NEW GRAVITATIONAL INSTANTONS AND UNIVERSAL SPIN STRUCTURES ☆

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An infinite class of new gravitational instantons for the axial anomaly is found. It consists entirely of algebraic spin-manifolds. In theories that allow manifolds without ordinary spin structure we find the presence of spinorial matter fields to require the existence of a "universal" gauged SU(2) or higher internal symmetry (e.g., SU(2) × SU(2) × G) and of an "internal-spin"-statistics connection. The possible relation of this to the gauge theory of weak and electromagnetic interactions is explored.

1 . Introduction. The transition from classical to quantum gravity is performed by a functional integration over matter and gravitational fields, thus over "world manifolds". In Einstein theory "world manifolds" are assumed (pseudo-)riemannian, and as such do not always possess a spin structure. The existence of spinorial matter fields being beyond doubt, a consistent quantum theory then either:

(A) restricts the functional integral to manifolds that have a spin structure: spin manifolds, or

(B) requires all matter fields to appear in suitable multiplets of a gauged symmetry that permits the definition of a generalized spin structure on *all* 4-dimensional (pseudo-)riemann manifolds.

We shall explore both these possibilities. ‡1. We find possibility (A) automatically realized in all forms of supergravity theory (ordinary, extended conformal).

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‡1 Alternative (A) can be readily implemented for both Riemann and pseudo-Riemann manifolds (i.e., for lorentzian and "euclidean" metric) as it is a simple topological criterion: integrate only over manifolds with null second Stiefel–Whitney class. Concerning alternative (B) we shall discuss it for the Riemann (++++ metric) case. All manifolds are assumed orientable.

Possibility (B), on the other hand, leads to "universal" internal symmetries and multiplet assignments quite similar to those appearing in unified weak–electromagnetic gauge theory models currently in circulation. These two alternatives differ considerably as to the instantons responsible for the gravitational contribution to the axial anomaly. In particular, Eguchi and Freund [1] have proposed that the complex projective plane $P_2(C)$ – a four-dimensional real manifold – is such an instanton. But $P_2(C)$ is not a spin manifold, so that it does not qualify under alternative (A). Under alternative (B) it does qualify as in this case there even exists a U(1) extended Spin^c structure [2] as has been noted by Hawking and Pope [3] (these authors have also contemplated nonabelian extensions for spin structures although not of the universal type to be presented below). For alternative (A) one is then robbed of the only known gravitational instanton (for the axial anomaly). We therefore find new instantons corresponding to certain algebraic hypersurfaces in $P_3(C)$. These are endowed with spin structure and as such are instantons for both alternatives (A) and (B).

2. New gravitational instantons. As a motivation for alternative (A) we consider supergravity theories that gauge the supergroups $OSp(N|4)$ [4] and $SU(N|2,2)$ [5] with $N \leqslant 8$. The Lorentz group is contained in $OSp(N|4)$ via the $Sp(4)$ de Sitter group and in $SU(N|2,2)$ via the $SU(2,2)$ conformal group. $Sp(4)$ and $SU(2,2)$ both contain Spin (3,1) which thus acts in the fibers of the supergravity bundle. The corresponding space–time manifolds are then automatically constrained to be spin manifolds and alternative (A) must hold. In any theory that implements alternative (A) a gravitational instanton (in this paper always for the axial anomaly) corresponds to a 4-dimensional compact spin manifold with positive metric form and first Pontryagin number $p_1 \neq 0$. To make significant contribution to vacuum tunnelling the manifold should also be Einstein (i.e., a solution of Einstein's equations with cosmological term). $P_2(C)$ is not an instanton (for alternative (A)), as it is not a spin manifold. We therefore construct here a new – and in a certain sense fundamental – class of gravitational instantons that are spin manifolds. Consider complex projective 3-space $P_3(C)$. Let z_0, z_1, z_2, z_3 be its homogeneous (complex) coordinates, and $F_m(z_i)$ a homogeneous polynomial of degree m in the z_i. The equation $F_m(z_i) = 0$ defines a degree m hypersurface V_m in $P_3(C)$. If the four vector $\partial F/\partial z^i$ is nonzero for points $z \neq 0$, then V_m will itself be a Kähler surface (i.e., real dimension 4). If X is the multiple of the Kähler form of $P_3(C)$ that integrates to 1 over $P_1(C)$ and i is the inclusion of V_m into $P_3(C)$, then the pullback i^*X gives a multiple of the Kähler form of V_m. Using standard procedures [6] (the first Chern class of the normal line bundle to V_m is Pioncaré dual to the 4-dimensional fundamental class of V_m [6]) the total Chern class of V_m is represented by

$$1 + c_1 + c_2 = i^*[1 + X)^4(1 + mX)^{-1}]$$
$$= 1 + (4 - m)i^*X + (m^2 - 4m + 6)i^*X^2 \,.$$

