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THE POISSON BRACKET FOR POISSON FORMS IN MULTISYMPLECTIC FIELD THEORY

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We present a general definition of the Poisson bracket between differential forms on the extended multiphase space appearing in the geometric formulation of first order classical field theories and, more generally, on exact multisymplectic manifolds. It is well defined for a certain class of differential forms that we propose to call Poisson forms and turns the space of Poisson forms into a Lie superalgebra.

Keywords: Geometric field theory; multisymplectic geometry; Poisson brackets.

1. Introduction

The multiphase space approach to classical field theory, whose origins can be traced back to the early work of Hermann Weyl on the calculus of variations, has recently undergone a rapid development, but a number of conceptual questions is still open.

The basic idea behind all attempts to extend the covariant formulation of classical field theory from the Lagrangian to the Hamiltonian domain is to treat spatial derivatives on the same footing as time derivatives. This requires associating to each field component φ^i not just its standard canonically conjugate momentum π_i but rather n conjugate momenta π_i^μ, where n is the dimension of space-time. If one starts out from a Lagrangian $\mathcal{L}$ depending on the field and its first partial

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derivatives, these are obtained by the covariant Legendre transformation

$$\pi_i^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^i} \,.$$

This allows one to rewrite the standard Euler-Lagrange equations of field theory,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^i} - \frac{\partial \mathcal{L}}{\partial \varphi^i} = 0$$

as a covariant first order system, the covariant Hamiltonian equations or De Donder-Weyl equations

$$\frac{\partial \mathcal{H}}{\partial \pi_i^\mu} = \partial_\mu \varphi^i \,, \quad \frac{\partial \mathcal{H}}{\partial \varphi^i} = -\partial_\mu \pi_i^\mu$$

where

$$\mathcal{H} = \pi_i^\mu \partial_\mu \varphi^i - \mathcal{L}$$

is the covariant Hamiltonian density or De Donder-Weyl Hamiltonian.

Multiphase space (ordinary as well as extended) is the geometric environment built by appropriately patching together local coordinate systems of the form (q^i, p_i^μ) — instead of the canonically conjugate variables (q^i, p_i) of mechanics — together with space-time coordinates x^μ and, in the extended version, a further energy type variable that we shall denote by p (without any index). In the recent literature on the subject, special attention has been devoted to the so-called multisymplectic form ω which is, except for a sign, the exterior derivative of another form θ that we propose to call the multicanonical form: both are naturally defined on extended multiphase space and are the geometric objects replacing, respectively, the symplectic form $\omega = dq^i \wedge dp_i$ and the canonical form $\theta = p_i dq^i$ of Hamiltonian mechanics (on cotangent bundles), or more precisely, the symplectic form $\omega = dq^i \wedge dp_i + dt \wedge dE$ and the canonical form $\theta = p_i dq^i + E\, dt$ of Hamiltonian mechanics (on cotangent bundles) for non-autonomous systems. Additional motivation and precise definitions will be given in the next section, and a table confronting the most relevant concepts of the field theoretical formalism with their counterparts in Hamiltonian mechanics can be found at the end of the paper.

The advantage of such an approach as compared to the orthodox strategy of treating field theoretical models as infinite-dimensional dynamical systems is threefold. First, general covariance (and in particular, Lorentz covariance) is trivially achieved. Second, by working on multiphase space which is a finite-dimensional manifold, one automatically avoids all the functional analytic complications that plague the orthodox method. Third, space-time locality is also automatically guaranteed, since one works with the field variables and their first derivatives or conjugates of these at single points of space-time, rather than with fields defined over entire hypersurfaces: integration is deferred to the very last step of every procedure. Of course, there is also a price to be paid for all these benefits, namely that the obvious duality of classical mechanics between coordinates and momenta is lost.

As a result, there is no evident multiphase space quantization procedure. What seems to be needed is a new and more sophisticated concept of "multi-duality" to replace the standard duality underlying the canonical commutation relations.

Certainly, an important step towards a better understanding of what might be the nature of this "multi-duality" and that of a multiphase space quantization procedure is the construction of Poisson brackets within this formalism. After all, the Poisson bracket should be the classical limit of the commutator of quantum theory. Surprisingly, this is to a large extent still an open problem. Our approach to the question has been motivated by the work of Kanatchikov [1, 2], who seems to have been the first to propose a Poisson bracket between differential forms of arbitrary degree in multimomentum variables and to analyze the restrictions that must be imposed on these forms in order to make this bracket well-defined: he uses the term "Hamiltonian form" in this context, although the concept as such is of course much older. It must be pointed out, however, that Kanatchikov's approach is essentially local and makes extensive use of features that have no invariant geometric meaning, such as a systematic splitting into horizontal and vertical parts; moreover, his definition of Hamiltonian forms is too restrictive. We avoid all these problems by working exclusively within the multisymplectic framework and on the extended multiphase space, instead of the ordinary one: this leads naturally to a definition of the concept of a Poisson form which is more general than Kanatchikov's notion of a Hamiltonian form, as well as to a coordinate-independent definition of the Poisson bracket between any two such forms. In fact, most of the concepts involved do not even depend on the explicit construction of extended multiphase space but only on its structure as an exact multisymplectic manifold, and we shall make use of this fact in order to simplify the treatment whenever possible.

The paper is organized as follows. In Sec. 2, we give a brief review of some salient features of the multiphase space approach to the geometric formulation of first order classical field theories, following Ref. [3] and, in particular, Ref. [4], to which the reader is referred for more details and for the discussion of many relevant examples; this material is included here mainly in order to fix notation and make our presentation reasonably self-contained. The main point is to show that the extended multiphase space of field theory does carry the structure of an exact multisymplectic manifold (in fact it seems to be the only known example of a multisymplectic manifold). In Sec. 3, we introduce the concept of a Poisson form on a general multisymplectic manifold, specify the notion of an exact multisymplectic manifold, define the Poisson bracket between Poisson forms on exact multi symplectic manifolds and prove our main theorem, which states that this bracket satisfies the usual axioms of a Lie superalgebra. The construction generalizes the corresponding one for Hamiltonian $(n-1)$-forms on the extended multiphase space of field theory given by two of the present authors in a previous paper [5]: the idea is to modify the standard formula that had been adopted for decades [6–11], even though it fails to satisfy the Jacobi identity, by adding a judiciously chosen exact form that turns out to cure the defect. Here, we show that the same trick works for forms of arbitrary degree, provided one introduces appropriate sign factors. In both cases, it is the structure of the correction term that requires the underlying manifold to be exact multisymplectic and not just multisymplectic. In Sec. 4, we define the notion of an exact Hamiltonian multivector field on an exact multisymplectic manifold and show that by contraction with the multicanonical form θ, any such multivector field gives rise to a Poisson form; moreover, this simple prescription yields an antiSchouten bracket of multivector fields and the Poisson bracket of Poisson forms introduced here). It can be viewed as an extension, from vector fields to multivector fields, of the universal part of the covariant momentum map [4], which is the geometric version of the construction of Noether currents and the energy-momentum tensor in field theory, and we shall therefore refer to it as the universal multimomentum map. In Sec. 5, we return to the case of extended multiphase space and discuss other examples for the construction of Poisson forms. More specifically, we show that arbitrary functions are Poisson forms (of degree 0) and find that Kanatchikov's Hamiltonian forms, when pulled back from ordinary to extended multiphase space by means of the appropriate projection, constitute a special class of Poisson forms. The complete determination of the space of Poisson forms of arbitrary degree > 0 on extended multiphase space, together with that of exact Hamiltonian and locally Hamiltonian multivector fields of arbitrary degree $< n$, is a technically demanding problem whose solution will be presented elsewhere [12]. The paper concludes with two appendices: the first presents a number of important formulas from the multivector calculus on manifolds, related to the definition and main properties of the Schouten bracket and the Lie derivative of differential forms along multivector fields, while the second shows how, given a connection in a fiber bundle, one can construct induced connections in various other fiber bundles derived from it, including the multiphase spaces of geometric field theory; this possibility is important for the comparison of the multisymplectic formalism with other approaches that have been proposed in the literature and to a certain extent depend on the *a priori* choice of a connection. Recently, the problem of constructing Poisson brackets has also been addressed in the context of other formalisms such as the one based on n-symplectic manifolds [13] (see [14] for a recent overview) or that of Lepage-Dedecker which is more general than that of De Donder-Weyl [15].

Finally, we would like to point out that there exists another construction of a covariant Poisson bracket in classical field theory, based on the same functional approach that underlies the construction of "covariant phase space" of Crnkovic-Witten [16, 17] and Zuckerman [18]. This bracket, originally due to Peierls [19] and further elaborated by de Witt [20, 21] (see also [22] for a recent exposition), has been adapted to the multiphase space approach by Romero [23] and shown to be precisely the Poisson bracket associated with the symplectic form on covariant phase space introduced in Refs. [16, 17] and [18]; these results will be presented elsewhere [24]. It would be interesting to identify the relation between that bracket and the one introduced here; this question is presently under investigation.

2. Multiphase Spaces in Geometric Field Theory

The starting point for the geometric formulation of classical field theory is the choice of a *configuration bundle*, which in general will be a fiber bundle over space-time whose sections are the fields of the theory under consideration. In what follows, we shall denote its total space by E, its base space by M, its typical fiber by Q and the projection from E to M by π; the dimensions are

$$\dim M = n, \quad \dim Q = N, \quad \dim E = n + N\,. \tag{2.1}$$

In field theoretical models, M is interpreted as *space-time* whereas Q is the *configuration space* of the theory — a manifold whose (local) coordinates describe internal degrees of freedom.[a] The total space E is locally but not necessarily globally isomorphic to the Cartesian product $M \times Q$, but it must be stressed that even when the configuration bundle is globally trivial, there will in general not exist any preferred trivialization, and it is precisely the freedom to change trivialization that allows one to incorporate gauge theories into the picture. Another point that deserves to be emphasized is that the configuration bundle does not in general carry any additional structures: these only appear when one focusses on special classes of field theories.

- *Vector bundles* arise naturally in theories with *linear matter fields* and also in general relativity: the metric tensor is an example.
- *Affine bundles* can be employed to incorporate *gauge fields*, since connections in a principal G-bundle P over space-time M can be viewed as sections of the *connection bundle* of P — an affine bundle CP over M constructed from P.
- *General fiber bundles* are used to handle *nonlinear matter fields*, in particular those corresponding to maps from space-time M to some target manifold Q: a standard example are the nonlinear sigma models.

In order to cover this variety of situations, the general constructions on which the geometric formulation of classical field theory is based must not depend on the choice of any additional structure on the configuration bundle. This requirement is naturally satisfied in the multiphase space formalism — in contrast to the majority of similar approaches that have over the last few decades found their way into the literature: most of these depend on the *a priori* choice of a connection in the configuration bundle, thus excluding gauge theories in which connections must be treated as dynamical variables and not as fixed background fields.

The multiphase space approach to first order classical field theory follows the same general pattern as the standard formalism of classical mechanics on the tangent and cotangent bundle of a configuration space Q [25, 26].[b] However, the

[a]This interpretation is turned around in the theory of strings and membranes.

[b]The term "first order" refers to the fact that the Lagrangian is supposed to be a pointwise defined function of the coordinates or fields and of their derivatives or partial derivatives of no more than first order; higher order derivatives should be eliminated, e.g., by introducing appropriate auxiliary variables.

correspondence between the objects and concepts underlying the geometric formulation of mechanics and that of field theory becomes fully apparent only when one reformulates mechanics so as to incorporate the time dimension. (This is standard practice, e.g., in the study of non-autonomous systems, that is, mechanical systems whose Lagrangian/Hamiltonian depends explicitly on time, such as systems of particles in time-dependent external fields. Additional motivation is provided by relativistic mechanics where Newton's concept of absolute time is abandoned and hence there is no place for an extraneous, absolute time variable that can be kept entirely separate from the arena where the dynamical phenomena take place.) In its simplest version, this reformulation amounts to replacing the configuration space Q by the extended configuration space $\mathbb{R} \times Q$ and the velocity phase space TQ (the tangent bundle of Q) by the extended velocity phase space $\mathbb{R} \times TQ$, where $\mathbb{R}$ stands for the time axis. The usual momentum phase space T^*Q (the cotangent bundle of Q) admits two different extensions: the simply extended phase space $\mathbb{R} \times T^*Q$, where $\mathbb{R}$ represents the time variable, and the doubly extended phase space $\mathbb{R} \times T^*Q \times \mathbb{R}$, where the first copy of $\mathbb{R}$ represents the time variable whereas the second copy of $\mathbb{R}$ represents an energy variable. This second extension is required if one wants to maintain a symplectic structure, rather than just a contact structure, for extended phase space, since energy is the physical quantity canonically conjugate to time. A further generalization appears when one considers mechanical systems in external gauge fields, since time-dependent gauge transformations do not respect the direct product structure of the extended configuration and phase spaces mentioned above. What does remain invariant under such transformations are certain projections, namely the projection from the extended configuration space onto the time axis, the projections from the various extended phase spaces onto extended configuration space and, finally, the projection from the doubly extended to the simply extended phase space which amounts to "forgetting the additional energy variable".

In passing to field theory, we must replace the time axis $\mathbb{R}$ by the space-time manifold M, the extended configuration space $\mathbb{R} \times Q$ by the configuration bundle E over M introduced above and the extended velocity phase space $\mathbb{R} \times TQ$ by the *jet bundle* JE of E.[c] It is well known that JE is — unlike the tangent bundle of a manifold — in general only an affine bundle over E (of fiber dimension Nn) and not a vector bundle; the corresponding difference vector bundle over E (also of fiber dimension Nn) will be called the *linearized jet bundle* of E and be denoted by $\vec{J}E$. This leads to the possibility of forming two kinds of dual: the linear dual of $\vec{J}E$, denoted here by $\vec{J}^*E$, and the affine dual of JE, denoted here by $J^\star E$; both of them are vector bundles over E (of fiber dimension Nn and $Nn + 1$, respectively). Even more important are their twisted versions, obtained by taking the tensor product with the line bundle of volume forms on M, pulled back to E via π: this gives rise to the twisted linear dual of $\vec{J}E$, called *ordinary multiphase space* and

[c]We consider only first order jet bundles and therefore omit the index "1" used by many authors.

denoted here by $\vec{J}^{\circledast}E$, and the twisted affine dual of JE, called *extended multiphase space* and denoted here by $J^{\circledast}E$; both of them, once again, are vector bundles over E (of fiber dimension Nn and $Nn+1$, respectively). The former replaces the simply extended phase space $\mathbb{R} \times T^*Q$ of mechanics whereas the latter replaces the doubly extended phase space $\mathbb{R} \times T^*Q \times \mathbb{R}$ of mechanics. Moreover, in both cases (twisted or untwisted), there is a natural projection η that, as in mechanics, can be interpreted as "forgetting the additional energy variable": it turns $J^{\circledast}E$ into an affine line bundle over $\vec{J}^{\circledast}E$ and, similarly, $J^{\star}E$ into an affine line bundle over $\vec{J}^{*}E$. The most remarkable property of extended multiphase space is that it is an *exact multisymplectic manifold*: it carries a naturally defined *multicanonical form* θ, of degree n, whose exterior derivative is the *multisymplectic form* ω, of degree $n+1$, replacing the canonical form θ and the symplectic form ω, respectively, on the doubly extended phase space $\mathbb{R} \times T^*Q \times \mathbb{R}$ of mechanics.

