U(1)$  is always possible, e.g., over compact Riemann surfaces [21, Theorem 10 & Corollary, p. 63], so the restriction made here really only concerns the connection form  $A$ , not the bundle  $P$ . Moreover, in this situation, the truncated adjoint bundle  $P \times_{U(2)} \mathbb{R}^3$  will be trivial and so the Higgs field  $\phi$  with its injective (covariant = ordinary) derivative will provide an immersion of  $M$  into  $\mathbb{R}^3$ : it is only slightly more restrictive to assume that this immersion is in fact an embedding. Finally, an important observation that applies to both settings is that, as already mentioned, the polysymplectic form  $\hat{\omega}$  of Example 6.2 and the multisymplectic form  $\omega$  of Example 6.3 are both invariant under the action of the respective structure groups  $SU(2)$  and  $U(2)$  on the total space  $P$ : expressed in physical language, this means that they are *gauge invariant*.

We expect it to be possible to apply similar procedures, with  $SU(2)$  replaced by some compact three-dimensional Lie group which is solvable and  $U(2)$  replaced by a compact four-dimensional Lie group which is a suitable one-dimensional extension thereof, to construct polysymplectic and multisymplectic fiber bundles  $P$  where  $P$  is still a compact manifold but  $L$  is now involutive. However, in our view the real challenge would be to come up with examples of polysymplectic and multisymplectic fiber bundles that appear naturally as multiphase spaces of physically realistic covariant hamiltonian field theories but, just as in the above cases, cannot be obtained by taking the affine dual of the first order jet bundle of some configuration bundle. This question has recently been answered and unfortunately (or perhaps fortunately, depending on interpretation), the outcome is essentially negative; see [25, 26] for more details.

## 7. The Darboux Theorem

Now we are able to prove the Darboux theorem for poly- and multisymplectic forms. Here, the specific algebraic structure of poly- and multilagrangian subspaces identified in the first two sections turns out to be crucial, in the sense that this

central theorem can, in all cases, be proved by appropriately adapting the procedure used to prove the classical Darboux theorem for symplectic forms. (See, for instance, [20]).

**Theorem 7.1 (Polysymplectic Darboux theorem).** *Let  $P$  be a polypresymplectic fiber bundle over an  $n$ -dimensional manifold  $M$  with polypresymplectic  $(k + 1)$ -form  $\hat{\omega}$  of rank  $N$  taking values in a fixed  $\hat{n}$ -dimensional vector bundle  $\hat{T}$  over the same manifold  $M$  and with polylagrangian distribution  $L$ , which is assumed to be involutive (recall this is automatic if  $\hat{n} \geq 3$ ), and let  $\{\hat{e}_a \mid 1 \leq a \leq \hat{n}\}$  be a basis of local sections of  $\hat{T}$ . Then around any point of  $P$  (within the domain of the given basis of local sections), there exists a system of local coordinates  $(x^\mu, q^i, p_{i_1 \dots i_k}^\alpha, r^\kappa)$  ( $1 \leq \mu \leq n, 1 \leq \alpha \leq \hat{n}, 1 \leq i \leq N, 1 \leq i_1 < \dots < i_k \leq N, 1 \leq \kappa \leq \dim \ker \hat{\omega}$ ), called **Darboux coordinates** or **canonical coordinates**, such that*

$$\hat{\omega} = \frac{1}{k!} dp_{i_1 \dots i_k}^\alpha \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k} \otimes \hat{e}_\alpha \quad (7.1)$$

and such that (locally)  $L$  is spanned by the vector fields  $\partial/\partial p_{i_1 \dots i_k}^\alpha$  and  $\partial/\partial r^\kappa$  while  $\ker \hat{\omega}$  is spanned by the vector fields  $\partial/\partial r^\kappa$ . In the “standard” case of main interest where  $\hat{\omega}$  is a 2-form, i.e.,  $k = 1$ , this expansion takes the form

$$\hat{\omega} = (dq^i \wedge dp_i^\alpha) \otimes \hat{e}_\alpha. \quad (7.2)$$

Similarly, we have

**Theorem 7.2 (Multisymplectic Darboux theorem).** *Let  $P$  be a multipresymplectic fiber bundle over an  $n$ -dimensional manifold  $M$  with multipresymplectic  $(k + 1)$ -form  $\omega$  of rank  $N$  and horizontality degree  $k + 1 - r$ , where  $1 \leq r \leq k$  and  $k + 1 - r \leq n$ , and with multilagrangian distribution  $L$ , which is assumed to be involutive (recall this is automatic if  $\binom{n}{k+1-r} \geq 3$ ). Then around any point of  $P$ , there exists a system of local coordinates  $(x^\mu, q^i, p_{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}}^\alpha, r^\kappa)$  ( $0 \leq s \leq r - 1, 1 \leq \mu \leq n, 1 \leq i \leq N, 1 \leq i_1 < \dots < i_s \leq N, 1 \leq \mu_1 < \dots < \mu_{k-s} \leq N, 1 \leq \kappa \leq \dim \ker \omega$ ), called **Darboux coordinates** or **canonical coordinates**, such that*

$$\omega = \sum_{s=0}^{r-1} \frac{1}{s!} \frac{1}{(k-s)!} dp_{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}}^\alpha \wedge dq^{i_1} \wedge \dots \wedge dq^{i_s} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{k-s}} \quad (7.3)$$

and such that (locally)  $L$  is spanned by the vector fields  $\partial/\partial p_{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}}^\alpha$  and  $\partial/\partial r^\kappa$  while  $\ker \omega$  is spanned by the vector fields  $\partial/\partial r^\kappa$ . In these coordinates, its symbol is given by

$$\hat{\omega} = \frac{1}{(r-1)!} \frac{1}{(k+1-r)!} dp_{i_1 \dots i_{r-1}; \mu_1 \dots \mu_{k+1-r}}^\alpha \wedge dq^{i_1} \wedge \dots \wedge dq^{i_{r-1}} \otimes dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{k+1-r}}. \quad (7.4)$$