Computing the first Pontryagin class $p_1 = c_1^2 - 2c_2$ and noting that the surface is of degree m, we find that the signature (or index) τ of V_m is

$$\tau(V_m) = \tfrac{1}{3}\int_{V_m} p_1 = 16(1 + \tfrac{1}{2}m)(\tfrac{1}{2}m)(1 - \tfrac{1}{2}m)/6 \,.$$

For V_m to be a spin manifold, its second Stiefel–Whitney class w_2 must vanish. But w_2 is the Z_2 (Z_2 = integers modulo 2) reduction of c_1, so $w_2 = 0$ requires that c_1 be even. Thus for all even $m = 2n$, V_m is a spin manifold. Notice in this case,

$$\tau(V_m) \equiv -8\hat{A}(V_m) = -16(n+1)n(n-1)/6 \,,$$

where the $\hat{A}$ genus $\hat{A}(V_{2n})$ is (as always for an $8k$+4-dimensional spin manifold [7]) an *even* integer. By the Atiyah–Singer index theorem for the Dirac operator,

$$n_R - n_L = -\frac{1}{384\pi^2}\int_{V_{2n}} R_{\mu\nu\alpha\beta}\,{}^*R^{\mu\nu\alpha\beta}\,d^4X$$
$$= -\tfrac{1}{8}\tau(V_{2n}) = \hat{A}(V_{2n})$$

(where ${}^*R^{\mu\nu\alpha\beta} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta}/\sqrt{g}$, and $n_{R(L)}$ is the number of right- (left-) handed bound states) is an even integer. To have an instanton, τ must not vanish: $n \geqslant 2$. For $n = 2$ ($m = 4$), $c_1 = 0$ and V_4 is known in algebraic geometry as a K3 surface. Since the first Chern class of a Kähler manifold is represented by the Ricci form, Yau's recent affirmative solution [8] of the Calabi conjecture show that K3 surfaces can be endowed with a metric that is a solution of Einstein's equations without cosmological term. Unfortunately, the Einstein metric on V_4 is *not* the induced $P_3(C)$ metric [9], and in fact has yet to be found. For $m = 2n > 4$, V_m is not an Einstein manifold in the induced metric from $P_3(C)$ either, but is "cohomologically" Einstein (i.e., as cohomology classes, the first Chern class is a multiple of the Kähler form induced from $P_3(C)$). V_m also satisfies Hitchin's necessary conditions [10] for the existence of an Einstein metric.

V_4 is in some sense a fundamental 4-dimensional spin manifold. Namely [11], if M is any 4-dimensional spin manifold, then there will be a 5-dimensional spin manifold W whose boundary is the union of M and of r copies of the K3 surface where r is $-\tau(M)/16$, i.e., M is spin-cobordant to rV_4.

3. *Universal spin structures.* In a theory like unextended $OSp(1|4)$-supergravity there is no internal symmetry and therefore there are no vector gauge fields. There are spinors so that one is forced into the alternative (A) discussed in the previous section. Yet, supergravity is far from being experimentally established and one may want to consider alternative ways of accomodating spinors in a quantum theory of gravity. Take again the case of $P_2(C)$. This is no spin manifold, yet it can be given a Spin^c structure. Essentially, this is

done [3] by having the matter fields carry a conserved charge, the corresponding current serving as the source of an abelian gauge field. The extra gauge phase freedom then allows the consistent definition of spinors on $P_2(C)$, provided one has a "charge-statistics" connection: fermions (i.e., spinors) carry odd (integer) values while bosons (i.e., tensors) carry even values of the charge. We then ask whether there exists a larger (non-abelian) gauge group G such that coupling all matter fields to gauge fields of G, *all* 4-dimensional Riemann manifolds can be given a generalized spin structure, again provided a certain "internal-spin"-statistics connection is enforced. We now proceed to answer this question in the affirmative and will show that in 4 dimensions the group G must be at least SU(2) or more realistically SU(2) × SU(2) or SU(2) × SU(2) × U(1).