The global construction of the first order jet bundle JE and the linearized first order jet bundle $\vec{J}E$ associated with a given fiber bundle E over a manifold M, as well as that of the various duals mentioned above, is quite easy to understand. (Higher order jet bundles are somewhat harder to deal with, but we won't need them in this paper.) Given a point e in E with base point $x = \pi(e)$ in M, the fiber J_eE of JE at e consists of all linear maps from the tangent space T_xM of the base space M at x to the tangent space T_eE of the total space E at e whose composition with the tangent map $T_e\pi : T_eE \to T_xM$ to the projection $\pi : E \to M$ gives the identity on T_xM:

$$J_eE = \{u_e \in L(T_xM, T_eE) / T_e\pi \circ u_e = \mathrm{id}_{T_xM}\}\,. \tag{2.2}$$

Thus the elements of J_eE are precisely the candidates for the tangent maps at x to (local) sections φ of the bundle E satisfying $\varphi(x) = e$. Obviously, J_eE is an affine subspace of the vector space $L(T_xM, T_eE)$ of all linear maps from T_xM to the tangent space T_eE, the corresponding difference vector space being the vector space of all linear maps from T_xM to the vertical subspace V_eE:

$$\vec{J}_eE = L(T_xM, V_eE)\,. \tag{2.3}$$

The jet bundle JE thus defined admits two different projections, namely the *target projection* $\tau_{JE} : JE \to E$ and the *source projection* $\sigma_{JE} : JE \to M$ which is simply its composition with the original projection π, that is, $\sigma_{JE} = \pi \circ \tau_{JE}$. It is easily shown that JE is a fiber bundle over M with respect to σ_{JE}, in general without any additional structure, but it is an affine bundle over E with respect to τ_{JE}, the corresponding difference vector bundle being the vector bundle over E of linear maps from the pull-back of the tangent bundle of the base space by the projection π to the vertical bundle of E:

$$\vec{J}E = L(\pi^*TM, VE)\,. \tag{2.4}$$

The affine structure of the jet bundle JE over E, as well as the linear structure of the linearized jet bundle $\vec{J}E$ over E, can also be read off directly from local coordinate expressions. Namely, choosing local coordinates x^μ for M, local coordinates q^i for Q and a local trivialization of E induces naturally a local coordinate system (x^μ, q^i, q^i_μ) for JE, as well as a local coordinate system $(x^\mu, q^i, \vec{q}^{\,i}_\mu)$ for $\vec{J}E$: such coordinates will simply be referred to as *adapted local coordinates*. Moreover, a transformation to new local coordinates x'^κ for M, new local coordinates q'^k for Q and a new local trivialization of E, according to

$$x'^\kappa = x'^\kappa(x^\mu)\,, \quad q'^k = q'^k(x^\mu, q^i) \tag{2.5}$$

induces naturally a transformation to new adapted local coordinates $(x'^\kappa, q'^k, q'^k_\kappa)$ for JE and $(x'^\kappa, q'^k, \vec{q}'^k_\kappa)$ for $\vec{J}E$ given by Eq. (2.5) and

$$q'^k_\kappa = q'^k_\kappa(x^\mu, q^i, q^i_\mu)\,, \quad \vec{q}'^k_\kappa = \vec{q}'^k_\kappa(x^\mu, q^i, \vec{q}^{\,i}_\mu)\,, \tag{2.6}$$

where

$$q'^k_\kappa = \frac{\partial x^\mu}{\partial x'^\kappa}\,\frac{\partial q'^k}{\partial q^i}\,q^i_\mu + \frac{\partial x^\mu}{\partial x'^\kappa}\,\frac{\partial q'^k}{\partial x^\mu}\,, \quad \vec{q}'^k_\kappa = \frac{\partial x^\mu}{\partial x'^\kappa}\,\frac{\partial q'^k}{\partial q^i}\,\vec{q}^{\,i}_\mu\,. \tag{2.7}$$

Before going on, we pause to fix some notation concerning differential forms, for which we shall in terms of local coordinates x^μ use the following conventions:

$$d^n x = dx^1 \wedge \cdots \wedge dx^n\,, \tag{2.8}$$

$$d^n x_\mu = i_{\partial_\mu}\, d^n x = (-1)^{\mu-1}\, dx^1 \wedge \cdots \wedge dx^{\mu-1} \wedge dx^{\mu+1} \wedge \cdots \wedge dx^n\,, \tag{2.9}$$

$$d^n x_{\mu\nu} = i_{\partial_\nu} i_{\partial_\mu}\, d^n x \;\ldots\; d^n x_{\mu_1 \ldots \mu_r} = i_{\partial_{\mu_r}} \ldots i_{\partial_{\mu_1}} d^n x\,. \tag{2.10}$$

Then

$$i_{\partial_\mu}\, d^n x_{\mu_1 \ldots \mu_r} = d^n x_{\mu_1 \ldots \mu_r \mu}\,, \tag{2.11}$$

whereas

$$dx^\kappa \wedge d^n x_\mu = \delta^\kappa_\mu\, d^n x\,, \tag{2.12}$$

$$dx^\kappa \wedge d^n x_{\mu\nu} = \delta^\kappa_\nu d^n x_\mu - \delta^\kappa_\mu\, d^n x_\nu\,, \tag{2.13}$$

$$dx^\kappa \wedge d^n x_{\mu_1 \ldots\, \mu_r} = \sum_{p=1}^{r} (-1)^{r-p} \delta^\kappa_{\mu_p}\, d^n x_{\mu_1 \ldots \mu_{p-1} \mu_{p+1} \ldots\, \mu_r}\,. \tag{2.14}$$

Moreover, these (local) forms on M are lifted to (local) forms on E by pull-back with the projection π_E, and later (local) forms on E will be lifted to (local) forms on total spaces of bundles over E by pull-back with the respective projection, without change of notation.

The dual $J^\star E$ of the jet bundle JE and the dual $\vec{J}^* E$ of the linearized jet bundle $\vec{J}E$ are obtained according to the standard rules for defining the dual of an affine space and of a vector space, respectively. In particular, these rules state that if A is an affine space of dimension k over $\mathbb{R}$, its dual $A^\star$ is the space $A(A, \mathbb{R})$ of affine maps from A to $\mathbb{R}$, which is a vector space of dimension $k+1$. Thus the dual or, more precisely, *affine dual* $J^\star E$ of the jet bundle JE and the dual or, more precisely, *linear dual* $\vec{J}^* E$ of the linearized jet bundle $\vec{J}E$ are obtained by defining their fiber over any point e in E to be the vector space

$$J_e^\star E = \{z_e : J_e E \to \mathbb{R} \text{ affine}\}\,, \tag{2.15}$$

and the vector space

$$\vec{J}_e^* E = \{\vec{z}_e : \vec{J}_e E \to \mathbb{R} \text{ linear}\}\,, \tag{2.16}$$

respectively. However, as mentioned before, the multiphase spaces of field theory are defined with an additional twist, replacing the real line by the one-dimensional space of volume forms on the base manifold M at the appropriate point. Thus the *twisted* (*affine*) *dual* $J^\circledast E$ of the jet bundle JE and the *twisted* (*linear*) *dual* $\vec{J}^\circledast E$ of the linearized jet bundle $\vec{J}E$ are obtained from the corresponding ordinary (untwisted) duals by taking the tensor product with the line bundle of volume forms on the base manifold M, pulled back to the total space E via the projection π, i.e. we put

$$J^\circledast E = J^\star E \otimes \pi^*(\textstyle\bigwedge^n T^* M)\,, \tag{2.17}$$

and

$$\vec{J}^\circledast E = \vec{J}^* E \otimes \pi^*(\textstyle\bigwedge^n T^* M)\,, \tag{2.18}$$

respectively, which means that if $x = \pi(e)$, we set

$$J_e^\circledast E = \{z_e : J_e E \to \textstyle\bigwedge^n T_x^* M \text{ affine}\}\,, \tag{2.19}$$

and

$$\vec{J}_e^\circledast E = \{\vec{z}_e : \vec{J}_e E \to \textstyle\bigwedge^n T_x^* M \text{ linear}\}\,, \tag{2.20}$$

respectively. As is the case for the jet bundle itself, the linearized jet bundle and the various types of dual bundles introduced here all admit two different projections, namely the *target projection* $\tau_{\dots}$ onto E and the *source projection* $\sigma_{\dots}$ onto M which is simply its composition with the original projection π, that is, $\sigma_{\dots} = \pi \circ \tau_{\dots}$. It is easily shown that all of them are fiber bundles over M with respect to $\sigma_{\dots}$, in general without any additional structure, but — as stated before — they are vector bundles over E with respect to $\tau_{\dots}$. The global linear structure of these bundles over E also becomes clear in local coordinates. Namely, choosing local coordinates x^μ for M, local coordinates q^i for Q and a local trivialization of E induces naturally not only local coordinate systems (x^μ, q^i, q_μ^i) for JE and $(x^\mu, q^i, \vec{q}_\mu^{\,i})$ for $\vec{J}E$ but also local coordinate systems (x^μ, q^i, p_i^μ, p) both for $J^\star E$ and for $J^\circledast E$, as well as local coordinate systems (x^μ, q^i, p_i^μ) both for $\vec{J}^* E$ and for $\vec{J}^\circledast E$, respectively: all these will again be referred to as *adapted local coordinates*. They are defined by requiring the dual pairing between a point in $J^\star E$ or in $J^\circledast E$ with coordinates (x^μ, q^i, p_i^μ, p) and a point in JE with coordinates (x^μ, q^i, q_μ^i) to be given by

$$p_i^\mu q_\mu^i + p \tag{2.21}$$

in the ordinary (untwisted) case and by

$$(p_i^\mu q_\mu^i + p) d^n x \tag{2.22}$$

in the twisted case, whereas the dual pairing between a point in $\vec{J}^* E$ or in $\vec{J}^\circledast E$ with coordinates (x^μ, q^i, p_i^μ) and a point in $\vec{J}E$ with coordinates $(x^\mu, q^i, \vec{q}_\mu^{\,i})$ should be given by

$$p_i^\mu \vec{q}_\mu^{\,i} \tag{2.23}$$

in the ordinary (untwisted) case and by

$$p_i^\mu \vec{q}_\mu^{\,i} \; d^n x \tag{2.24}$$

in the twisted case. Moreover, a transformation to new local coordinates x'^κ for M, new local coordinates q'^k for Q and a new local trivialization of E, according to Eq. (2.5), induces naturally not only a transformation to new adapted local coordinates $(x'^\kappa, q'^k, q'^k_\kappa)$ for JE and $(x'^\kappa, q'^k, \vec{q}'^k_\kappa)$ for $\vec{J}E$, as given by Eqs. (2.6) and (2.7), but also a transformation to new adapted local coordinates $(x'^\kappa, q'^k, p'^\kappa_k, p')$ both for $J^\star E$ and for $J^\circledast E$, as well as a transformation to new adapted local coordinates $(x'^\kappa, q'^k, p'^\kappa_k)$ both for $\vec{J}^* E$ and for $\vec{J}^\circledast E$, respectively: they are given by

$$p'^\kappa_k = p'^\kappa_k(x^\mu, q^i, p_i^\mu, p) \,, \quad p' = p'(x^\mu, q^i, p_i^\mu, p) \,, \tag{2.25}$$

where

$$p'^\kappa_k = \frac{\partial x'^\kappa}{\partial x^\mu} \frac{\partial q^i}{\partial q'^k} \, p_i^\mu \,, \quad p' = p - \frac{\partial q'^k}{\partial x^\mu} \, \frac{\partial q^i}{\partial q'^k} \, p_i^\mu \tag{2.26}$$

in the ordinary (untwisted) case and

$$p'^\kappa_k = \det\left(\frac{\partial x}{\partial x'}\right) \frac{\partial x'^\kappa}{\partial x^\mu} \frac{\partial q^i}{\partial q'^k} p_i^\mu \,, \quad p' = \det\left(\frac{\partial x}{\partial x'}\right) \left(p - \frac{\partial q'^k}{\partial x^\mu} \frac{\partial q^i}{\partial q'^k} p_i^\mu\right) \tag{2.27}$$

in the twisted case. Finally, it is worth noting that the affine duals $J^\star E$ and $J^\circledast E$ of JE contain line subbundles $J_0^\star E$ and $J_0^\circledast E$ whose fiber over any point e in E consists of the constant (rather than affine) maps from $J_e E$ to $\mathbb{R}$ and to $\bigwedge^n T_x^* M$ respectively, and the corresponding quotient vector bundles over E can be naturally identified with the respective duals $\vec{J}^* E$ and $\vec{J}^\circledast E$ of $\vec{J}E$, i.e. we have

$$J^\star E / J_0^\star E \cong \vec{J}^* E \cong L(VE, \pi^* TM) \,, \tag{2.28}$$

and

$$J^\circledast E / J_0^\circledast E \cong \vec{J}^\circledast E \cong L(VE, \pi^*(\textstyle\bigwedge^{n-1} T^* M)) \,, \tag{2.29}$$

respectively. This shows that, in both cases, the corresponding projection onto the quotient amounts to "forgetting the additional energy variable" since it takes a point with coordinates (x^μ, q^i, p_i^μ, p) to the point with coordinates (x^μ, q^i, p_i^μ); it will be denoted by η (as a reminder for the fact that it projects the extended multiphase space to the ordinary one) and is easily seen to turn $J^\star E$ and $J^\circledast E$ into affine line bundles over $\vec{J}^* E$ and over $\vec{J}^\circledast E$, respectively.