In the “standard” case of main interest, where  $\omega$  is an  $(n - 1)$ -horizontal  $(n + 1)$ -form, i.e.,  $k = n$  and  $r = 2$ , these expansions take the form

$$\omega = dq^i \wedge dp_i^\mu \wedge d^n x_\mu - dp \wedge d^n x \quad (7.5)$$

and

$$\hat{\omega} = dq^i \wedge dp_i^\mu \otimes d^n x_\mu, \tag{7.6}$$

where

$$d^n x_\mu = i_{\partial_\mu} d^n x = \frac{1}{(n-1)!} \epsilon_{\mu\mu_1 \dots \mu_{n-1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{n-1}}. \tag{7.7}$$

**Proof.** For the sake of definiteness, we concentrate on the multisymplectic case: the proof for the other case is entirely analogous, requiring only small and rather obvious modifications. Also, due to the local character of this theorem and since the kernel of  $\omega$ , the multilagrangian subbundle  $L$  and the vertical bundle  $VP$  are all involutive, with  $\ker \omega \subset L \subset VP$ , we can without loss of generality work in a local chart of the manifold  $P$  around the chosen reference point in which the corresponding foliations are “straightened out”, so we may assume that  $P \cong \mathbb{R}^n \oplus \mathbb{R}^N \oplus L_0 \oplus K_0$  with  $VP \cong \mathbb{R}^N \oplus L_0 \oplus K_0$ ,  $L \cong L_0 \oplus K_0$  and  $\ker \omega \cong K_0$  with fixed subspaces  $L_0$  and  $K_0$  and such that the aforementioned reference point corresponds to the origin. We also take  $\omega_0$  to be the constant multipresymplectic form, with multilagrangian distribution  $L$ , obtained by spreading  $\omega(0)$ , the value of the multipresymplectic form  $\omega$  at the origin, all over the space  $P$ ; then the existence of canonical coordinates for  $\omega_0$ , in the form given by Eq. (7.3), is already guaranteed by the algebraic Darboux theorem of the previous chapter (Theorem 3.3).

Now consider the family of  $(k+1)$ -forms given by  $\omega_t = \omega_0 + t(\omega - \omega_0)$ , for every  $t \in \mathbb{R}$ . Obviously,  $\omega_t(0) = \omega_0$  for every  $t \in \mathbb{R}$ , which is non-degenerate on  $K'_0 = \mathbb{R}^n \oplus \mathbb{R}^N \oplus L_0$  (a complement of  $K_0$  in  $P$ ). Since non-degeneracy is an open condition, and using a compactness argument with respect to the parameter  $t$ , we conclude that there is an open neighborhood of 0 such that, for all  $t$  satisfying  $0 \leq t \leq 1$  and all points  $p$  in this neighborhood,  $\omega_t(p)$  is non-degenerate on  $K'_0 = \mathbb{R}^n \oplus \mathbb{R}^N \oplus L_0$ , that is, its kernel equals  $K_0$ . Moreover, for all  $t$  satisfying  $0 \leq t \leq 1$  and all points  $p$  in this neighborhood, the subspace  $L_0$ , being isotropic for  $\omega_0$  as well as for  $\omega(p)$ , is also isotropic for  $\omega_t(p)$  and, since it contains the kernel of  $\omega_t(p)$  and has the right dimension as given by Eq. (3.17), is even multilagrangian for  $\omega_t(p)$ , according to Proposition 3.2. On the other hand, we have  $d\omega_0 = 0$  (trivially) and  $d\omega = 0$  (by hypothesis), so we can apply an appropriate version of the Poincaré lemma (see Appendix B) to prove, in some open neighborhood of the point 0 in  $P$  (contained in the previous one), existence of a  $k$ -form  $\alpha$  satisfying  $d\alpha = \omega_0 - \omega$  and  $\alpha^\flat(L) = 0$ . Now take  $X_t$  to be the unique time dependent vector field on  $P$  taking values in  $L_0$  defined by<sup>r</sup>

$$i_{X_t} \omega_t = \alpha.$$

Let  $F_t \equiv F_{(0,t)}$  be its flux beginning at 0, once again defined, for  $0 \leq t \leq 1$ , in some open neighborhood of the point 0 in  $P$  (contained in the previous one). Then

<sup>r</sup>It is at this point that we make essential use of the hypothesis that  $L_0$  is multilagrangian and not just isotropic or even maximal isotropic (with respect to  $\omega_t(p)$ , in this case).

it follows that

$$\begin{aligned}
 \frac{d}{ds} \Big|_{s=t} F_s^* \omega_s &= F_t^* \left( \frac{d}{ds} \Big|_{s=t} \omega_s \right) + \frac{d}{ds} \Big|_{s=t} F_s^* \omega_t \\
 &= F_t^* (\omega - \omega_0 + L_{X_t} \omega_t) \\
 &= F_t^* (\omega - \omega_0 + d(i_{X_t} \omega_t)) \\
 &= F_t^* (\omega - \omega_0 + d\alpha) \\
 &= 0.
 \end{aligned}$$

Therefore,  $F_1$  is the desired coordinate transformation, since

$$F_1^* \omega = F_1^* \omega_1 = F_0^* \omega_0 = \omega_0. \quad \square$$

## 8. Conclusions and Outlook

In this paper we have presented a concise definition of a new class of geometric structures which we propose to call polysymplectic or multisymplectic structures and which include the ones encountered in the hamiltonian formulation of classical field theory as special cases. All of them are defined in terms of differential forms satisfying an algebraic condition that amounts to postulating the existence of an isotropic subbundle which is “sufficiently large” — a condition which, when combined with the standard integrability conditions that the pertinent differential form should be closed and the aforementioned “sufficiently large” isotropic subbundle should be involutive (in most cases, this additional requirement is automatically satisfied), allows to derive a Darboux theorem assuring the existence of “canonical” local coordinates around each point. Another characteristic feature of all these structures is that they are naturally defined on the total spaces of fiber bundles whose base space is interpreted as the space-time manifold of field theory. Moreover, there is a standard class of examples defined by bundles of partially horizontal forms over the total space of another fiber bundle (the so-called configuration bundle), which includes the multiphase spaces of interest in physics and is the analogue of cotangent bundles of manifolds, regarded as a standard class of examples of symplectic manifolds. To our knowledge, this is the first example of a natural geometric structure with important physical applications that is defined by a differential form (or even a tensor field) of degree strictly larger than 2 and strictly smaller than the dimension of the underlying manifold.<sup>s</sup>