To present our argument, let us briefly recall [2] the Spin^c structure of $P_2(C)$, as it clearly illustrates the reasoning involved. As a 4-dimensional real Riemann manifold $P_2(C)$ has a principal SO(4) bundle of orthonormal frames. Being also a 2-dimensional complex manifold it has a U(2) bundle of frames. There exists a natural homomorphism ν of U(2) into SO(4). Consider the group Spin(4) × U(1). For every $g \in$ Spin(4), $h \in$ U(1), the relation $(g, h) \sim (-g, -h)$ between elements of Spin(4) × U(1) defines a set of equivalence classes endowed with group structure. The corresponding group is $\mathrm{Spin}^c(4) \equiv \mathrm{Spin}(4) \times_{Z_2} \mathrm{U}(1) \equiv (\mathrm{Spin}(4) \times \mathrm{U}(1))/Z_2$ (Z_2 indicates the effect of the Z_2 equivalence relation). There exists a homomorphism κ: U(2) → $\mathrm{Spin}^c(4)$ which allows the U(2)-bundle to be extended to a $\mathrm{Spin}^c(4)$-bundle. But the Spin(4) part of $\mathrm{Spin}^c(4)$ can be projected down to SO(4) by a two-to-one map π. It can be checked that the composition map $\pi \circ \kappa$ from U(2) to SO(4) is the same as the natural map ν. One has thus upgraded the SO(4)-bundle to a $\mathrm{Spin}^c(4)$-bundle and the key to all this was the complex structure of $P_2(C)$. The price for this Spin^c structure is a remarkable "charge-statistics" connection in the multiplet spectrum, noticed by Hawking and Pope [3]. It emerges from the following consistency requirement: if ρ is a representation of $\mathrm{Spin}^c(4)$ on a vector space V, then pulling it back to a Spin(4) × U(1) = SU(2) × SU(2) × U(1) representation, the (−1, −1, −1) element of SU(2) × SU(2) × U(1) must map to the identity within the representation ρ. Label ρ by the spins j_1 and j_2 of the two SU(2) factors, and the charge q of the U(1) factor. The consistency requirement is then $j_1 + j_2 + q/2$ = integer so that fermions ($j_1 + j_2$ = half odd integer) must carry odd (integer) charge q, and bosons ($j_1 + j_2$ = integer) even charge. This is the "charge-statistics" connection.

Unlike $P_2(C)$, a general 4-dimensional Riemann manifold is not endowed with a complex structure. So, the U(2)-bundle is not available and one must make do with the SO(4)-bundle of orthonormal frames. There being no nontrivial homomorphism of SO(4) into U(1) it is unlikely that a $\mathrm{Spin}^c(4)$ structure will in general be possible. We therefore attempt a more general structure of the form $\mathrm{Spin}(4) \times_{Z_2} G$ and choose the nonabelian group G so that a homomorphism Spin(4) → G exists. The simplest choice G = SU(2) works. Indeed, there are two homomorphisms from $\overline{\mathrm{Spin}}(4)$ = SU(2) × SU(2) to Spin(4) × SU(2) involving the identity map to Spin (4) and projection on one of the two SU(2) factors as the map to SU(2). SO(4) and $\overline{\mathrm{Spin}}(4) \equiv \mathrm{Spin}(4) \times_{Z_2} \mathrm{SU}(2)$ differ from Spin(4) and Spin(4) × SU(2) each by a Z_2 factor. So, this way we also have two homomorphisms SO(4) → $\overline{\mathrm{Spin}}(4)$ with left inverse the projection π introduced above from the Spin(4) part of $\overline{\mathrm{Spin}}(4)$ to SO(4). Either of these homomorphisms upgrades the SO(4)-bundle into a $\overline{\mathrm{Spin}}(4)$-bundle. Thus, with a gauged internal SU(2)-symmetry one can define spinors on any 4-dimensional Riemann manifold. Physically one may attempt to identify this with the "weak isospin" SU(2) factor of the unified electromagnetic–weak gauge group. The corresponding "weak isospin-statistics" connection requires bosons (fermions) to have integer (half-odd integer) weak isospin, and one could accomodate neither the Higgs fields nor the right-handed leptons of the Weinberg–Salam model [12]. But, one can choose a larger group in constructing the universal spin structure For instance, instead of the "minimal" SU(2) we choose G = SU(2) × SU(2) and obtain $\overline{\overline{\mathrm{Spin}}}(4) = \mathrm{Spin}(4) \times_{Z_2} (\mathrm{SU}(2) \times \mathrm{SU}(2))$. The diagonal map Spin(4) → Spin(4) × Spin(4) produces a homomorphism SO(4) → $\overline{\overline{\mathrm{Spin}}}(4)$. Identify this internal SU(2) × SU(2) with the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ of electromagnetic–weak gauge models. The "weak isospin-statistics" connection then requires fermions (bosons) to belong to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ multiplets (k_L, k_R) with $k_L + k_R$ half odd integer (integer). This allows fermions in (1/2, 0), (0, 1/2), bosons in (1, 0), (0, 1), (0, 0), (1/2, 1/2), but for instance no fermion singlets. This differs somewhat from various phenomenological schemes [13]. Yet G

may be further enlarged by $SU(3)_{color}$ and one or more U(1) factors, with new "internal-spin"-statistics connections in each case.

4. Conclusions. We found a new family of gravitational instantons the V_{2n} spin manifolds. With alternative (A) they are, in a sense, fundamental as explained in section 3. Theories that naturally implement alternative (A) are the various supergravity theories. They are known to determine the gauged internal symmetry and the multiplet spectrum. Remarkably, with alternative (B) we again obtained information on the gauged internal symmetry and on the multiplet spectrum, this time in the form of an "internal-spin"-statistics connection.

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