An alternative but equivalent description of the extended multiphase space of field theory is as a certain bundle of differential forms on the total space E of the configuration bundle, namely the bundle $\bigwedge_{n-1}^n T^*E$ of $(n-1)$-horizontal n-forms on E, that is, of n-forms on E that vanish whenever one inserts at least two vertical vectors. In fact, there is a canonical isomorphism

$$\Phi : \bigwedge\nolimits_{n-1}^n T^*E \xrightarrow{\cong} J^\circledast E \tag{2.30}$$

of vector bundles over E that can be defined explicitly as follows: given any point e in E with base point $x = \pi(e)$ in M and any $(n-1)$-horizontal n-form $\alpha_e \in \bigwedge_{n-1}^n T_e^* E$, together with a jet $u_e \in J_e E$, we can use u_e, which is a linear map from $T_x M$ to $T_e E$, to pull back the n-form α_e on $T_e E$ to an n-form $u_e^* \alpha_e$ on $T_x M$. Obviously, $u_e^* \alpha_e$ is an affine function of u_e as u_e varies over the affine space $J_e E$ because it is actually a linear function of u_e when u_e is allowed to vary over the entire vector space $L(T_x M, T_e E)$ (the restriction of a linear map between two vector spaces to an affine subspace of its domain is an affine map). Thus putting

$$\Phi_e(\alpha_e) \cdot u_e = u_e^* \alpha_e \tag{2.31}$$

defines a map $\Phi_e : \bigwedge_{n-1}^n T_e^* E \to J_e^\star E$ which is evidently linear and, as e varies over E, provides the desired isomorphism (2.30). Further details can be found in Ref. [4]. The importance of this canonical isomorphism is due to the fact that it provides a natural way to introduce a *multicanonical form* θ and a *multisymplectic form* ω on extended multiphase space which play a similar role in field theory as the canonical form θ and the symplectic form ω on cotangent bundles in mechanics. Namely, θ is an n-form that can be defined intrinsically by using the tangent map $T\tau_{J^\circledast E} : T(J^\circledast E) \to TE$ to the bundle projection $\tau_{J^\circledast E} : J^\circledast E \to E$, as follows. Given a point $z \in J^\circledast E$ with base point $e = \tau_{J^\circledast E}(z)$ in E and n tangent vectors $w_1, \ldots, w_n$ to $J^\circledast E$ at z, put

$$\theta_z(w_1, \ldots, w_n) = (\Phi_e^{-1}(z))(T_z \tau_{J^\circledast E} \cdot w_1, \ldots, T_z \tau_{J^\circledast E} \cdot w_n)\,. \tag{2.32}$$

Moreover, ω is an $(n+1)$-form which, as in mechanics, is defined to be the negative of the exterior derivative of θ:

$$\omega = -d\theta\,. \tag{2.33}$$

Another important object that can be defined globally both on extended and ordinary multiphase space is the *scaling* or *Euler vector field* which we shall denote here by Σ. Its definition is based exclusively on the fact that $J^\circledast E$ and $\vec{J}^\circledast E$ are total spaces of vector bundles over E. In fact, given any vector bundle V over E, Σ_V (which we shall simply denote by Σ when there is no danger of confusion) is defined to be the fundamental vector field associated with the action of $\mathbb{R}$, considered as a commutative group under addition, by scaling transformations on the fibers:

$$\begin{array}{ccc} \mathbb{R} \times V & \rightarrow & V \\ (\lambda, v) & \mapsto & \exp(\lambda) v \end{array} .$$

Thus Σ is simply that vertical vector field on V which, under identification of the vertical tangent spaces to V with the fibers of V itself typical for vector bundles, becomes the identity on V:

$$\Sigma(v) = \frac{d}{d\lambda} \exp(\lambda) v \bigg|_{\lambda=0} = v \,.$$

In adapted local coordinates, the isomorphism Φ can be defined by the requirement that the $(n-1)$-horizontal n-form on E corresponding to the point in $J^{\circledast}E$ with coordinates (x^μ, q^i, p_i^μ, p) is explicitly given by

$$p_i^\mu \, dq^i \wedge d^n x_\mu + p \, d^n x \,. \tag{2.34}$$

The tautological nature of the definition of θ then becomes apparent by realizing that exactly the same expression represents the multicanonical form θ:

$$\theta = p_i^\mu \, dq^i \wedge d^n x_\mu + p \, d^n x \,. \tag{2.35}$$

Taking the exterior derivative yields

$$\omega = dq^i \wedge dp_i^\mu \wedge d^n x_\mu - dp \wedge d^n x \,. \tag{2.36}$$

Moreover, the scaling vector fields on $J^{\circledast}E$ and on $\vec{J}^{\circledast}E$ are given by

$$\Sigma = p_i^\mu \frac{\partial}{\partial p_i^\mu} + p \frac{\partial}{\partial p} \tag{2.37}$$

and by

$$\Sigma = p_i^\mu \frac{\partial}{\partial p_i^\mu} \tag{2.38}$$

respectively. Finally, we note the following relations, which will be used later.

Proposition 2.1. *The multicanonical form θ, the multisymplectic form ω and the scaling or Euler vector field Σ on extended multiphase space $J^{\circledast}E$ satisfy the following relations*:

$$L_\Sigma \theta = \theta \,. \tag{2.39}$$

$$L_\Sigma \omega = \omega \,. \tag{2.40}$$

$$i_\Sigma \theta = 0 \,. \tag{2.41}$$

$$i_\Sigma \omega = -\theta \,. \tag{2.42}$$

Proof. Let $(\varphi_\lambda)_{\lambda \in \mathbb{R}}$ denote the one-parameter group of scaling transformations on $J^{\circledast}E$ given by $\varphi_\lambda(z) = \mathrm{e}^\lambda z$. Then by the formula relating the Lie derivative of a differential form along a vector field to the derivative of its pull-back under the flow of that vector field (see, e.g., [25, p. 91]) and the definition of θ, we have

$$
\begin{aligned}
(L_\Sigma\theta)_z(w_1,\ldots,w_n) &= \frac{\partial}{\partial\lambda}(\varphi_\lambda^*\theta)_z(w_1,\ldots,w_n)\bigg|_{\lambda=0} \\
&= \frac{\partial}{\partial\lambda}\theta_{\varphi_\lambda(z)}(T_z\varphi_\lambda\cdot w_1,\ldots,T_z\varphi_\lambda\cdot w_n)\bigg|_{\lambda=0} \\
&= \frac{\partial}{\partial\lambda}\Phi_e^{-1}(\varphi_\lambda(z))(T_{\varphi_\lambda(z)}\tau_{J^\circledast E}\cdot(T_z\varphi_\lambda\cdot w_1),\ldots,T_{\varphi_\lambda(z)}\tau_{J^\circledast E}\cdot(T_z\varphi_\lambda\cdot w_n))\bigg|_{\lambda=0} \\
&= \frac{\partial}{\partial\lambda}\Phi_e^{-1}(\mathrm{e}^\lambda z)(T_z(\tau_{J^\circledast E}\circ\varphi_\lambda)\cdot w_1,\ldots,T_z(\tau_{J^\circledast E}\circ\varphi_\lambda)\cdot w_n)\bigg|_{\lambda=0} \\
&= \frac{\partial}{\partial\lambda}\mathrm{e}^\lambda\Phi_e^{-1}(z)(T_z\tau_{J^\circledast E}\cdot w_1,\ldots,T_z\tau_{J^\circledast E}\cdot w_n)\bigg|_{\lambda=0} \\
&= \frac{\partial}{\partial\lambda}\mathrm{e}^\lambda\theta_z(w_1,\ldots,w_n)\bigg|_{\lambda=0} \\
&= \theta_z(w_1,\ldots,w_n)\,,
\end{aligned}
$$

which proves Eq. (2.39) and also Eq. (2.40) since L_Σ commutes with the exterior derivative. Next, observe that with respect to the target projection of $J^\circledast E$ onto E, Σ is vertical whereas θ is horizontal, which implies Eq. (2.41). Combining these two equations, we finally get

$$\theta = L_\Sigma\theta = d(i_\Sigma\theta) + i_\Sigma d\theta = -i_\Sigma\omega\,,$$

proving Eq. (2.42). □

We note here that the existence of the canonically-defined forms θ and ω is what distinguishes the twisted affine dual $J^\circledast E$ from the ordinary affine dual $J^\star E$ of JE.

Using the jet bundle JE and the multiphase spaces $\vec{J}^\circledast E$ and $J^\circledast E$ associated with a given fiber bundle E over space-time M, one can develop a general covariant Lagrangian and Hamiltonian formalism for field theories whose configurations are sections of E. For example, the Lagrangian function of mechanics is replaced by a *Lagrangian density* $\mathcal{L}$, which is a function on JE with values in the volume forms on space-time, so that one can integrate it to compute the action functional and formulate a variational principle. It gives rise to a *covariant Legendre transformation* which replaces that of mechanics and comes in two variants, both defined by an appropriate notion of vertical derivative or fiber derivative: one of them is a fiber preserving smooth map $\vec{\mathbb{F}}\mathcal{L} : JE \to \vec{J}^\circledast E$ and the other a fiber preserving smooth map $\mathbb{F}\mathcal{L} : JE \to J^\circledast E$; of course, the former is obtained from the latter by composition with the natural projection η from $J^\circledast E$ onto $\vec{J}^\circledast E$ mentioned above. When $\vec{\mathbb{F}}\mathcal{L}$ is a local/global diffeomorphism, the Lagrangian $\mathcal{L}$ is called *regular/hyperregular*.

On the other hand, the Hamiltonian function of mechanics is replaced by a *Hamiltonian density* $\mathcal{H}$, which is a section of extended multiphase space $J^{\circledast}E$ as an affine line bundle over ordinary multiphase space $\vec{J}^{\circledast}E$. Once again, any such section gives rise to a *covariant Legendre transformation*, defined by an appropriate notion of vertical derivative or fiber derivative: it is a fiber preserving smooth map $\mathbb{F}\mathcal{H} : \vec{J}^{\circledast}E \to JE$. When $\mathbb{F}\mathcal{H}$ is a local/global diffeomorphism, the Hamiltonian $\mathcal{H}$ is called *regular/hyperregular*. In any case, pulling back θ and ω from $J^{\circledast}E$ to JE via $\mathbb{F}\mathcal{L}$ generates the *Poincaré-Cartan forms* $\theta_{\mathcal{L}}$ and $\omega_{\mathcal{L}}$ on JE, and similarly, pulling them back from $J^{\circledast}E$ to $\vec{J}^{\circledast}E$ via $\mathcal{H}$ generates the forms $\theta_{\mathcal{H}}$ and $\omega_{\mathcal{H}}$ on $\vec{J}^{\circledast}E$. As in mechanics, the Lagrangian and Hamiltonian formulations turn out to be completely equivalent in the hyperregular case, with $\vec{\mathbb{F}}\mathcal{L}$ and $\mathbb{F}\mathcal{H}$ being each other's inverse. For more details on these and related matters, the reader may consult Ref. [3] and, in particular, Ref. [4] — except for the direct construction of the Legendre transformation $\vec{\mathbb{F}}\mathcal{H}$ associated with a Hamiltonian $\mathcal{H}$, which was first derived in Ref. [23]; see also Ref. [24]. There is also a generalization of the Hamilton-Jacobi equation to the field theoretical situation; the reader may consult the extensive review by Kastrup [27] as a starting point for this direction.

3. Poisson Forms and Their Poisson Brackets

The constructions exposed in the previous section have identified the extended multiphase space of field theory as an example of a multisymplectic manifold.

Definition 3.1. A **multisymplectic manifold** is a manifold P equipped with a non-degenerate closed $(n+1)$-form ω, called the **multisymplectic form**.

Remark. This definition is deliberately vague as to the meaning of the term "non-degenerate", at least when $n > 1$. The standard interpretation is that the kernel of ω on vectors should vanish, that is,

$$i_X\omega = 0 \;\Rightarrow\; X = 0 \quad \text{for vector fields } X\,. \tag{3.1}$$

Note that, of course, no such conclusion holds for multivector fields, that is, the kernel of ω on multivectors is non-trivial. (This is true even for symplectic forms which vanish on certain bivectors, for example on those that represent two-dimensional isotropic subspaces.) However, the condition (3.1) alone is too weak and it is not clear what additional algebraic constraints should be imposed on ω. A first attempt in this direction has been made by Martin [28, 29], but his conditions are too restrictive and do not seem to agree with what is needed in applications to field theory. More recently, a promising proposal has been made by Cantrijn, Ibort and de León [30] which seems to come close to a convincing definition of the concept of a multisymplectic manifold. Fortunately, there is no need to enter this discussion here since the "minimal" requirement of non-degeneracy formulated in Eq. (3.1) is sufficient for our purposes and will be used here to provide a working definition.

In what follows, we shall make extensive use of the basic operations of calculus on manifolds involving multivector fields and differential forms, namely the Schouten bracket between multivector fields, the contraction of differential forms with multivector fields and the Lie derivative of differential forms along multivector fields. For the convenience of the reader, the relevant formulae are summarized in Appendix A; in particular, Eqs. (A.9) and (A.11) will be used constantly and often without further mention.

On multisymplectic manifolds, there are special classes of multivector fields and of differential forms:

Definition 3.2. An r-multivector field X on a multisymplectic manifold P is called **locally Hamiltonian** if $i_X \omega$ is closed, or equivalently, if

$$L_X \omega = 0\,, \tag{3.2}$$

and it is called **globally Hamiltonian** or simply **Hamiltonian** if $i_X \omega$ is exact, i.e. if there exists an $(n-r)$-form f on P such that

$$i_X \omega = df\,. \tag{3.3}$$

In this case, we say that f is **associated** with X or **corresponds** to X.

Conversely, an $(n-r)$-form f on a multisymplectic manifold P is called **Hamiltonian** if there exists an r-multivector field X on P such that

$$i_X \omega = df\,. \tag{3.4}$$

In this case, we say that X is **associated** with f or **corresponds** to f.

Remark. As mentioned before, the kernel of ω on multivectors is non-trivial, so the correspondence between Hamiltonian multivector fields and Hamiltonian forms is not unique (in either direction). Moreover, by far not every form is Hamiltonian. In particular, as first shown in special examples by Kijowski [8] and then more systematically by Kanatchikov [1], although in a somewhat different context, there are restrictions on the allowed multimomentum dependence of the coefficient functions. Of course, every closed form is Hamiltonian (the corresponding Hamiltonian multivector field vanishes identically). Below we will give more interesting examples to show that the definition is not empty.