Starting from this basis, there is a number of rather obvious questions that arise, most of which are closely interrelated. One of them that has been answered

<sup>s</sup>Recently, we have learnt about one other example:  $G_2$ -structures on 7-dimensional manifolds can be defined naturally in terms of a certain closed 3-form. However, it is not clear where this concept can be applied in physics.

recently [25, 26] concerns the quest for other classes of examples besides the standard ones — in particular, examples that would provide an analogue of the coadjoint orbit construction of symplectic geometry. The result of this investigation is negative: there are no such examples. And the underlying reason is relatively simple to understand if one observes that the correct symplectic analogue of a polysymplectic or multisymplectic structure is not simply a symplectic structure but rather a symplectic structure with a fixed lagrangian foliation, and this additional restriction essentially eliminates the coadjoint orbit construction.

Other questions that come to mind naturally refer to the nature of the underlying Lie group  $G$  when these structures are regarded as  $G$ -structures, the definition of Poisson brackets (see the discussion in [5–7, 22]), the definition of actions of Lie groups and, more generally, of Lie groupoids on polysymplectic or multisymplectic fiber bundles, the construction of a corresponding momentum map (which would provide a general framework for the construction of Noether currents and the energy-momentum tensor within a direct and manifestly covariant Hamiltonian approach) and the formulation of a Marsden-Weinstein reduction procedure. All these problems are still largely open and certainly will provide a fertile ground for research in the future.

## Acknowledgments

This work has been done in partial fulfillment of the requirements for the degree of Doctor in Science of Leandro G. Gomes and has been supported by CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) and by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Brazil.

## Appendix A. Some Counterexamples

In this appendix, we wish to provide additional evidence for the conceptual simplicity and usefulness of our definition of polysymplectic and multisymplectic structures by investigating, in a purely algebraic setting and for the simplest case of vector-valued 2-forms, various structural properties of polysymplectic forms that can be introduced directly for general vector-valued 2-forms. By constructing explicit counterexamples, we will show, however, that none of them is sufficiently strong to replace the requirement of existence of a polylagrangian subspace: this condition must therefore be imposed separately and is then sufficient to imply all the others, so that — in contrast to what is done, e.g., in [23] — we have refrained from including any of them into our definition of a polysymplectic structure.

Suppose, as in Sec. 1, that  $V$  and  $\hat{T}$  are finite-dimensional real vector spaces, with  $\dim \hat{T} \equiv \hat{n}$ , and assume that  $\hat{\omega} \in \bigwedge^2 V^* \otimes \hat{T}$  is an arbitrary  $\hat{T}$ -valued 2-form on  $V$ . Given any linear form  $\hat{t}^* \in \hat{T}^*$  on  $\hat{T}$ , we consider the projection  $\omega_{\hat{t}^*} = \langle \hat{t}^*, \hat{\omega} \rangle$  of  $\hat{\omega}$  along  $\hat{t}^*$ , which is an ordinary 2-form on  $V$ , and we define its **rank** to be equal to half the dimension of its **support**, which in turn can be defined as the annihilator

of its kernel [24]:<sup>t</sup>

$$\text{rk}(\omega_{\hat{t}^*}) = \frac{1}{2} \dim \text{supp } \omega_{\hat{t}^*} = \frac{1}{2} (\dim V - \dim \ker \omega_{\hat{t}^*}). \quad (\text{A.1})$$

Now note that the linear mapping

$$\begin{aligned} \hat{T}^* &\longrightarrow \bigwedge^2 V^* \\ \hat{t}^* &\longmapsto \omega_{\hat{t}^*} \end{aligned} \quad (\text{A.2})$$

induces, for every integer  $k \geq 1$ , a canonically defined linear mapping

$$\begin{aligned} \bigvee^k \hat{T}^* &\longrightarrow \bigwedge^{2k} V^* \\ P &\longmapsto P(\hat{\omega}) \end{aligned} \quad (\text{A.3})$$

where we have identified the space  $\bigvee^k \hat{T}^*$  of covariant symmetric tensors of degree  $k$  over  $\hat{T}$  with the space of homogeneous polynomials  $P$  of degree  $k$  on  $\hat{T}$ . Explicitly, in terms of a basis  $\{\hat{e}_a \mid 1 \leq a \leq \hat{n}\}$  of  $\hat{T}$ , with dual basis  $\{\hat{e}^a \mid 1 \leq a \leq \hat{n}\}$  of  $\hat{T}^*$ , we write  $\omega^a = \omega_{\hat{e}^a} = \langle \hat{e}^a, \hat{\omega} \rangle$  ( $1 \leq a \leq \hat{n}$ ) and obtain

$$P = P_{a_1 \dots a_k} \hat{e}^{a_1} \vee \dots \vee \hat{e}^{a_k} \implies P(\hat{\omega}) = P_{a_1 \dots a_k} \omega^{a_1} \wedge \dots \wedge \omega^{a_k}. \quad (\text{A.4})$$

This allows us to introduce the following terminology:

**Definition A.1.** Let  $V$  and  $\hat{T}$  be finite-dimensional vector spaces and let  $\hat{\omega}$  be a  $\hat{T}$ -valued 2-form on  $V$ . We say that  $\hat{\omega}$  has **constant rank**  $N$  if  $\text{rk}(\omega_{\hat{t}^*}) = N$  for every  $\hat{t}^* \in \hat{T}^* \setminus \{0\}$  and that  $\hat{\omega}$  has **uniform rank**  $N$  if the linear mapping (A.3) is injective for  $k = N$  and identically zero for  $k = N + 1$ .