Proposition 3.3. *The Schouten bracket of any two locally Hamiltonian multivector fields X and Y on a multisymplectic manifold P is a globally Hamiltonian multivector field $[X,Y]$ on P whose associated Hamiltonian form can, up to sign, be chosen to be the double contraction $i_X i_Y \omega$. More precisely, assuming X to be of degree r and Y to be of degree s, we have*

$$i_{[X,Y]}\omega = (-1)^{(r-1)s}\, d(i_X i_Y \omega)\,. \tag{3.5}$$

In particular, this implies that under the Schouten bracket, the space $\mathfrak{X}^{\wedge}_{LH}(P)$ of locally Hamiltonian multivector fields on P is a subalgebra of the Lie superalgebra

$\mathfrak{X}^\wedge(P)$ of all multivector fields on P, containing the space $\mathfrak{X}^\wedge_H(P)$ of globally Hamiltonian multivector fields, as well as the (smaller) space $\mathfrak{X}^\wedge_0(P)$ of multivector fields taking values in the kernel of ω, as ideals: if X is locally Hamiltonian, then

$$i_\xi \omega = 0 \quad \Rightarrow \quad i_{[\xi,X]}\omega = 0\,. \tag{3.6}$$

Proof. According to Eqs. (A.11) and (A.9), we have for any two multivector fields X of degree r and Y of degree s,

$$\begin{aligned} i_{[X,Y]}\omega &= (-1)^{(r-1)s}\, L_X i_Y \omega - i_Y L_X \omega \\ &= (-1)^{(r-1)s}\, d(i_X i_Y \omega) + (-1)^{(r-1)(s-1)}\, i_X\, d(i_Y\omega) - i_Y L_X \omega \\ &= (-1)^{(r-1)s}\, d(i_X i_Y \omega) + (-1)^{(r-1)(s-1)}\, i_X L_Y \omega - i_Y L_X \omega\,, \end{aligned}$$

since $d\omega = 0$, showing that if X and Y are both locally Hamiltonian, then $[X,Y]$ is globally Hamiltonian and Eq. (3.5) holds. □

Definition 3.4. A Hamiltonian form f on a multisymplectic manifold P is called a **Poisson form** if its contraction with any multivector field ξ on P taking values in the kernel of ω vanishes:

$$i_\xi \omega = 0 \quad \Rightarrow \quad i_\xi f = 0\,. \tag{3.7}$$

Remark. For the Poisson bracket introduced below to be well-defined, it would be sufficient to impose the apparently weaker condition that the contraction of f with any multivector field ξ on P taking values in the kernel of ω should be a closed form:

$$i_\xi \omega = 0 \quad \Rightarrow \quad d(i_\xi f) = 0\,. \tag{3.8}$$

However, it turns out that this condition is already sufficient to imply the previous one. To see this, observe that if f is a differential form on P satisfying Eq. (3.8) and ξ is any multivector field on P taking values in the kernel of ω, then for any function φ on P, $\varphi\xi$ will be a multivector field on P taking values in the kernel of ω as well and hence

$$0 = d(i_{\varphi\xi} f) = d(\varphi i_\xi f) = d\varphi \wedge i_\xi f + \varphi d(i_\xi f) = d\varphi \wedge i_\xi f\,.$$

But this means that the exterior product of $i_\xi f$ with any one-form on P must vanish, which is only possible if $i_\xi f$ itself vanishes.

Definition 3.5. An **exact multisymplectic manifold** is a multisymplectic manifold whose multisymplectic form ω is the exterior derivative of a Poisson form:

$$\omega = -d\theta\,. \tag{3.9}$$

$$i_\xi \omega = 0 \quad \Rightarrow \quad i_\xi \theta = 0\,. \tag{3.10}$$

We shall call θ the **multicanonical form**.

Remark. It is an immediate consequence of Proposition 2.1, in particular of Eq. (2.42), that the extended multiphase space of field theory is an exact multisymplectic manifold. However, the condition that the kernel of θ should contain that of ω is non-trivial in the sense that it is not always possible to modify a potential of an exact form by adding an appropriate closed form so as to achieve the desired inclusion of the kernels, as the following counterexample will show.[d] Consider the three-sphere S^3 as the total space of the Hopf bundle, a principal $U(1)$-bundle over the two-sphere S^2, and let ξ be the fundamental vector field of the $U(1)$ group action on S^3 and α be the canonical connection 1-form on S^3. Then $i_\xi \alpha = 1$ and $i_\xi d\alpha = 0$. We want to modify α by some closed form β so that $i_\xi(\alpha + \beta) = 0$. But S^3 is simply connected, so $d\beta = 0$ implies that there is a function f with $df = \beta$. Hence we are looking for a function f on S^3 that satisfies $i_\xi df = -1$. But S^3 is compact, so f must have at least two critical points (a maximum and a minimum), and we arrive at a contradiction. In other words, we cannot modify the potential α of $d\alpha$ in such a way that the kernel of $d\alpha$ is contained in the kernel of the modified potential.

Definition 3.6. Let P be an exact multisymplectic manifold. Given any two Poisson forms f of degree $n-r$ and g of degree $n-s$ on P, their **Poisson bracket** is defined to be the $(n+1-r-s)$-form on P given by

$$\{f, g\} = -L_X g + (-1)^{(r-1)(s-1)} L_Y f - (-1)^{(r-1)s} L_{X \wedge Y} \theta \,, \tag{3.11}$$

or equivalently,

$$\begin{aligned} \{f, g\} = {} & (-1)^{r(s-1)} i_Y i_X \omega \\ & + d\big((-1)^{(r-1)(s-1)} i_Y f - i_X g - (-1)^{(r-1)s} i_Y i_X \theta\big) \,, \end{aligned} \tag{3.12}$$

where X and Y are Hamiltonian multivector fields associated with f and with g, respectively.

Remark. This Poisson bracket is an extension of the one between Hamiltonian $(n-1)$-forms introduced by two of the present authors in an earlier article [5], except for the fact that when f and g are $(n-1)$-forms, X and Y are vector fields and are uniquely determined by f and g, so there is no need to impose restrictions on the contraction of f and g with multivector fields taking values in the kernel of ω: the definition given in Ref. [5] works for all Hamiltonian $(n-1)$-forms and not just for Poisson $(n-1)$-forms.

Proposition 3.7. *The Poisson bracket introduced above closes and is well-defined, i.e. when f and g are Poisson forms, $\{f, g\}$ is again a Poisson form which does not depend on the choice of the Hamiltonian multivector fields X and Y used in its definition. Moreover, we have*

$$i_{[Y,X]}\omega = d\{f, g\} \,, \tag{3.13}$$

[d]This example is due to M. Bordemann.

i.e. if X is a Hamiltonian multivector field associated with f and Y is a Hamiltonian multivector field associated with g, then $[Y,X]$ is a Hamiltonian multivector field associated with $\{f,g\}$.

Proof. We begin by using Eq. (A.9) to show that, for any two Hamiltonian forms f of degree $n-r$ and g of degree $n-s$ with associated Hamiltonian multivector fields X and Y, respectively, the expressions on the right-hand side of Eqs. (3.11) and (3.12) coincide:

$$\begin{aligned}
&-L_X g + (-1)^{(r-1)(s-1)} L_Y f - (-1)^{(r-1)s} L_{X\wedge Y}\theta \\
&\quad = -d(i_X g) + (-1)^r i_X dg \\
&\qquad + (-1)^{(r-1)(s-1)} d(i_Y f) - (-1)^{(r-1)(s-1)+s} i_Y df \\
&\qquad - (-1)^{(r-1)s} d(i_{X\wedge Y}\theta) - (-1)^{r(s-1)} i_{X\wedge Y}\omega \\
&\quad = -d(i_X g) + (-1)^{rs+r} i_Y i_X \omega \\
&\qquad + (-1)^{(r-1)(s-1)} d(i_Y f) + (-1)^{rs-r} i_Y i_X \omega \\
&\qquad - (-1)^{(r-1)s} d(i_Y i_X \theta) - (-1)^{rs-r} i_Y i_X \omega \\
&\quad = (-1)^{r(s-1)} i_Y i_X \omega \\
&\qquad + d\big((-1)^{(r-1)(s-1)} i_Y f - i_X g - (-1)^{(r-1)s} i_Y i_X \theta\big)\,.
\end{aligned}$$

In order for the bracket to be well-defined, it is necessary and sufficient that this expression vanishes whenever X or Y takes its values in the kernel of ω: this is guaranteed by the requirement that f, g and θ should be Poisson forms. Moreover, in view of Eq. (3.5), Eq. (3.13) follows immediately from Eq. (3.12), proving that the Poisson bracket $\{f,g\}$ of two Poisson forms is a Hamiltonian form. To check that it is in fact a Poisson form, assume ξ to be a multivector field taking values in the kernel of ω, say of degree k, and consider the expressions obtained by contracting each of the four terms in Eq. (3.12) with ξ. The first obviously vanishes, whereas the fourth can be seen to vanish due to Eqs. (3.6) and (3.10):

$$\begin{aligned}
i_\xi d(i_Y i_X \theta) &= (-1)^s i_\xi i_Y d(i_X\theta) + i_\xi L_Y i_X \theta \\
&= (-1)^{s(k-1)} i_Y i_\xi L_X \theta + (-1)^{r+s(k-1)} i_Y i_\xi i_X d\theta \\
&\quad - i_{[Y,\xi]} i_X \theta + (-1)^{(s-1)k} L_Y i_\xi i_X \theta \\
&= -(-1)^{s(k-1)} i_Y i_{[X,\xi]}\theta + (-1)^{(r-1)k+s(k-1)} i_Y L_X i_\xi \theta \\
&\quad - (-1)^{r+s(k-1)} i_Y i_\xi i_X \omega \\
&\quad - i_{[Y,\xi]} i_X \theta + (-1)^{(s-1)k} L_Y i_\xi i_X \theta \\
&= 0\,.
\end{aligned}$$

Similarly, the second and third can be handled by using Eqs. (3.6) and (3.7) which imply that

$$\begin{aligned} i_\xi d(i_Y f) &= (-1)^s i_\xi i_Y df + i_\xi L_Y f \\ &= (-1)^s i_\xi i_Y i_X \omega - i_{[Y,\xi]} f + (-1)^{(s-1)k} L_Y i_\xi f\,, \end{aligned}$$

and

$$\begin{aligned} i_\xi d(i_X g) &= (-1)^r i_\xi i_X dg + i_\xi L_X g \\ &= (-1)^r i_\xi i_X i_Y \omega - i_{[X,\xi]} g + (-1)^{(r-1)k} L_X i_\xi g\,. \end{aligned}$$

vanish since f and g are Poisson forms. □

Now we can formulate the main theorem of this paper:

Theorem 3.8. *Let P be an exact multisymplectic manifold. The Poisson bracket introduced above is bilinear over $\mathbb{R}$, is graded antisymmetric, which means that for any two Poisson forms f of degree $n-r$ and g of degree $n-s$ on P, we have*

$$\{g,f\} = -(-1)^{(r-1)(s-1)}\{f,g\}\,, \tag{3.14}$$

and satisfies the graded Jacobi identity, which means that for any three Poisson forms f of degree $n-r$, g of degree $n-s$ and h of degree $n-t$ on P, we have

$$(-1)^{(r-1)(t-1)}\{f,\{g,h\}\} + \text{cyclic perm.} = 0\,, \tag{3.15}$$

thus turning the space of Poisson forms on P into a Lie superalgebra.

Remark. Bilinearity over $\mathbb{R}$ and the graded antisymmetry (3.14) being obvious, the main statement of the theorem is of course the validity of the graded Jacobi identity (3.15), which depends crucially on the exact correction terms, that is, the last three terms in the defining equation (3.12). To prove this, we need the following two lemmas:

Lemma 3.9. *Let P be a multisymplectic manifold. For any three locally Hamiltonian multivector fields X of degree r, Y of degree s and Z of degree t on P, we have the cyclic identity*

$$(-1)^{r(t-1)} i_X d(i_Y i_Z \omega) + \text{cyclic perm.} = (-1)^{rt} d(i_X i_Y i_Z \omega)\,, \tag{3.16}$$

Proof. This is obtained by calculating

$$\begin{aligned} i_X d(i_Y i_Z \omega) &= (-1)^{(s-1)t} i_X i_{[Y,Z]}\omega = (-1)^{(s-1)t+r(s+t-1)} i_{[Y,Z]} i_X \omega \\ &= (-1)^{r(s+t-1)} (L_Y i_Z - (-1)^{(s-1)t} i_Z L_Y) i_X \omega \\ &= (-1)^{r(s+t-1)} d(i_Y i_Z i_X \omega) + (-1)^{r(s+t-1)+s-1} i_Y d(i_Z i_X \omega) \\ &\quad -(-1)^{r(s+t-1)+(s-1)t} i_Z d(i_Y i_X \omega)\,, \end{aligned}$$

and multiplying by $(-1)^{rt-r}$. □

Lemma 3.10. *Let P be an exact multisymplectic manifold. For any three locally Hamiltonian multivector fields X of degree r, Y of degree s and Z of degree t on P, we have the cyclic identity*

$$\begin{aligned}(-1)^{r(t-1)} i_X d(i_Y i_Z \theta) - (-1)^{r(t-1)+s} i_X i_Y d(i_Z \theta) + \text{cyclic perm.} \\ = (-1)^{rt+r+s+t} i_X i_Y i_Z \omega + (-1)^{rt} d(i_X i_Y i_Z \theta)\,. \end{aligned} \tag{3.17}$$

Proof. This is obtained by calculating

$$\begin{aligned} & i_X d(i_Y i_Z \theta) + (-1)^{s-1} i_X i_Y d(i_Z \theta) - (-1)^{(s-1)t} i_X i_Z d(i_Y \theta) + (-1)^{(s-1)(t-1)} i_X i_Z i_Y \omega \\ &\quad = i_X (L_Y i_Z - (-1)^{(s-1)t} i_Z L_Y) \theta \\ &\quad = (-1)^{(s-1)t} i_X i_{[Y,Z]} \theta = (-1)^{(s-1)t + r(s+t-1)} i_{[Y,Z]} i_X \theta \\ &\quad = (-1)^{r(s+t-1)} (L_Y i_Z - (-1)^{(s-1)t} i_Z L_Y) i_X \theta \\ &\quad = (-1)^{r(s+t-1)} d(i_Y i_Z i_X \theta) + (-1)^{r(s+t-1)+s-1} i_Y d(i_Z i_X \theta) \\ &\qquad - (-1)^{r(s+t-1)+(s-1)t} i_Z d(i_Y i_X \theta) - (-1)^{r(s+t-1)+(s-1)(t-1)} i_Z i_Y d(i_X \theta)\,, \end{aligned}$$

and multiplying by $(-1)^{rt-r}$. □

Proof of Theorem 3.8. Given any three Poisson forms f of degree $n-r$, g of degree $n-s$ and h of degree $n-t$ and fixing three Hamiltonian multivector fields X of degree r, Y of degree s and Z of degree t associated with f, with g and with h, respectively, we compute the double Poisson bracket

$$\begin{aligned} & (-1)^{(r-1)(t-1)} \{f, \{g, h\}\} \\ &\quad = (-1)^{(r-1)(t-1)+r(s+t)} i_{[Z,Y]} i_X \omega \\ &\qquad + (-1)^{(r-1)(t-1)+(r-1)(s+t)} d(i_{[Z,Y]} f) \\ &\qquad - (-1)^{(r-1)(t-1)} d(i_X \{g, h\}) \\ &\qquad - (-1)^{(r-1)(t-1)+(r-1)(s+t-1)} d(i_{[Z,Y]} i_X \theta) \\ &\quad = -(-1)^{(rs+r+t)+r(s+t-1)+(st+s+t)} i_X i_{[Y,Z]} \omega \\ &\qquad + (-1)^{(r-1)(s-1)+(t-1)s} d(L_Z i_Y f) - (-1)^{(r-1)(s-1)} d(i_Y L_Z f) \\ &\qquad - (-1)^{(r-1)(t-1)+s(t-1)} d(i_X i_Z i_Y \omega) \\ &\qquad - (-1)^{(r-1)(t-1)+(s-1)(t-1)} d(i_X d(i_Z g)) + (-1)^{(r-1)(t-1)} d(i_X d(i_Y h)) \\ &\qquad + (-1)^{(r-1)(t-1)+(s-1)t} d(i_X d(i_Z i_Y \theta)) \\ &\qquad - (-1)^{(r-1)s+(t-1)s} d(L_Z i_Y i_X \theta) + (-1)^{(r-1)s} d(i_Y L_Z i_X \theta) \end{aligned}$$

$$
\begin{aligned}
&= -(-1)^{rt+s+t} i_X d(i_Y i_Z \omega) \\
&\quad + (-1)^{rs+st+r+t} d(i_Z d(i_Y f)) + (-1)^{rs+r+s} d(i_Y d(i_Z f)) \\
&\quad - \underline{(-1)^{rs+r+s+t} d(i_Y i_Z i_X \omega)} \\
&\quad + \underline{(-1)^{rt+st+r+s+t} d(i_X i_Z i_Y \omega)} \\
&\quad - (-1)^{rt+st+r+s} d(i_X d(i_Z g)) - (-1)^{rt+r+t} d(i_X d(i_Y h)) \\
&\quad - (-1)^{rt+r} d(i_X d(i_Y i_Z \theta)) \quad \leftarrow \\
&\quad + (-1)^{st+t} d(i_Z d(i_X i_Y \theta)) \quad \leftarrow \\
&\quad + (-1)^{rs+s} d(i_Y d(i_Z i_X \theta)) - (-1)^{rs+s+t} d(i_Y i_Z d(i_X \theta))\,.
\end{aligned}
$$

In the last expression, the underlined terms cancel each other. Moreover, under the cyclic sum, the terms marked by an arrow cancel each other and the terms containing derivatives of contractions of f, g, h cancel pairwise, i.e. the expression

$$
\begin{aligned}
&+(-1)^{rs+st+r+t} d(i_Z d(i_Y f)) + (-1)^{rs+r+s} d(i_Y d(i_Z f)) \\
&-(-1)^{rt+st+r+s} d(i_X d(i_Z g)) - (-1)^{rt+r+t} d(i_X d(i_Y h)) \\
&+(-1)^{st+tr+s+r} d(i_X d(i_Z g)) + (-1)^{st+s+t} d(i_Z d(i_X g)) \\
&-(-1)^{sr+tr+s+t} d(i_Y d(i_X h)) - (-1)^{sr+s+r} d(i_Y d(i_Z f)) \\
&+(-1)^{tr+rs+t+s} d(i_Y d(i_X h)) + (-1)^{tr+t+r} d(i_X d(i_Y h)) \\
&-(-1)^{ts+rs+t+r} d(i_Z d(i_Y f)) - (-1)^{ts+t+s} d(i_Z d(i_X g))
\end{aligned}
$$

vanishes. Finally, using the cyclic identities (3.16) and (3.17), we see that the remaining terms sum up as follows:

$$
\begin{aligned}
&(-1)^{(r-1)(t-1)}\{f,\{g,h\}\} + \text{cyclic perm.} \\
&\quad = -(-1)^{r+s+t}\big((-1)^{r(t-1)} i_X d(i_Y i_Z \omega) + \text{cyclic perm.}\big) \\
&\qquad + d\big((-1)^{r(t-1)} i_X d(i_Y i_Z \theta) - (-1)^{r(t-1)+s} i_X i_Y d(i_Z \theta) + \text{cyclic perm.}\big) \\
&\quad = -(-1)^{r+s+t}(-1)^{rt} d(i_X i_Y i_Z \omega) \\
&\qquad + d\big((-1)^{r+s+t}(-1)^{rt} i_X i_Y i_Z \omega + (-1)^{rt} d(i_X i_Y i_Z \theta)\big) \\
&\quad = 0\,.
\end{aligned}
$$

This completes the proof of the main theorem. □

Remark. From the definition given in Eq. (3.12), it is obvious that the Poisson bracket between an arbitrary Poisson form f and a closed Poisson form g is exact, since in this case the Hamiltonian multivector field Y associated with g may be chosen to vanish identically, so that one gets $\{f, g\} = -d(i_X g)$. Therefore, the space of closed Poisson forms is an ideal in the Lie superalgebra of all Poisson forms.

Concluding, it must not go unnoticed that the Poisson bracket between Poisson forms introduced in this paper should be looked upon with a certain amount of caution, for a variety of reasons. One of these is that the space of Poisson forms is a Lie superalgebra but apparently not a Poisson superalgebra, since the Poisson bracket does not act as a superderivation in its second argument with respect to the exterior product of forms, nor does there seem to exist any other naturally defined associative supercommutative product between Poisson forms with that property: this is in contrast to the situation for multivector fields which do form a Poisson superalgebra with respect to the exterior product and the Schouten bracket. There is also a degree problem, since for example, the Poisson bracket between functions would be a form of negative degree, which is always zero: this is, at least at first sight, rather odd. Finally, the question about the relation to the covariant Poisson bracket of Peierls and de Witt mentioned at the end of the introduction remains open.

4. The Universal Multimomentum Map

On exact multisymplectic manifolds, Definition 3.2 can be complemented as follows.

Definition 4.1. A multivector field X on an exact multisymplectic manifold P is called **exact Hamiltonian** if

$$L_X \theta = 0\,. \tag{4.1}$$

The terminology is consistent with that introduced before because exact Hamiltonian multivector fields are Hamiltonian: this is an immediate consequence of Proposition 4.3 below. Thus Proposition 3.3 can be complemented as follows.

Proposition 4.2. *The Schouten bracket of any two exact Hamiltonian multivector fields X and Y on an exact multisymplectic manifold P is an exact Hamiltonian multivector field $[X, Y]$ on P. This means that the space $\mathfrak{X}^\wedge_{EH}(P)$ of exact Hamiltonian multivector fields on P is a subalgebra of the Lie superalgebra $\mathfrak{X}^\wedge(P)$ of all multivector fields on P which, according to Eq.* (3.6), *contains the space $\mathfrak{X}^\wedge_0(P)$ of multivector fields taking values in the kernel of ω as an ideal.*

Proof. The proposition follows directly from Eq. (A.12). □

Exact Hamiltonian multivector fields generate Poisson forms, by contraction with the multicanonical form.

Proposition 4.3. *Let P be an exact multisymplectic manifold. For every exact Hamiltonian r-multivector field X on P, the formula*

$$J(X) = (-1)^{r-1} i_X \theta \tag{4.2}$$

defines a Poisson $(n-r)$*-form* $J(X)$ *on* P *whose associated Hamiltonian multivector field is* X *itself. In particular,* X *is Hamiltonian.*

Proof. Using Eq. (A.9), we see that the condition (4.1) implies

$$d(J(X)) = (-1)^{r-1} d(i_X \theta) = (-1)^{r-1} L_X \theta - i_X d\theta = i_X \omega \,, \tag{4.3}$$

so $J(X)$ is a Hamiltonian form whose associated Hamiltonian multivector field is X itself. Moreover, the kernel of $J(X)$ on multivectors contains that of θ which in turn contains that of ω, so $J(X)$ is a Poisson form. □

Proposition 4.4. *Let* P *be an exact multisymplectic manifold. The linear map* J *from the space* $\mathfrak{X}^{\wedge}_{EH}(P)$ *of exact Hamiltonian multivector fields on* P *to the space of Poisson forms on* P *defined by Eq.* (4.2) *is an antihomomorphism of Lie superalgebras, i.e. we have*

$$\{J(X), J(Y)\} = J([Y, X]) \,. \tag{4.4}$$

Proof. For any two exact Hamiltonian multivector fields X of degree r and Y of degree s, we have, according to the defining Eqs. (3.12) and (4.2),

$$\begin{aligned}
\{J(X), J(Y)\} &= (-1)^{r(s-1)} i_Y i_X \omega + (-1)^{(r-1)(s-1)+r-1} d(i_Y i_X \theta) \\
&\quad - (-1)^{s-1} d(i_X i_Y \theta) - (-1)^{(r-1)s} d(i_Y i_X \theta) \\
&= (-1)^{r(s-1)} i_Y i_X \omega + (-1)^{(r-1)s} d(i_Y i_X \theta) \,,
\end{aligned}$$

whereas combining Eqs. (A.11), (A.9) and (4.3) gives

$$\begin{aligned}
J([Y, X]) &= (-1)^{r+s} i_{[Y,X]} \theta \\
&= (-1)^{r+s+r(s-1)} L_Y i_X \theta \qquad \text{since} \;\; L_Y \theta = 0 \\
&= (-1)^{r(s-1)} d(i_Y i_X \theta) - (-1)^{r(s-1)+s} i_Y d(i_X \theta) \\
&= (-1)^{r(s-1)} d(i_Y i_X \theta) + (-1)^{r(s-1)} i_Y i_X \omega \,.
\end{aligned}$$

Obviously, these two expressions coincide. □

Remark. This proposition, even when restricted to vector fields and $(n-1)$-forms, constitutes a remarkable improvement over the corresponding Proposition 4.5 of Ref. [4] where, due to an inadequate definition of the Poisson bracket (omitting the exact correction terms, that is, the last three terms in Eq. (3.12)), Eq. (4.4) must be modified by an exact correction term.

Definition 4.5. Let P be an exact multisymplectic manifold. The linear map J from the space $\mathfrak{X}^{\wedge}_{EH}(P)$ of exact Hamiltonian multivector fields on P to the space of Poisson forms on P defined by Eq. (4.2) will be called the **universal multimomentum map** and its restriction to the space $\mathfrak{X}_{EH}(P)$ of exact Hamiltonian vector fields on P the **universal momentum map**.

Remark. The term "universal momentum map" can be justified in the context of Noether's theorem, dealing with the derivation of conservation laws from symmetries. In classical field theory, conserved quantities are described by Noether currents which depend on the fields of the theory and are $(n-1)$-forms on n-dimensional space-time, so that they can be integrated over compact regions in spacelike hyper-surfaces in order to provide Noether charges associated with each such region: Noether's theorem then asserts that when the fields satisfy the equations of motion of the theory, these Noether currents are closed forms. In the multiphase space approach, the Noether currents on space-time are obtained from corresponding Noether current forms defined on (extended) multiphase space via pull-back of differential forms, their entire field dependence being induced by this pull-back. Moreover, there is an explicit procedure to construct these Noether current forms on (extended) multiphase space: it is the field theoretical analogue of the momentum map of Hamiltonian mechanics on cotangent bundles and, in Ref. [4], is called the "special covariant momentum map". Briefly, given a Lie group G, with Lie algebra $\mathfrak{g}$, the statement that G is a symmetry group of a specific theory supposes that we are given an action of G on the configuration bundle E over M by bundle automorphisms, which of course induces actions of G on JE and on $\vec{J}E$, as well as on all of their duals, including $\vec{J}^{\circledast}E$ and $J^{\circledast}E$, by bundle automorphisms. (In order to speak of a symmetry, we must also assume the Lagrangian or Hamiltonian density to be invariant, or rather equivariant, under the action of G, but this aspect is not relevant for the present discussion.) As usual, each of these actions induces an antihomomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on the corresponding manifold, taking each generator X in $\mathfrak{g}$ to the corresponding fundamental vector field X_M, X_E, X_{JE}, $X_{\vec{J}E} \ldots X_{\vec{J}^{\circledast}E}$, $X_{J^{\circledast}E}$, all of which (except X_M) are projectable: for example, X_E projects to X_M under the tangent map $T\pi : TE \to TM$ to the projection $\pi : E \to M$. Moreover, the vector fields X_{JE}, $X_{\vec{J}E} \ldots X_{\vec{J}^{\circledast}E}$, $X_{J^{\circledast}E}$ can all be obtained from the vector field X_E by a canonical lifting process. In particular, the projectable vector fields $X_{J^{\circledast}E}$ on $J^{\circledast}E$ obtained from projectable vector fields X_E on E by lifting are exact Hamiltonian, and conversely, it turns out that all exact Hamiltonian vector fields on $J^{\circledast}E$ are obtained in this way. (The last statement, analogous to a corresponding statement for cotangent bundles, is not proved in Ref. [4]; it will be derived in Ref. [12].) Now the "special covariant momentum map" of Ref. [4] associated with the symmetry under G is simply given by composing the antihomomorphism that takes generators X in $\mathfrak{g}$ to exact Hamiltonian fundamental vector fields $X_{J^{\circledast}E}$ on $J^{\circledast}E$ with the universal momentum map introduced above. Therefore, the universal momentum map comprises that part of the construction of the momentum map in field theory which does not depend on the *a priori* choice of a symmetry group or its action on the dynamical variables of the theory, and the universal multimomentum map extends that from vector fields to multivector fields.

5. Poisson Forms on Multiphase Space

Our aim in this final section is to give a series of examples for Poisson forms on the extended multiphase space $J^{\circledast}E$ of field theory. A full, systematic treatment of the subject will be given in a forthcoming separate paper [12].