Using multi-indices  $\alpha = (\alpha_1, \dots, \alpha_{\hat{n}}) \in \mathbb{N}^{\hat{n}}$ , we set

$$\hat{e}^\alpha = (\hat{e}^1)^{\alpha_1} \vee \dots \vee (\hat{e}^{\hat{n}})^{\alpha_{\hat{n}}} \quad \text{where} \quad (\hat{e}^a)^{\alpha_a} = \hat{e}^a \vee \dots \vee \hat{e}^a \quad (\alpha_a \text{ times})$$

and

$$\omega^\alpha = (\omega^1)^{\alpha_1} \wedge \dots \wedge (\omega^{\hat{n}})^{\alpha_{\hat{n}}} \quad \text{where} \quad (\omega^a)^{\alpha_a} = \omega^a \wedge \dots \wedge \omega^a \quad (\alpha_a \text{ times})$$

to rewrite Eq. (A.4) in the form

$$P = \sum_{|\alpha|=k} P_\alpha \hat{e}^\alpha \implies P(\hat{\omega}) = \sum_{|\alpha|=k} P_\alpha \omega^\alpha. \quad (\text{A.5})$$

Since  $\{\hat{e}^\alpha \mid |\alpha|=k\}$  is a basis of  $\bigvee^k \hat{T}^*$ , requiring  $\hat{\omega}$  to have uniform rank  $N$  amounts to imposing the following conditions:

$$\begin{aligned} \{\omega^\alpha \mid |\alpha| = N\} &\text{ is linearly independent,} \\ \omega^\alpha &= 0 \quad \text{for } |\alpha| = N + 1. \end{aligned} \quad (\text{A.6})$$

It is in this form that the requirement of uniform rank appears in the definition of a polysymplectic form adopted in [23].

<sup>t</sup>Thus our definition of rank differs from that of [24] by a factor of 2.



To gain a better understanding for the conditions of constant rank and of uniform rank introduced above, we note first of all that they both generalize the standard notion of rank for ordinary forms. Indeed, when  $\hat{n} = 1$ , that is, given an ordinary 2-form  $\omega$  of rank  $N$  on  $V$ , we can choose a canonical basis  $\{e_1, \dots, e_N, f^1, \dots, f^N\}$  of a subspace of  $V$  complementary to the kernel of  $\omega$ , with dual basis  $\{e^1, \dots, e^N, f_1, \dots, f_N\}$  of the subspace  $\text{supp } \omega$  of  $V^*$ , to conclude that  $\omega = e^i \wedge f_i$  and therefore

$$\omega^N = \pm e^1 \wedge \dots \wedge e^N \wedge f_1 \wedge \dots \wedge f_N \neq 0, \quad \omega^{N+1} = 0.$$

In other words, the rank of  $\omega$  can be characterized as that positive integer  $N$  for which  $\omega^N \neq 0$  but  $\omega^{N+1} = 0$ . From this observation, it follows that, in the general case considered before, the requirement of uniform rank implies that of constant rank because it guarantees that for every  $\hat{t}^* \in \hat{T}^* \setminus \{0\}$ , we have  $\omega_{\hat{t}^*}^N \neq 0$  and  $\omega_{\hat{t}^*}^{N+1} = 0$ , that is,  $\text{rk}(\omega_{\hat{t}^*}) = N$ . However, the converse does not hold, as shown by the following

**Example A.1** ( $\hat{n} = 2, N = 2, \dim V = 4, \ker \hat{\omega} = \{0\}$ ). Let  $V = \mathbb{R}^4, \hat{T} = \mathbb{R}^2$  and consider the  $\mathbb{R}^2$ -valued 2-form  $\hat{\omega}$  built from the following two ordinary 2-forms:

$$\omega^1 = dx \wedge dy + du \wedge dv, \quad \omega^2 = dx \wedge du - dy \wedge dv.$$

Then for  $\hat{t}^* = \hat{t}_a^* \hat{e}^a \in (\mathbb{R}^2)^*$ , we have

$$\omega_{\hat{t}^*} = \hat{t}_a^* \omega^a = dx \wedge (\hat{t}_1^* dy + \hat{t}_2^* du) + dv \wedge (\hat{t}_2^* dy - \hat{t}_1^* du).$$

Thus we obtain, for every  $\hat{t}^* \neq 0$ ,

$$(\omega_{\hat{t}^*})^2 \equiv \omega_{\hat{t}^*} \wedge \omega_{\hat{t}^*} = ((\hat{t}_1^*)^2 + (\hat{t}_2^*)^2) dx \wedge dy \wedge du \wedge dv \neq 0,$$

whereas, due to the fact that we are in a four-dimensional space,

$$(\omega_{\hat{t}^*})^3 \equiv \omega_{\hat{t}^*} \wedge \omega_{\hat{t}^*} \wedge \omega_{\hat{t}^*} = 0,$$

which guarantees that  $\hat{\omega}$  has constant rank 2. However,  $\hat{\omega}$  does not have uniform rank 2, since

$$\omega^1 \wedge \omega^2 = 0.$$

On the other hand, polysymplectic forms do have uniform rank:

**Proposition A.1.** *Let  $V$  and  $\hat{T}$  be finite-dimensional vector spaces and let  $\hat{\omega}$  be a  $\hat{T}$ -valued polysymplectic form of rank  $N$  on  $V$ . Then  $\hat{\omega}$  has uniform rank  $N$ .*