As a preliminary step, we observe that there is a natural, globally defined notion of vertical vectors and of horizontal covectors on $J^{\circledast}E$. In fact, there are two such notions, one referring to the "source" projection onto space-time M and the other to the "target" projection onto the total space E of the configuration bundle. In either case, the vertical vectors are those that vanish under the tangent to the projection, while the horizontal covectors are those that vanish on all vertical vectors. In adapted local coordinates,

$$\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i^\mu} \quad \text{and} \quad \frac{\partial}{\partial p} \quad \text{are vertical with respect to the source projection,} \tag{5.1}$$

$$\frac{\partial}{\partial p_i^\mu} \quad \text{and} \quad \frac{\partial}{\partial p} \quad \text{are vertical with respect to the target projection,} \tag{5.2}$$

while

$$dx^\mu \quad \text{are horizontal with respect to the source projection,} \tag{5.3}$$

$$dx^\mu \quad \text{and} \quad dq^i \quad \text{are horizontal with respect to the target projection.} \tag{5.4}$$

This can be extended to multivectors and exterior forms, as follows. Given positive integers r and s with $s \leqslant r$, an exterior r-form is said to be s-horizontal if it vanishes whenever one inserts at least $r - s + 1$ vertical vectors (this includes the standard notion of horizontal forms by taking $s = r$), and an r-multivector is said to be s-vertical if it is annihilated by all $(r - s + 1)$-horizontal exterior forms. Using the standard expansion of multivectors and of exterior forms in adapted local coordinates, it is not difficult to see that an r-form is s-horizontal if and only if it is a linear combination of terms each of which is an exterior product containing at least s horizontal covectors and that an r-multivector is s-vertical if and only if it is a linear combination of terms each of which is an exterior product containing at least s vertical vectors. Thus for example, Eqs. (2.35)–(2.37) show that θ and ω are both $(n-1)$-horizontal with respect to the source projection and even n-horizontal with respect to the target projection, while Σ is vertical with respect to both projections.

In what follows, the terms "vertical" and "horizontal" will always refer to the source projection, except when explicitly stated otherwise.

For later use, we first write down the expansion of a general multivector field X of degree r in terms of adapted local coordinates, as follows:

$$X = \frac{1}{r!} X^{\mu_1 \ldots \mu_r} \frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} + \frac{1}{(r-1)!} X^{i,\mu_2 \ldots \mu_r} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}}$$
$$+ \frac{1}{r!} X_i^{\mu_1 \ldots \mu_r} \frac{\partial}{\partial p_i^{\mu_1}} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} + \frac{1}{(r-1)!} X_0^{\mu_2 \ldots \mu_r} \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} + \xi \,. \tag{5.5}$$

Here, all coefficients are assumed to be totally antisymmetric in their space-time indices, whereas ξ is assumed to take values in the kernel of ω. (This can always be achieved without loss of generality, because if we begin by supposing instead that ξ should contain all other terms of the standard expansion, that is, all 2-vertical terms, then ξ would contain just one group of terms that are not obviously annihilated under contraction with ω, namely the terms of the form

$$\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_k^\kappa} \wedge \frac{\partial}{\partial x^{\mu_3}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} \,.$$

However, this part of ξ can be decomposed into the sum of a term which is annihilated under contraction with ω and a linear combination of the 1-vertical terms

$$\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \frac{\partial}{\partial x^{\mu_3}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} \,,$$

so that by a redefinition of the coefficents $X_0^{\mu_2 \ldots \mu_r}$ and of ξ, we arrive at the expression for X given in Eq. (5.5), with ξ now taking values in the kernel of ω. For a more detailed discussion, see Ref. [12].) Explicitly, the contraction of ω with X then reads

$$\begin{aligned} i_X \omega = {} & \frac{1}{r!} X^{\mu_1 \ldots \mu_r} dq^i \wedge dp_i^\mu \wedge d^n x_{\mu\mu_1 \ldots \mu_r} - \frac{(-1)^r}{r!} X^{\mu_1 \ldots \mu_r} dp \wedge d^n x_{\mu_1 \ldots \mu_r} \\ & + \frac{(-1)^{r-1}}{(r-1)!} X^{i,\mu_2 \ldots \mu_r} dp_i^\mu \wedge d^n x_{\mu\mu_2 \ldots \mu_r} \\ & + \frac{(-1)^r}{r!} X_i^{\mu_1 \ldots \mu_r} dq^i \wedge d^n x_{\mu_1 \ldots \mu_r} \\ & - \frac{1}{(r-1)!} X_0^{\mu_2 \ldots \mu_r} d^n x_{\mu_2 \ldots \mu_r} \,, \end{aligned} \tag{5.6}$$

while that of θ with X reads

$$\begin{aligned} i_X \theta = {} & \frac{(-1)^r}{r!} X^{\mu_1 \ldots \mu_r} p_i^\mu dq^i \wedge d^n x_{\mu\mu_1 \ldots \mu_r} + \frac{1}{r!} X^{\mu_1 \ldots \mu_r} p \, d^n x_{\mu_1 \ldots \mu_r} \\ & + \frac{1}{(r-1)!} X^{i,\mu_2 \ldots \mu_r} p_i^\mu d^n x_{\mu\mu_2 \ldots \mu_r} \,, \end{aligned} \tag{5.7}$$

where, in each of the last two equations, the first term is to be omitted if $r = n$, whereas only the last term in the first equation remains and $i_X \theta$ vanishes identically if $r = n + 1$.

With these preliminaries out of the way, we can easily deal with the simplest case, which is that of functions.

Proposition 5.1. *A function f on $J^{\circledast}E$ is always a Poisson* 0*-form. Moreover, in adapted local coordinates, the corresponding Hamiltonian n-multivector field X is, modulo terms taking values in the kernel of ω, given by*

$$X = -\frac{1}{(n-1)!}\epsilon^{\mu_2\ldots\mu_n\mu}\left(\frac{\partial f}{\partial x^\mu}\frac{\partial}{\partial p} - \frac{1}{n}\frac{\partial f}{\partial p}\frac{\partial}{\partial x^\mu}\right)\wedge\frac{\partial}{\partial x^{\mu_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_n}}$$
$$+\frac{1}{(n-1)!}\epsilon^{\mu_2\ldots\mu_n\mu}\left(\frac{\partial f}{\partial p_i^\mu}\frac{\partial}{\partial q^i} - \frac{1}{n}\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i^\mu}\right)\wedge\frac{\partial}{\partial x^{\mu_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_n}}\,. \quad (5.8)$$

Proof. First of all, observe that for functions f, the kernel condition (3.7) is void. Next, we simplify the expression (5.6), with $r = n$, by noting that due to our conventions (2.8), (2.9) and (2.10), we have

$$d^n x_{\mu_1\ldots\mu_n} = \epsilon_{\mu_1\ldots\mu_n}\,, \quad d^n x_{\mu_2\ldots\mu_n} = \epsilon_{\mu_2\ldots\mu_n\mu}\,dx^\mu\,. \quad (5.9)$$

Thus

$$i_X\omega = -\frac{(-1)^n}{n!}\epsilon_{\mu_1\ldots\mu_n}X^{\mu_1\ldots\mu_n}\,dp + \frac{1}{(n-1)!}\epsilon_{\mu_2\ldots\mu_n\mu}X^{i,\mu_2\ldots\mu_n}\,dp_i^\mu$$
$$-\frac{1}{(n-1)!}\epsilon_{\mu_2\ldots\mu_n\mu}X_i^{\mu,\mu_2\ldots\mu_n}\,dq^i - \frac{1}{(n-1)!}\epsilon_{\mu_2\ldots\mu_n\mu}X_0^{\mu_2\ldots\mu_n}\,dx^\mu\,. \quad (5.10)$$

Equating this expression with the exterior derivative of f, we obtain the following system of equations

$$X^{\mu_1\ldots\mu_n} = (-1)^{n-1}\epsilon^{\mu_1\ldots\mu_n}\frac{\partial f}{\partial p}\,, \quad (5.11)$$

$$X^{i,\mu_2\ldots\mu_n} = \epsilon^{\mu_2\ldots\mu_n\mu}\frac{\partial f}{\partial p_i^\mu}\,, \quad (5.12)$$

$$X_i^{\mu,\mu_2\ldots\mu_n} = -\epsilon^{\mu_2\ldots\mu_n\mu}\frac{1}{n}\frac{\partial f}{\partial q^i}\,, \quad (5.13)$$

$$X_0^{\mu_2\ldots\mu_n} = -\epsilon^{\mu_2\ldots\mu_n\mu}\frac{\partial f}{\partial x^\mu}\,. \quad (5.14)$$

Inserting this back into Eq. (5.5), with $r = n$, and rearranging the terms, we arrive at Eq. (5.8). □

Remark. It has been shown in Ref. [31] that for functions h on $J^{\circledast}E$ of the special form

$$h(x^\mu, q^i, p_i^\mu, p) = -H(x^\mu, q^i, p_i^\mu) - p\,, \quad (5.15)$$

the associated Hamiltonian multivector field X can be chosen so that it defines an n-dimensional distribution in $J^{\circledast}E$ because it is locally decomposable, that is, locally there exist vector fields $X_1, \ldots, X_n$ such that $X = X_1 \wedge \cdots \wedge X_n$ satisfies the equation $i_X\omega = dh$. Indeed, setting

$$X_\mu = -\frac{\partial}{\partial x^\mu} + \frac{\partial h}{\partial p_i^\mu}\frac{\partial}{\partial q^i} - \frac{1}{n}\frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i^\mu} - \left(\frac{\partial h}{\partial x^\mu} - \frac{1}{n}\frac{\partial h}{\partial q^i}\frac{\partial h}{\partial p_i^\mu}\right)\frac{\partial}{\partial p}\,, \quad (5.16)$$

we can convince ourselves that this choice of X and the choice of X made in Eq. (5.8) differ by a term taking values in the kernel of ω. Under additional assumptions, this distribution will be integrable and its integral manifolds will be the images of sections of $J^{\circledast}E$ over M satisfying the covariant Hamiltonian equations of motion, or De Donder-Weyl equations.

Another method for constructing Poisson forms on the extended multiphase space $J^{\circledast}E$ is from Hamiltonian forms on the ordinary multiphase space $\vec{J}^{\circledast}E$, as introduced by Kanatchikov [1, 2], pulling these back to $J^{\circledast}E$ via the appropriate projection.

To describe the salient features of Kanatchikov's construction, one must first of all introduce a structure on $\vec{J}^{\circledast}E$ similar to the multisymplectic form ω that exists naturally on $J^{\circledast}E$. This requires the choice of a connection in E and of a linear connection in TM which, for the sake of convenience, will be assumed to be torsion free. Together, they induce connections in all the other bundles that are important in the multiphase space approach to field theory, including the multiphase spaces $\vec{J}^{\circledast}E$ and $J^{\circledast}E$; for the convenience of the reader, the relevant formulas in adapted local coordinates are collected in Appendix B. In the case of $\vec{J}^{\circledast}E$, this induced connection can be used to define a "vertical multisymplectic form" ω^V which is however not closed; instead, it is annihilated under the action of a "vertical exterior derivative" d^V for differential forms. In adapted local coordinates, these objects can be written in the form

$$\omega^V = e^i \wedge e_i^\mu \wedge d^n x_\mu + \cdots \tag{5.17}$$

and

$$d^V = e^i \wedge \frac{\partial}{\partial q^i} + e_i^\mu \wedge \frac{\partial}{\partial p_i^\mu} \tag{5.18}$$

respectively, where $e^i = dq^i + \Gamma_\nu^i dx^\nu$ and $e_i^\mu = dp_i^\mu - (\partial_i \Gamma_\kappa^j\, p_j^\mu - \Gamma_{\kappa\lambda}^\mu p_i^\lambda + \Gamma_{\kappa\rho}^\rho p_i^\mu) dx^\kappa$ are vertical 1-forms (with respect to the aforementioned induced connection): the dots in the definition of ω^V indicate n-horizontal terms that are not important here, while the partial derivatives in the definition of d^V are meant to act on the coefficient functions. As shown by one of the present authors [32], d^V is still a cohomology operator, i.e. it has square zero. Then the Hamiltonian forms as defined by Kanatchikov can be shown to be precisely the horizontal forms $\tilde{f}$ on $\vec{J}^{\circledast}E$ satisfying the equation

$$i_{\tilde{X}} \omega^V = d^V \tilde{f}\,, \tag{5.19}$$

where $\tilde{X}$ is a multivector field on $\vec{J}^{\circledast}E$; this relation is of course completely analogous to our equation (3.3/3.4). Moreover, Kanatchikov introduces a Poisson bracket between Hamiltonian forms $\tilde{f}$ of degree $n-r$ and $\tilde{g}$ of degree $n-s$, with multivector fields $\tilde{X}$ of degree r and $\tilde{Y}$ of degree s corresponding to $\tilde{f}$ and to $\tilde{g}$ according to Eq. (5.19), by setting

$$\{\tilde{f}, \tilde{g}\}^V = (-1)^{r(s-1)} i_{\tilde{Y}} i_{\tilde{X}} \omega^V\,. \tag{5.20}$$

This Poisson bracket satisfies the analogue of the graded Jacobi identity (3.15).

We will now show how this approach can be naturally incorporated into the multisymplectic framework used in the present paper.

Proposition 5.2. *Under the canonical projection from extended multiphase space* $J^{\circledast}E$ *to ordinary multiphase space* $\vec{J}^{\circledast}E$, *every Hamiltonian form* $\tilde{f}$ *on* $\vec{J}^{\circledast}E$ *as defined by Kanatchikov pulls back to a horizontal Poisson form* f *on* $J^{\circledast}E$. *Conversely, every horizontal Poisson form* f *of degree* > 0 *on* $J^{\circledast}E$ *is obtained in this way. Moreover, the Hamiltonian multivector field* X *on* $J^{\circledast}E$ *corresponding to* f *can be chosen so as to project to a Hamiltonian multivector field* $\tilde{X}$ *on* $\vec{J}^{\circledast}E$ *corresponding to* $\tilde{f}$.

Proof. We begin by analyzing the properties of Poisson forms f of degree $n-r$ $(0 < r < n)$ on $J^{\circledast}E$ which are horizontal. Being horizontal, such a form trivially satisfies the kernel condition (3.7) and its expansion in adapted local coordinates is

$$f = \frac{1}{r!} f^{\mu_1 \dots \mu_r} d^n x_{\mu_1 \dots \mu_r} \,,$$

implying

$$\begin{aligned} df = {} & \frac{1}{(r-1)!} \frac{\partial f^{\mu_2 \dots \mu_r \nu}}{\partial x^\nu} d^n x_{\mu_2 \dots \mu_r} + \frac{1}{r!} \frac{\partial f^{\mu_1 \dots \mu_r}}{\partial q^i} dq^i \wedge d^n x_{\mu_1 \dots \mu_r} \\ & + \frac{1}{r!} \frac{\partial f^{\mu_1 \dots \mu_r}}{\partial p_k^\kappa} dp_k^\kappa \wedge d^n x_{\mu_1 \dots \mu_r} + \frac{1}{r!} \frac{\partial f^{\mu_1 \dots \mu_r}}{\partial p} dp \wedge d^n x_{\mu_1 \dots \mu_r} \,. \end{aligned}$$

Comparing this formula with Eq. (5.6), we see that f being a Hamiltonian form implies first of all that X must be 1-vertical since the coefficients $X^{\mu_1 \cdots \mu_r}$ give a contribution to $i_X \omega$ proportional to $dq^i \wedge dp_i^\mu \wedge d^n x_{\mu \mu_1 \dots \mu_r}$ which is absent from df. But this implies that $i_X \omega$ contains no terms proportional to $dp \wedge d^n x_{\mu_1 \dots \mu_r}$ either and hence the coefficients $f^{\mu_1 \cdots \mu_r}$ cannot depend on the energy variable p; the same then goes for all the coefficients of X. Therefore, f is the pull-back of a horizontal form $\tilde{f}$ on $\vec{J}^{\circledast}E$ whereas X projects onto a 1-vertical multivector field $\tilde{X}$ on $\vec{J}^{\circledast}E$ whose expansion in terms of adapted local coordinates is given by the second and third term in Eq. (5.5). Finally, we see that with these relations between the various objects involved, Eq. (3.3, 3.4) becomes equivalent to Eq. (5.19) plus the relation

$$X_0^{\mu_2 \dots \mu_r} = -\frac{\partial f^{\mu_2 \dots \mu_r \nu}}{\partial x^\nu} \,,$$

which has no counterpart in $\vec{J}^{\circledast}E$ but also does not convey any additional information. □

Finally, the fact that the Poisson bracket (5.20) introduced by Kanatchikov, when pulled back from $\vec{J}^{\circledast}E$ to $J^{\circledast}E$, coincides with the Poisson bracket defined by Eq. (3.12) follows from the following simple observation.

Proposition 5.3. *Let* f *and* g *be two horizontal Poisson forms on* $J^{\circledast}E$ *of respective degrees* $n-r$ *and* $n-s$, *with corresponding* 1*-vertical Hamiltonian multivector* *fields X and Y of respective degrees r and s. Then the definition* (3.12) *of their Poisson bracket reduces to the pull-back of Eq.* (5.20):

$$\{f, g\} = (-1)^{r(s-1)} i_Y i_X \omega \,. \tag{5.21}$$

Proof. As we have seen in the proof of the preceding proposition, f and g being horizontal forces X and Y to be 1-vertical, so $i_Y f$ and $i_X g$ vanish. Similarly, Eq. (5.7) shows that $i_X \theta$ and $i_Y \theta$ are horizontal, so $i_Y i_X \theta$ and $i_X i_Y \theta$ vanish. Therefore, the exact correction term of Eq. (3.12) does not contribute in this case. Finally, $X \wedge Y$ will be 2-vertical, so contraction of the pull-back of ω^V or of ω with X and Y gives the same result, implying that Eq. (5.21) is really the pull-back of Eq. (5.20). □

Remark. In the case of horizontal Poisson forms, one can also introduce an associative product, which has been found by Kanatchikov [2]:

$$f \bullet g = *^{-1}(*f \wedge *g) \,, \tag{5.22}$$

where $*$ is the Hodge star operator on M associated to some metric which can be transported to horizontal forms on $J^{\circledast}E$ in an obvious manner. With respect to this product, the Poisson bracket (5.21) satisfies a graded Leibniz rule

$$\{f, g \bullet h\} = \{f, g\} \bullet h + (-1)^{(r-1)s} g \bullet \{f, h\} \,. \tag{5.23}$$

However, this product cannot be extended in any natural way to arbitrary Poisson forms. To see this, suppose we had such an extension at hand. Then we could define a space of vertical covectors at every point of $J^{\circledast}E$ by requiring it to consist of all covectors that vanish when multiplied by a horizontal $(n-1)$-form, which would be equivalent to the choice of a connection.

Appendix

A. Multivector calculus on manifolds

The extension of the usual calculus on manifolds from vector fields to multivector fields is by now well known, although it does not seem to be treated in any of the standard textbooks on the subject. Moreover, there is a certain amount of ambiguity concerning sign conventions. Our sign conventions follow those of Tulczyjew [33], but for the sake of completeness we shall briefly expose the structural properties that naturally motivate these choices.

Multivector fields of degree r on a manifold are sections of the rth exterior power of its tangent bundle: they are the dual objects to differential forms of degree r, which are sections of the rth exterior power of its cotangent bundle. Every known natural operation involving vector fields, such as the contraction on differential forms, the Lie bracket and the Lie derivative, has a natural extension to multivector fields: this is the subject of an area of differential geometry that we simply refer to as "multivector calculus". The most important and the ones that we need in this paper are (a) the Schouten bracket between multivector fields, (b) the contraction of a differential form with a multivector field and (c) the Lie derivative of a differential form along a multivector field.

Throughout this appendix, let M be an n-dimensional manifold, $\mathfrak{F}(M)$ the commutative algebra of functions on M (with respect to pointwise multiplication), $\mathfrak{X}(M)$ the space of vector fields on M and

$$\mathfrak{X}^{\wedge}(M) = \bigoplus_{r=0}^{n} \bigwedge^{r} \mathfrak{X}(M)$$

the supercommutative superalgebra of multivector fields on M (with respect to pointwise exterior multiplication).

A.1. *The Schouten bracket*

The Schouten bracket between multivector fields constitutes the natural, canonical extension both of the Lie bracket between vector fields and of the Lie derivative of multivector fields (as special tensor fields) along vector fields. Starting from the Lie derivative of multivector fields along vector fields, it can be defined by imposing a Leibniz rule with respect to the exterior product of multivector fields, as in Eq. (A.4) below.

Proposition A.1. *There exists a unique* $\mathbb{R}$*-bilinear map*

$$[\cdot\,,\cdot] : \mathfrak{X}^{\wedge}(M) \times \mathfrak{X}^{\wedge}(M) \to \mathfrak{X}^{\wedge}(M) \tag{A.1}$$

called the ***Schouten bracket***, *with the following properties.*

1. *It is homogeneous of degree* -1 *with respect to the standard tensor degree, i.e.*
$$\deg X = r\,, \quad \deg Y = s \;\Rightarrow\; \deg[X,Y] = r+s-1\,. \tag{A.2}$$
2. *It is graded antisymmetric: if* X *has tensor degree* r *and* Y *has tensor degree* s, *then*
$$[Y,X] = -(-1)^{(r-1)(s-1)}[X,Y]\,. \tag{A.3}$$
3. *It coincides with the standard Lie bracket on vector fields.*
4. *It satisfies the graded Leibniz rule: if* X *has tensor degree* r, Y *has tensor degree* s *and* Z *has tensor degree* t, *then*
$$[X, Y \wedge Z] = [X,Y] \wedge Z + (-1)^{(r-1)s}\, Y \wedge [X,Z]\,. \tag{A.4}$$
5. *It satisfies the graded Jacobi identity: if* X *has tensor degree* r, Y *has tensor degree* s *and* Z *has tensor degree* t, *then*
$$(-1)^{(r-1)(t-1)}[X,[Y,Z]] + \text{cyclic perm.} = 0\,. \tag{A.5}$$

We shall not prove this proposition here but just point out that uniqueness of an operation with the properties stipulated above follows from the required $\mathbb{R}$-bilinearity (not $\mathfrak{F}(M)$-bilinearity, of course), the homogeneity (A.2), the graded antisymmetry (A.3) and the graded Leibniz rule (A.4) alone; existence can then be proved, for example, by showing that the resulting local coordinate formula satisfies all these requirements. Moreover, the validity of the graded Jacobi identity (A.5) can be derived from the standard Jacobi identity for the Lie bracket of vector fields by means of the graded Leibniz rule (A.4), using induction on the degree.

An explicit formula which is slightly more general than the local coordinate formula just mentioned and often useful in practical applications is that for the Schouten bracket between decomposable multivector fields; it follows directly from the same kind of argument and states that for any $r+s$ vector fields $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$, we have

$$\begin{aligned}
&[X_1 \wedge \cdots \wedge X_r,\ Y_1 \wedge \cdots \wedge Y_s] \\
&\quad = \sum_{i=1}^{r} \sum_{j=1}^{s} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge X_{i-1} \wedge X_{i+1} \wedge \cdots \wedge X_r \\
&\qquad\qquad \wedge Y_1 \wedge \cdots \wedge Y_{j-1} \wedge Y_{j+1} \wedge \cdots \wedge Y_s\,. \qquad \text{(A.6)}
\end{aligned}$$

Note also that there is a graded Leibniz rule in the other factor as well: it follows from the one written down above by using graded antisymmetry and reads

$$[X \wedge Y, Z] = (-1)^{(t-1)s} [Z, X] \wedge Y + X \wedge [Y, Z]\,. \qquad \text{(A.7)}$$

Finally, a word seems in order on the adequate choice of signs and degrees. Indeed, one recognizes Eqs. (A.2), (A.3) and (A.5) as the graded homogeneity, the graded antisymmetry and the graded Jacobi identity familiar from the definition of a Lie superalgebra, provided one assigns to every multivector field X of tensor degree r the parity $(-1)^{r-1}$: this means that X is even with respect to the Schouten bracket if it has odd tensor degree and is odd with respect to the Schouten bracket if it has even tensor degree! This switch can be better understood by realizing that the operator $\mathrm{ad}(X) = [X, .]$ lowers the tensor degree of any multivector field that it operates on by $r-1$. The same argument explains the sign that appears in the graded Leibniz identity (A.4), which can be thought of as stating that the operator $\mathrm{ad}(X) = [X, .]$ should be a superderivation with respect to the exterior product and, more precisely, an even or odd superderivation according to whether X is even or odd with respect to the Schouten bracket. We can also think of this operator as defining the Lie derivative L_X of multivector fields along X (possibly up to signs, which are a matter of convention), but this will not be needed here.

Algebraically, the situation can be summarized by stating that $\mathfrak{X}^\wedge(M)$ is a *Poisson superalgebra*, the supersymmetric analogue of a Poisson algebra — the structure encountered, for example, on the space of functions on a symplectic manifold or, more generally, a Poisson manifold. The surprising aspect is that this intricate structure requires *no* additional structure whatsoever on the underlying manifold.

A.2. Lie derivative of differential forms along multivector fields

We now come to the other two operations of multivector calculus mentioned at the beginning of this appendix, namely the contraction of differential forms with multivector fields and the Lie derivative of differential forms along multivector fields.

The case of contraction is easy. First, the contraction of a differential form α with a decomposable multivector field $X_1 \wedge \cdots \wedge X_r$ is simply defined as repeated contraction with its constituents (which by convention should be performed in the opposite order):

$$i_{X_1 \wedge \cdots \wedge X_r} \alpha = i_{X_r} \ldots i_{X_1} \alpha \,. \tag{A.8}$$

This is then extended to arbitrary (non-decomposable) multivector fields X by $\mathfrak{F}(M)$-linearity. (Here, of course, one uses that contraction is a purely algebraic operation; it would not work so naively if we were dealing with a differential operator.)

The Lie derivative $L_X \alpha$ of a differential form α along a multivector field X is most conveniently defined by a generalization of a well known formula for vector fields.

Definition A.2. On differential forms, the Lie derivative L_X along a multivector field X is defined as the supercommutator of the exterior derivative d and the contraction operator i_X:

$$L_X \alpha = d i_X \alpha - (-1)^r i_X d\alpha \,. \tag{A.9}$$

According to the rules of supersymmetry, the sign of the second term is fixed by observing that d is an odd operator (it is of degree 1 since it raises the tensor degree of forms by 1) while i_X is an even/odd operator if r is even/odd (it is of degree $-r$ since it lowers the tensor degree of forms by r).

Proposition A.3. *Given any two multivector fields X of tensor degree r and Y of tensor degree s, we have for any differential form α*

$$d L_X \alpha = (-1)^{r-1} L_X d\alpha \,, \tag{A.10}$$

$$i_{[X,Y]} \alpha = (-1)^{(r-1)s} L_X i_Y \alpha - i_Y L_X \alpha \,. \tag{A.11}$$

$$L_{[X,Y]} \alpha = (-1)^{(r-1)(s-1)} L_X L_Y \alpha - L_Y L_X \alpha \,. \tag{A.12}$$

$$L_{X \wedge Y} \alpha = (-1)^s i_Y L_X \alpha + L_Y i_X \alpha \,. \tag{A.13}$$

Proof. The first formula is an immediate consequence of the definition (A.9), since $d^2 = 0$. Next, the last formula can be proved by direct calculation:

$$\begin{aligned} L_{X \wedge Y}\alpha &= d(i_{X \wedge Y}\alpha) - (-1)^{r+s} i_{X \wedge Y} d\alpha \\ &= d(i_Y i_X \alpha) - (-1)^{r+s} i_Y i_X d\alpha \\ &= d(i_Y i_X \alpha) - (-1)^s i_Y d(i_X \alpha) \\ &\quad + (-1)^s i_Y d(i_X\alpha) - (-1)^{r+s} i_Y i_X d\alpha \\ &= L_Y i_X \alpha + (-1)^s i_Y L_X \alpha\,. \end{aligned}$$

Next, observe that the first formula is well known to be true when X and Y are vector fields. The general case follows by induction on the tensor degree of both factors. Indeed, if X, Y and Z are multivector fields of tensor degree r, s and t, respectively, such that the above equation holds for $[X,Y]$ and for $[X,Z]$, one can use the graded Leibniz rule (A.4) to derive that it also holds for $[X, Y \wedge Z]$:

$$\begin{aligned} i_{[X, Y \wedge Z]}\alpha &= i_{[X,Y] \wedge Z}\alpha + (-1)^{(r-1)s} i_{Y \wedge [X,Z]}\alpha \\ &= i_Z i_{[X,Y]}\alpha + (-1)^{(r-1)s} i_{[X,Z]} i_Y \alpha \\ &= (-1)^{(r-1)s} i_Z L_X i_Y \alpha - i_Z i_Y L_X \alpha \\ &\quad + (-1)^{(r-1)s+(r-1)t} L_X i_Z i_Y \alpha - (-1)^{(r-1)s} i_Z L_X i_Y \alpha \\ &= (-1)^{(r-1)(s+t)} L_X i_{Y \wedge Z}\alpha - i_{Y \wedge Z} L_X \alpha\,. \end{aligned}$$

Similarly, if X, Y and Z are multivector fields of tensor degree r, s and t, respectively, such that the above equation holds for $[X,Z]$ and for $[Y,Z]$, one can use the graded Leibniz rule (A.7) together with Eq. (A.13) to derive that it also holds for $[X \wedge Y, Z]$:

$$\begin{aligned} i_{[X \wedge Y, Z]}\alpha &= (-1)^{(t-1)s} i_{[X,Z] \wedge Y}\alpha + i_{X \wedge [Y,Z]}\alpha \\ &= (-1)^{(t-1)s} i_Y i_{[X,Z]}\alpha + i_{[Y,Z]} i_X \alpha \\ &= (-1)^{(t-1)s+(r-1)t} i_Y L_X i_Z \alpha - (-1)^{(t-1)s} i_Y i_Z L_X \alpha \\ &\quad + (-1)^{(s-1)t} L_Y i_Z i_X \alpha - i_Z L_Y i_X \alpha \\ &= (-1)^{(r+s-1)t+s} i_Y L_X i_Z \alpha - (-1)^s i_Z i_Y L_X \alpha \\ &\quad + (-1)^{(r+s-1)t} L_Y i_X i_Z \alpha - i_Z L_Y i_X \alpha \\ &= (-1)^{(r+s-1)t} L_{X \wedge Y} i_Z \alpha - i_Z L_{X \wedge Y}\alpha\,. \end{aligned}$$

Finally, the second formula can now again be proved by direct calculation:

$$\begin{aligned} L_{[X,Y]}\alpha &= d i_{[X,Y]}\alpha + (-1)^{r+s} i_{[X,Y]} d\alpha \\ &= (-1)^{(r-1)s} d L_X i_Y \alpha - d i_Y L_X \alpha \\ &\quad + (-1)^{r(s-1)} L_X i_Y d\alpha - (-1)^{r+s} i_Y L_X d\alpha \end{aligned}$$

$$= (-1)^{(r-1)(s-1)} L_X d i_Y \alpha - d i_Y L_X \alpha$$
$$+ (-1)^{r(s-1)} L_X i_Y d\alpha + (-1)^s i_Y d L_X \alpha$$
$$= (-1)^{(r-1)(s-1)} L_X L_Y \alpha - L_Y L_X \alpha \,. \qquad \square$$

B. Induced connections

In this appendix we want to describe briefly the construction of various induced connections in jet bundle language.