**Proof.** Introducing a (polysymplectic) canonical basis in which

$$\hat{\omega} = e_i^a \wedge e^i \otimes \hat{e}_a,$$

or equivalently

$$\omega^a = e_i^a \wedge e^i \quad (1 \leq a \leq \hat{n}),$$

suppose that  $\alpha = (\alpha_1, \dots, \alpha_{\hat{n}}) \in \mathbb{N}^{\hat{n}}$  is a multi-index of degree  $k$  (i.e., such that  $|\alpha| = \alpha_1 + \dots + \alpha_{\hat{n}} = k$ ) and consider the form

$$\omega^\alpha = \pm \left( (e^{i_1^1} \wedge \dots \wedge e^{i_{\alpha_1}^1}) \wedge \dots \wedge (e^{i_1^{\hat{n}}} \wedge \dots \wedge e^{i_{\alpha_{\hat{n}}}^{\hat{n}}}) \wedge (e_{i_1^1}^1 \wedge \dots \wedge e_{i_{\alpha_1}^1}^1) \wedge \dots \wedge (e_{i_1^{\hat{n}}}^{\hat{n}} \wedge \dots \wedge e_{i_{\alpha_{\hat{n}}}^{\hat{n}}}^{\hat{n}}) \right).$$

Obviously, any such form vanishes when  $k = N + 1$  since it then contains an exterior product of  $(N + 1)$  1-forms  $e^i$  belonging to an  $N$ -dimensional subspace. On the other hand, all these forms are linearly independent when  $k = N$  since  $\omega^\alpha$  then contains the exterior product  $e^1 \wedge \dots \wedge e^N$  multiplied by the exterior product of  $\alpha_1$  1-forms of type  $e_i^1$  with  $\dots$  with  $\alpha_{\hat{n}}$  1-forms of type  $e_i^{\hat{n}}$ ; thus  $\omega^\alpha$  and  $\omega^\beta$  belong to different subspaces of  $\bigwedge^{2N} V^*$  whenever  $\alpha \neq \beta$ .  $\square$

The converse statement, as we shall see shortly, is remote from being true. In fact, if it were true, then if  $\hat{n} \geq 2$ , it should be possible to construct the poly-lagrangian subspace as the sum of the kernels of the projected forms, according to Theorem 2.1. Therefore, it should be possible to show that the subspace defined as the sum of these kernels is isotropic. And indeed, as a partial result in this direction, we have the following

**Proposition A.2.** *Let  $V$  and  $\hat{T}$  be finite-dimensional vector spaces and let  $\hat{\omega}$  be a  $\hat{T}$ -valued 2-form of uniform rank  $N$  on  $V$ . Then for any  $\hat{t}_1^*, \hat{t}_2^* \in \hat{T}^* \setminus \{0\}$ , the kernel of  $\omega_{\hat{t}_1^*}$  is isotropic with respect to  $\omega_{\hat{t}_2^*}$ .*

**Proof.** Given  $u, v \in \ker \omega_{\hat{t}_1^*}$ , we have

$$i_u \omega_{\hat{t}_1^*}^N = N i_u \omega_{\hat{t}_1^*} \wedge \omega_{\hat{t}_1^*}^{N-1} = 0, \quad i_v \omega_{\hat{t}_1^*}^N = N i_v \omega_{\hat{t}_1^*} \wedge \omega_{\hat{t}_1^*}^{N-1} = 0,$$

and therefore

$$\omega_{\hat{t}_2^*}(u, v) \omega_{\hat{t}_1^*}^N = i_u i_v (\omega_{\hat{t}_2^*} \wedge \omega_{\hat{t}_1^*}^N) = 0.$$

Using that  $\omega_{\hat{t}_1^*}^N \neq 0$ , it follows that  $\omega_{\hat{t}_2^*}(u, v) = 0$ .  $\square$

However, isotropy of the subspace defined as the sum of the kernels of all the projected forms, which is equivalent to the (stronger) condition that for any  $\hat{t}_1^*, \hat{t}_2^*, \hat{t}_3^* \in \hat{T}^* \setminus \{0\}$ ,  $\ker \omega_{\hat{t}_1^*}$  and  $\ker \omega_{\hat{t}_2^*}$  are orthogonal under  $\omega_{\hat{t}_3^*}$ , i.e., that

$$\omega_{\hat{t}_3^*}(u_1, u_2) = 0 \quad \text{for } u_1 \in \ker \omega_{\hat{t}_1^*} \text{ and } u_2 \in \ker \omega_{\hat{t}_2^*},$$

cannot be derived from the condition of uniform rank. A nice counterexample is obtained by choosing  $V$  and  $\hat{T}$  to be the same space, supposing it to be a Lie algebra  $\mathfrak{g}$  and defining  $\hat{\omega}$  to be the commutator in  $\mathfrak{g}$ . Then for  $\hat{t}^* \in \mathfrak{g}^*$ , the kernel  $\ker \omega_{\hat{t}^*}$  and the support  $\text{supp } \omega_{\hat{t}^*}$  of the projected form  $\omega_{\hat{t}^*}$  are the isotropy algebra of  $\hat{t}^*$  and the tangent space to the coadjoint orbit passing through  $\hat{t}^*$ , respectively. There is one and only one semisimple Lie algebra for which  $\hat{\omega}$  has constant rank, since this condition states that all coadjoint orbits except the trivial one,  $\{0\}$ , should have

the same dimension: this is the algebra of type  $A_1$ , that is,  $\mathbb{R}^3$  equipped with the vector product  $\times$ .