First of all, if E is a fiber bundle over M, we shall view a connection in E as a section Γ_E of the first order jet bundle JE of E, considered as an affine bundle over E; see [34, Ch. IV.17]. In adapted local coordinates (x^μ, q^i) for E and (x^μ, q^i, q^i_μ) for JE, this section is given by

$$\Gamma_E : (x^\mu, q^i) \mapsto (x^\mu, q^i, \Gamma^i_\mu(x, q)) \,.$$

Next, if V is a vector bundle over M, a linear connection in V is given by a section Γ_V of JV over V that depends linearly on the fiber coordinates. In adapted local coordinates (x^μ, v^i) for V and (x^μ, v^i, v^i_μ) for JV, this section is given by

$$\Gamma_V : (x^\mu, v^i) \mapsto (x^\mu, v^i, \Gamma^i_{\mu j}(x)\, v^j) \,,$$

where the $\Gamma^i_{\mu,j}$ are of course the connection coefficients (gauge potentials) associated with the corresponding covariant derivative. In particular, a linear connection in the tangent bundle TM of the base manifold M corresponds to a section Γ_{TM} of $J(TM)$ over TM which, in adapted local coordinates $(x^\mu, \dot{x}^\kappa)$ for TM and $(x^\mu, \dot{x}^\kappa, \dot{x}^\kappa_\mu)$ for $J(TM)$ is given by

$$\Gamma_{TM} : (x^\mu, \dot{x}^\kappa) \mapsto (x^\mu, \dot{x}^\kappa, \Gamma^\kappa_{\mu\lambda}(x)\, \dot{x}\lambda) \,,$$

where the $\Gamma^\kappa_{\mu\lambda}$ are of course the corresponding Christoffel symbols.

Now given a fiber bundle E over M together with a connection in E and a linear connection in TM, we can introduce induced connections in all the various induced bundles that appear in this paper — regarded as fiber bundles over M, not over E. (This means that jets of sections will contain just one additional lower space-time index for counting partial derivatives with respect to the space-time variables.) The simplest way to describe them is by introducing adapted local coordinates (x^μ, q^i) for E as before; then the local coefficient functions of the induced connections with respect to the induced adapted local coordinates can be expressed directly in terms of the local coefficient functions Γ^i_μ and $\Gamma^\kappa_{\mu\lambda}$ of the original two connections with respect to the original adapted local coordinates, as follows.

- The vertical bundle VE of E:
  in adapted local coordinates $(x^\mu, q^i, \dot{q}^k)$ for VE and $(x^\mu, q^i, \dot{q}^k, q^i_\mu, \dot{q}^k_\mu)$ for $J(V^*E)$, the induced connection maps $(x^\mu, q^i, \dot{q}^k)$ to
  $$(x^\mu, q^i, \dot{q}^k, \Gamma^i_\mu(x, q), \partial_l \Gamma^k_\mu(x, q) \dot{q}^l) \,.$$

- The dual vertical bundle V^*E of E:
  in adapted local coordinates (x^μ, q^i, p_k) for V^*E and $(x^\mu, q^i, p_k, q^i_\mu, p_{\mu,k})$ for $J(V^*E)$, the induced connection maps (x^μ, q^i, p_k) to

$$(x^\mu, q^i, p_k, \Gamma^i_\mu(x,q), -\partial_k \Gamma^l_\mu(x,q) p_l)\,.$$

- The pull-back $\pi^*(TM)$ of the tangent bundle TM of M to E:
  in adapted local coordinates $(x^\mu, q^i, \dot{x}^\kappa)$ for $\pi^*(TM)$ and $(x^\mu, q^i, \dot{x}^\kappa, q^i_\mu, \dot{x}^\kappa_\mu)$ for $J(\pi^*(TM))$, the induced connection maps $(x^\mu, q^i, \dot{x}^\kappa)$ to

$$(x^\mu, q^i, \dot{x}^\kappa, \Gamma^i_\mu(x,q), \Gamma^\kappa_{\mu\lambda}(x)\dot{x}^\lambda)\,.$$

- The pull-back $\pi^*(T^*M)$ of the cotangent bundle T^*M of M to E:
  in adapted local coordinates $(x^\mu, q^i, \alpha_\kappa)$ for $\pi^*(T^*M)$ and $(x^\mu, q^i, \alpha_\kappa, q^i_\mu, \alpha_{\mu,\kappa})$ for $J(\pi^*(T^*M))$, the induced connection maps $(x^\mu, q^i, \alpha_\kappa)$ to

$$(x^\mu, q^i, \alpha_\kappa, \Gamma^i_\mu(x,q), -\Gamma^\lambda_{\mu\kappa}(x)\alpha_\lambda)\,.$$

- The pull-back $\pi^*(\bigwedge^n T^*M)$ of the bundle $\bigwedge^n T^*M$ of volume forms on M to E:
  in adapted local coordinates (x^μ, q^i, ϵ) for $\pi^*(\bigwedge^n T^*M)$ and $(x^\mu, q^i, \epsilon, q^i_\mu, \epsilon_\mu)$ for $J(\pi^*(\bigwedge^n T^*M))$, the induced connection maps (x^μ, q^i, ϵ) to

$$(x^\mu, q^i, \epsilon, \Gamma^i_\mu(x,q), -\Gamma^\rho_{\mu\rho}(x)\epsilon)\,.$$

- The linearized jet bundle $\vec{J}E$ of E:
  in adapted local coordinates $(x^\mu, q^i, \vec{q}^{\,k}_\kappa)$ for $\vec{J}E$ and $(x^\mu, q^i, \vec{q}^{\,k}_\kappa, q^i_\mu, \vec{q}^{\,k}_{\mu,\kappa})$ for $J(\vec{J}E)$, the induced connection maps $(x^\mu, q^i, \vec{q}^{\,k}_\kappa)$ to

$$(x^\mu, q^i, \vec{q}^{\,k}_\kappa, \Gamma^i_\mu(x,q), \partial_l \Gamma^k_\mu(x,q)\vec{q}^{\,l}_\kappa - \Gamma^\lambda_{\mu\kappa}(x)\vec{q}^{\,k}_\lambda)\,.$$

- The jet bundle JE of E:
  in adapted local coordinates (x^μ, q^i, q^k_κ) for JE and $(x^\mu, q^i, q^k_\kappa, q^i_\mu, q^k_{\mu,\kappa})$ for $J(JE)$, the induced connection maps (x^μ, q^i, q^k_κ) to

$$(x^\mu, q^i, q^k_\kappa, \Gamma^i_\mu(x,q), \partial_l \Gamma^k_\mu(x,q)(q^l_\kappa - \Gamma^l_\kappa(x,q)) - \Gamma^\lambda_{\mu\kappa}(x)(q^k_\lambda - \Gamma^k_\lambda(x,q)))\,.$$

- Ordinary multiphase space $\vec{J}^{\circledast}E$:
  in adapted local coordinates (x^μ, q^i, p^κ_k) for $\vec{J}^{\circledast}E$ and $(x^\mu, q^i, p^\kappa_k, q^i_\mu, p^\kappa_{\mu,k})$ for $J(\vec{J}^{\circledast}E)$, the induced connection maps (x^μ, q^i, p^κ_k) to

$$(x^\mu, q^i, p^\kappa_k, \Gamma^i_\mu(x,q), -\partial_k \Gamma^l_\mu(x,q)p^\kappa_l + \Gamma^\kappa_{\mu\lambda}(x)p^\lambda_k - \Gamma^\rho_{\mu\rho}(x)\,p^\kappa_k)\,.$$

- Extended multiphase space $J^{\circledast}E$:
  in adapted local coordinates $(x^\mu, q^i, p^\kappa_k, p)$ for $J^{\circledast}E$ and $(x^\mu, q^i, p^\kappa_k, p, q^i_\mu, p^\kappa_{\mu,k}, p_\mu)$ for $J(J^{\circledast}E)$, the induced connection maps $(x^\mu, q^i, p^\kappa_k, p)$ to

$$\begin{aligned}(x^\mu, q^i, p^\kappa_k, p, \Gamma^i_\mu(x,q), &-\partial_k \Gamma^l_\mu(x,q)p^\kappa_l + \Gamma^\kappa_{\mu\lambda}(x)p^\lambda_k - \Gamma^\rho_{\mu\rho}(x)p^\kappa_k\,,\\ &- \Gamma^\rho_{\mu\rho}(x)p - (\partial_\mu \Gamma^j_\nu(x,q) - \Gamma^k_\nu(x,q)\partial_k \Gamma^j_\mu(x,q) - \Gamma^\kappa_{\mu\nu}(x)\Gamma^j_\kappa(x,q))p^\nu_j)\,.\end{aligned}$$

Table 1. Correspondence of important concepts in the multiphase space approach: time-dependent mechanics versus field theory.

Mechanics	Field Theory
Extended configuration space $\mathbb{R} \times Q$, where $\mathbb{R}$ is the time axis	Configuration bundle E over M with typical fibre Q, where M is the space-time manifold
Extended velocity space $\mathbb{R} \times TQ$	Velocity bundle: jet bundle JE
Doubly extended phase space $\mathcal{P} = T^*(\mathbb{R} \times Q) = \mathbb{R} \times T^*Q \times \mathbb{R}$	Extended multiphase space: twisted affine dual of JE $\mathcal{P} = J^{\circledast}E = J^{\star}E \otimes \bigwedge^n T^*M$
Simply extended phase space $\mathcal{P}_0 = \mathbb{R} \times T^*Q$	Ordinary multiphase space: twisted linear dual of $\vec{J}E$ $\mathcal{P}_0 = \vec{J}^{\circledast}E = \vec{J}^*E \otimes \bigwedge^n T^*M$
Local coordinates for $\mathbb{R} \times Q$ t, q^i Local coordinates for $\mathbb{R} \times TQ$ $t, q^i, \dot{q}^i$ Local coordinates for $\mathcal{P}$ t, q^i, p_i, E Local coordinates for $\mathcal{P}_0$ t, q^i, p_i Projection from $\mathcal{P}$ to $\mathcal{P}_0$ $(t, q^i, p_i, E) \mapsto (t, q^i, p_i)$	Local coordinates for E x^μ, q^i Local coordinates for JE x^μ, q^i, q^i_μ Local coordinates for $\mathcal{P}$ x^μ, q^i, p^μ_i, p Local coordinates for $\mathcal{P}_0$ x^μ, q^i, p^μ_i Projection from $\mathcal{P}$ to $\mathcal{P}_0$ $(x^\mu, q^i, p^\mu_i, p) \mapsto (x^\mu, q^i, p^\mu_i)$
Canonical 1-form on $\mathcal{P}$ $\theta = p_i dq^i + E dt$	Multicanonical n-form on $\mathcal{P}$ $\theta = p^\mu_i dq^i \wedge d^n x_\mu + p\ d^n x$
Symplectic 2-form $\omega = -d\theta$ on $\mathcal{P}$, non-degenerate $\omega = dq^i \wedge dp_i - dE \wedge dt$	Multisymplectic $(n+1)$-form $\omega = -d\theta$ on $\mathcal{P}$, non-degenerate (on vector fields) $\omega = dq^i \wedge dp^\mu_i \wedge d^n x_\mu - dp \wedge d^n x$
Hamiltonian is a function on $\mathcal{P}_0$	Hamiltonian is a section of $\mathcal{P}$ (as an affine line bundle over $\mathcal{P}_0$)
$i_X \omega = df$ Hamiltonian vector fields X $\leftrightarrow$ functions f	$i_X \omega = df$ Hamiltonian r-multivector fields X $\leftrightarrow$ Hamiltonian or Poisson $(n-r)$-forms f
Poisson bracket for functions $f, g \in C^\infty(\mathcal{P})$ $\{f, g\} = L_Y f - L_X g$	Poisson bracket for Poisson forms $f \in \Omega^{n-r}_P(\mathcal{P})$, $g \in \Omega^{n-s}_P(\mathcal{P})$ $\{f, g\} = (-1)^{(r-1)(s-1)} L_Y f - L_X g$ $-(-1)^{(r-1)s} L_{X \wedge Y} \theta$
Hamiltonian equations $\frac{\partial H}{\partial p_i} = \dot{q}^i, \frac{\partial H}{\partial q^i} = -\dot{p}_i$	De Donder-Weyl equations $\frac{\partial H}{\partial p^\mu_i} = \frac{\partial q^i}{\partial x^\mu}, \frac{\partial H}{\partial q^i} = -\frac{\partial p^\mu_i}{\partial x^\mu}$

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