**Example A.2** ( $\hat{n} = 3$ ,  $N = 1$ ,  $\dim V = 3$ ,  $\ker \hat{\omega} = \{0\}$ ). Let  $V = \hat{T} = \mathbb{R}^3$  and consider the  $\mathbb{R}^3$ -valued 2-form  $\hat{\omega}$  built from the following three ordinary 2-forms:

$$\omega^1 = dy \wedge dz, \quad \omega^2 = dz \wedge dx, \quad \omega^3 = dx \wedge dy.$$

Then for  $\hat{t}^* = \hat{t}_a^* \hat{e}^a \in (\mathbb{R}^3)^*$ , we have

$$\omega_{\hat{t}^*} = \hat{t}_a^* \omega^a = \hat{t}_1^* dy \wedge dz + \hat{t}_2^* dz \wedge dx + \hat{t}_3^* dx \wedge dy.$$

Obviously,  $\omega^1, \omega^2$  and  $\omega^3$  are linearly independent and hence  $\hat{\omega}$  has uniform rank 1, since there exists no non-zero 4-form on a three-dimensional space. On the other hand, we have

$$\ker \omega_{\hat{t}^*} = \left\langle \hat{t}_1^* \frac{\partial}{\partial x} + \hat{t}_2^* \frac{\partial}{\partial y} + \hat{t}_3^* \frac{\partial}{\partial z} \right\rangle.$$

Therefore, the intersection of the three kernels  $\ker \omega^1$ ,  $\ker \omega^2$  and  $\ker \omega^3$  is  $\{0\}$  (i.e.,  $\hat{\omega}$  is non-degenerate). However,  $\ker \omega^1$  and  $\ker \omega^2$  are orthogonal under  $\omega^1$  and under  $\omega^2$  but not under  $\omega^3$ . Now if there existed a polylagrangian subspace it would have to coincide with the sum of the kernels of all the projected forms, but that is the whole space  $\mathbb{R}^3$ , which is not isotropic. Thus  $\hat{\omega}$  does not admit a polylagrangian subspace.

Finally, we observe that even if the sum of the kernels of all the projected forms is an isotropic subspace with respect to  $\hat{\omega}$ , it may still fail to be a polylagrangian subspace, as shown by the following

**Example A.3** ( $\hat{n} = 2$ ,  $N = 2$ ,  $\dim V = 5$ ,  $\ker \hat{\omega} = \{0\}$ ). Let  $V = \mathbb{R}^5$ ,  $\hat{T} = \mathbb{R}^2$  and consider the  $\mathbb{R}^2$ -valued 2-form  $\hat{\omega}$  built from the following two ordinary 2-forms:

$$\omega^1 = dx^1 \wedge dx^4 + dx^2 \wedge dx^3, \quad \omega^2 = dx^1 \wedge dx^3 - dx^2 \wedge dx^5.$$

Then for  $\hat{t}^* = \hat{t}_a^* \hat{e}^a \in (\mathbb{R}^2)^*$ , we have

$$\omega_{\hat{t}^*} = \hat{t}_a^* \omega^a = dx^1 \wedge (\hat{t}_1^* dx^4 + \hat{t}_2^* dx^3) + dx^2 \wedge (\hat{t}_1^* dx^3 - \hat{t}_2^* dx^5).$$

Obviously,  $\omega^1, \omega^2$  and the forms

$$\begin{aligned} (\omega^1)^2 &\equiv \omega^1 \wedge \omega^1 = 2 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4, \\ \omega^1 \wedge \omega^2 &= dx^1 \wedge dx^2 \wedge dx^4 \wedge dx^5, \\ (\omega^2)^2 &\equiv \omega^2 \wedge \omega^2 = 2 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^5, \end{aligned}$$

are linearly independent and hence  $\hat{\omega}$  has uniform rank 2, since there exists no non-zero 6-form on a five-dimensional space. On the other hand, we have

$$\ker \omega_{\hat{t}^*} = \left\langle \hat{t}_1^* \hat{t}_2^* \frac{\partial}{\partial x^3} - (\hat{t}_2^*)^2 \frac{\partial}{\partial x^4} + (\hat{t}_1^*)^2 \frac{\partial}{\partial x^5} \right\rangle.$$

The intersection of the two kernels  $\ker \omega^1$  and  $\ker \omega^2$  is  $\{0\}$  (i.e.,  $\hat{\omega}$  is non-degenerate). Note that their (direct) sum is the two-dimensional subspace of  $V$ ,

say  $L'$ , spanned by  $\partial/\partial x^4$  and  $\partial/\partial x^5$ , whereas the subspace spanned by all the kernels  $\ker \omega_{\hat{t}^*}$  ( $\hat{t}^* \in \hat{T}^* \setminus \{0\}$ ) is the three-dimensional subspace of  $V$ , say  $L''$ , spanned by  $\partial/\partial x^i$  with  $i = 3, 4, 5$ , and this is isotropic with respect to all the forms  $\omega_{\hat{t}^*}$  ( $\hat{t}^* \in \hat{T}^* \setminus \{0\}$ ). More than that: since its codimension is 2, it is maximal isotropic with respect to all the forms  $\omega_{\hat{t}^*}$  ( $\hat{t}^* \in \hat{T}^* \setminus \{0\}$ ). Now if there existed a polylagrangian subspace it would have to coincide with  $L'$  and also with  $L''$ , but these two are not equal and do not have the right dimension, which according to Eq. (2.12) would have to be 4: both of them are too small. Thus  $\hat{\omega}$  does not admit a polylagrangian subspace.

To summarize, the examples given above show that the hypothesis of existence of a polylagrangian subspace is highly non-trivial and very restrictive: as it seems, it cannot be replaced by any other hypothesis that is not obviously equivalent. The examples also show the great variety of possibilities for the “relative positions” of the kernels of the various projected forms that prevails when such a subspace does not exist. In this sense, the definition adopted in [23] is quite inconvenient, since it makes no reference to this subspace, thus hiding the central aspect of the theory.

To conclude, we want to add some remarks about the relation between the polylagrangian subspace, when it exists, and the more general class of maximal isotropic subspaces. First, we emphasize that in contrast with a polylagrangian subspace, maximal isotropic subspaces always exist. To prove this, it suffices to start out from an arbitrary one-dimensional subspace  $L_1$ , which is automatically isotropic, and construct a chain  $L_1 \subset L_2 \subset \dots$  of subspaces where  $L_{p+1}$  is defined as the direct sum of  $L_p$  and the one-dimensional subspace spanned by some non-zero vector in its 1-orthogonal complement  $L_p^{\hat{\omega},1}$ . For dimensional reasons, this process must stop at some point, which means that at this point we have succeeded in constructing a maximal isotropic subspace. However, nothing guarantees that maximal isotropic subspaces resulting from different chains must have the same dimension, nor that there must exist some chain leading to a maximal isotropic subspace of sufficiently high dimension to be polylagrangian: this happens only in the special case of ordinary forms ( $\hat{n} = 1$ ), where all maximal isotropic subspaces have the same dimension and where the notions of a polylagrangian subspace (or simply lagrangian subspace, in this case) and of a maximal isotropic subspace coincide.

Another important point concerns the relation between the notions of isotropic subspace and maximal isotropic subspace with respect to the form  $\hat{\omega}$  and with respect to its projections. First, it is obvious that a subspace of  $V$  is isotropic with respect to  $\hat{\omega}$  if and only if it is isotropic with respect to each of the projected forms  $\omega_{\hat{t}^*}$  ( $\hat{t}^* \in T^* \setminus \{0\}$ ) or  $\omega^a$  ( $1 \leq a \leq \hat{n}$ ). However, this no longer holds when we substitute the term “isotropic” by the term “maximal isotropic”: a subspace of  $V$  that is maximal isotropic with respect to each of the projections of  $\hat{\omega}$  is also maximal isotropic with respect to  $\hat{\omega}$ , but conversely, it can very well be maximal isotropic with respect to  $\hat{\omega}$ , and will then be isotropic with respect to each of the projections of  $\hat{\omega}$ , but even so may fail to be maximal isotropic with respect to some of them.

And finally, a polylagrangian subspace of  $V$  is maximal isotropic with respect to each of the projections of  $\hat{\omega}$  (this follows from Theorem 2.1), but as we have seen in the last example above, the converse is false: a subspace can be maximal isotropic with respect to each of the projections of  $\hat{\omega}$  without being polylagrangian. All these statements illustrate the special nature of the polylagrangian subspace, already in the case of vector-valued 2-forms.

### Appendix B. Poincaré Lemma

In this appendix we want to state the Poincaré lemma in the form in which it is used in the proof of the Darboux theorem in Sec. 7.

**Theorem B.1.** *Let  $\omega \in \Omega^k(P, \hat{T})$  be a closed form on a manifold  $P$  taking values in a fixed vector space  $\hat{T}$  and let  $L$  be an involutive distribution on  $P$ . Suppose that  $\omega$  is  $(k - r)$ -horizontal (with respect to  $L$ ), i.e., such that for any  $p \in P$  and all  $v_1, \dots, v_{r+1} \in L_p$ , we have*

$$i_{v_1} \dots i_{v_{r+1}} \omega_p = 0.$$

*Then for any point of  $P$  there exist an open neighborhood  $U$  of that point and a  $(k - 1)$ -form  $\theta \in \Omega^{k-1}(U, \hat{T})$  on  $U$  which is also  $(k - r)$ -horizontal (with respect to  $L$ ), i.e., such that for any  $p \in U$  and all  $v_1, \dots, v_r \in L_p$ , we have*

$$i_{v_1} \dots i_{v_r} \theta_p = 0,$$

*and such that  $\omega = d\theta$  on  $U$ .*

**Proof.** Due to the local character of this theorem and since the subbundle  $L$  of  $TP$  is involutive, we can without loss of generality work in a local chart of the manifold  $P$  around the chosen reference point in which the foliation defined by  $L$  is “straightened out”, so we may assume that  $P \cong K_0 \oplus L_0$  with  $L \cong L_0$  with fixed subspaces  $K_0$  and  $L_0$  and such that the aforementioned reference point corresponds to the origin. (In what follows, we shall omit the index 0.) We also suppose that  $\hat{T} = \mathbb{R}$ , since we may prove the theorem separately for each component of  $\omega$  and  $\theta$ , with respect to some fixed basis of  $\hat{T}$ .

Now for  $t \in \mathbb{R}$ , define the “ $K$ -contraction”  $F_t^K : P \rightarrow P$  and “ $L$ -contraction”  $F_t^L : P \rightarrow P$  by  $F_t^K(x, y) = (tx, y)$  and  $F_t^L(x, y) = (x, ty)$ ; obviously,  $F_t^K$  and  $F_t^L$  are diffeomorphisms if  $t \neq 0$  and are projections if  $t = 0$ . Associated with each of these families of mappings there is a time dependent vector field which generates it in the sense that, for  $t \neq 0$ ,

$$X_t^K(F_t^K(x, y)) = \left. \frac{d}{ds} F_s^K(x, y) \right|_{s=t} \quad \text{and} \quad X_t^L(F_t^L(x, y)) = \left. \frac{d}{ds} F_s^L(x, y) \right|_{s=t}.$$

Explicitly, for  $t \neq 0$ ,

$$X_t^K(x, y) = t^{-1}(x, 0) \quad \text{and} \quad X_t^L(x, y) = t^{-1}(0, y).$$

Define  $\omega_0 = (F_0^L)^*\omega$  and, for  $\epsilon > 0$ ,

$$\theta_\epsilon = \int_\epsilon^1 dt \left( (F_t^L)^*(i_{X_t^L}\omega) + (F_t^K)^*(i_{X_t^K}\omega_0) \right),$$

as well as

$$\theta = \lim_{\epsilon \rightarrow 0} \theta_\epsilon = \int_0^1 dt \left( (F_t^L)^*(i_{X_t^L}\omega) + (F_t^K)^*(i_{X_t^K}\omega_0) \right).$$

To see that the  $(k-1)$ -forms  $\theta$  and  $\theta_\epsilon$  are well defined, consider  $k-1$  vectors  $(u_i, v_i) \in K \oplus L$  ( $1 \leq i \leq k-1$ ) and observe that, for  $t \neq 0$ ,

$$\begin{aligned} & (F_t^L)^*(i_{X_t^L}\omega)_{(x,y)}((u_1, v_1), \dots, (u_{k-1}, v_{k-1})) \\ &= \omega_{(x,ty)}((0, y), (u_1, tv_1), \dots, (u_{k-1}, tv_{k-1})), \end{aligned}$$

and

$$\begin{aligned} & (F_t^K)^*(i_{X_t^K}\omega_0)_{(x,y)}((u_1, v_1), \dots, (u_{k-1}, v_{k-1})) \\ &= t^{k-1} \omega_{(tx,0)}((x, 0), (u_1, 0), \dots, (u_{k-1}, 0)). \end{aligned}$$

Here we see easily that both expressions are differentiable in  $t$  and provide  $(k-1)$ -forms which are  $(k-r)$ -horizontal and  $(k-1)$ -horizontal with respect to  $L$ , respectively. Thus,  $\theta_\epsilon$  and  $\theta$  are  $(k-1)$ -forms which are  $(k-r)$ -horizontal with respect to  $L$ . Moreover, since  $d\omega = 0$  and  $d\omega_0 = 0$ ,

$$\begin{aligned} d\theta_\epsilon &= \int_\epsilon^1 dt \left( (F_t^L)^*(i_{X_t^L}\omega) + (F_t^K)^*(i_{X_t^K}\omega_0) \right) \\ &= \int_\epsilon^1 dt \left( (F_t^L)^*(L_{X_t^L}\omega) + (F_t^K)^*(L_{X_t^K}\omega_0) \right) \\ &= \int_\epsilon^1 dt \left( \frac{d}{dt}((F_t^L)^*\omega) + \frac{d}{dt}((F_t^K)^*\omega_0) \right) \\ &= \omega - (F_\epsilon^L)^*\omega + \omega_0 - (F_\epsilon^K)^*\omega_0. \end{aligned}$$

Taking the limit  $\epsilon \rightarrow 0$ , we get  $(F_\epsilon^L)^*\omega \rightarrow \omega_0$  and  $(F_\epsilon^K)^*\omega_0 \rightarrow 0$  and hence  $d\theta = \omega$ .  $\square$

## References

- [1] Th. De Donder, *Théorie Invariante du Calcul des Variations* (Gauthier-Villars, Paris 1935).
- [2] H. Weyl, Geodesic fields in the calculus of variations for multiple integrals, *Ann. Math.* **36** (1935) 607-629.
- [3] M. J. Gotay, J. Isenberg, J. E. Marsden and R. Montgomery, Momentum maps and classical relativistic fields. Part I: Covariant field theory, arXiv: physics/9801019v2.
- [4] J. F. Cariñena, M. Crampin and L. A. Ibort, On the multisymplectic formalism for first order field theories, *Differential Geom. Appl.* **1** (1991) 345-374.
- [5] M. Forger and H. Römer, A Poisson bracket on multisymplectic phase space, *Rep. Math. Phys.* **48** (2001) 211-218.

- [6] M. Forger, C. Paufler and H. Römer, The Poisson bracket for Poisson forms in multisymplectic field theory, *Rev. Math. Phys.* **15** (2003) 705-743.
- [7] M. Forger, C. Paufler and H. Römer, Hamiltonian multivector fields and Poisson forms in multisymplectic field theory, *J. Math. Phys.* **46** (2005) 112903, 29pp.
- [8] G. Martin, A Darboux theorem for multisymplectic manifolds, *Lett. Math. Phys.* **16** (1988) 133-138.
- [9] F. Cantrijn, L. A. Ibort and M. de León, On the geometry of multisymplectic manifolds, *J. Austral. Math. Soc. (Ser. A)* **66** (1999) 303-330.
- [10] M. de León, D. M. de Diego and A. Santamaría-Merino, Tulczyjew's triples and Lagrangian submanifolds in classical field theories, in *Applied Differential Geometry and Mechanics* (Academic Press, 2003), pp. 21-47; arXiv: math-ph/0302002v1.
- [11] J. Kijowski, A finite-dimensional canonical formalism in the classical field theory, *Comm. Math. Phys.* **30** (1973) 99-128.
- [12] J. Kijowski, Multiphase spaces and gauge in calculus of variations, *Bull. Acad. Pol. Sci. SMAP* **22** (1974) 1219-1225.
- [13] J. Kijowski and W. Szczyrba, Multisymplectic manifolds and the geometrical construction of the Poisson brackets in the classical field theory, in *Géométrie Symplectique et Physique Mathématique* (C.N.R.S., Paris, 1975), pp. 347-379.
- [14] J. Kijowski and W. Szczyrba, Canonical structure for classical field theories, *Comm. Math. Phys.* **46** (1976) 183-206.
- [15] C. Günther, The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: The local case, *J. Diff. Geom.* **25** (1987) 23-53.
- [16] A. Awane,  $k$ -symplectic structures, *J. Math. Phys.* **33** (1992) 4046-4052.
- [17] L. G. Gomes, Polysymplectic and multisymplectic structures on manifolds and fiber bundles, PhD thesis, Institute for Mathematics and Statistics, University of São Paulo (2007); <http://www.ime.usp.br/~forger/pdffiles/teselgg.pdf>.
- [18] W. H. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology*, Vol. 2 (Academic Press, 1973).
- [19] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 1 (Interscience, 1963).
- [20] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, 2nd edn. (Benjamin-Cummings, 1978).
- [21] R. C. Gunning, *Lectures on Vector Bundles over Riemann Surfaces* (Princeton University Press, 1967).
- [22] M. Forger and S. V. Romero, Covariant Poisson brackets in geometric field theory, *Comm. Math. Phys.* **256** (2005) 375-410.
- [23] M. Grassi, Polysymplectic spaces,  $s$ -Kähler manifolds and Lagrangian fibrations, arXiv: math.DG/0006154v1.
- [24] J. Martinet, Sur les singularités des formes différentielles, *Ann. Inst. Four.* **20** (1970) 95-178.
- [25] S. Z. Yepes, Distribuições Lagrangianas e conexões em geometria simplética generalizada, PhD thesis, Institute for Mathematics and Statistics, University of São Paulo (2011).
- [26] M. Forger and S. Z. Yepes, Lagrangian distributions and connections in multisymplectic and polysymplectic geometry, to appear in *Differential Geom. Appl.*, arXiv:1202.5054.