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HIGHER CONSERVATION LAWS FOR TEN-DIMENSIONAL SUPERSYMMETRIC YANG-MILLS THEORIES

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It is shown that ten-dimensional supersymmetric Yang-Mills theories are integrable systems, in the (weak) sense of admitting a (superspace) Lax representation for their equations of motion This is achieved by means of an explicit proof that the equations of motion are not only a consequence of but in fact fully equivalent to the superspace constraint $F_{\alpha\beta} = 0$ Moreover, a procedure for deriving infinite series of non-local conservation laws is outlined

1. Introduction

One of the corner-stones in the development of mathematical physics has been the discovery of the inverse scattering method, which has opened the way to extending the notion of a completely integrable system from mechanics to field theory (and also to statistical mechanics) In two-dimensional space-time, many such systems have meanwhile been identified, and a great variety of exact results – most of them inaccessible from perturbation theory – has been obtained explicit solutions (e.g., solitons), infinite series of conservation laws (both local and non-local), Bäcklund transformations, hidden symmetries related to Kac-Moody algebras, etc Moreover, in spite of large differences between the various models, and even though, in field theory, the notion of integrability is not quite unambiguously defined, there is one generally accepted starting point which all integrable systems

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have in common this is the existence of a Lax representation, i e the possibility to rewrite the equations of motion of the model under consideration as the compatibility conditions for a certain system of linear partial differential equations. In most cases, such compatibility conditions take the form of zero-curvature constraints on a certain connection, defined in terms of the dynamical variables of the model, and usually depending on additional "spectral" parameters.

An especially interesting class of models that fit into this scenario are the two-dimensional non-linear σ models on symmetric spaces Although these models are unlikely to be completely integrable in the strict sense, a Lax representation does exist [1, 2], and can be used to derive infinite series of local [1, 3] as well as non-local [2, 4] conservation laws Moreover, for a certain class of symmetric spaces, the first non-local conservation law is known to survive the transition from the classical to the quantum theory [5, 6], and finally the existence of the resulting conserved quantum non-local charge is sufficient to determine the S-matrix [5, 7]

An important, but largely open question is whether, or rather to what extent, these techniques can be extended to higher dimensions In particular, in view of the many analogies between two-dimensional non-linear σ models and four-dimensional Yang-Mills theories [8, 9], one might suspect that the latter are also integrable. However, there are arguments (see, e g , ref [10], pp 469/470) which indicate that this is not the case, and indeed, a Lax representation for the Yang-Mills equations has not been found On the other hand, a Lax representation for the (more restrictive) self-dual Yang-Mills equations does exist [11–14], and in fact, self-dual Yang-Mills theories constitute the most popular examples for integrable systems in four dimensions, their integrability being intimately connected with the explicit construction [15] of their general (euclidean) solution Unfortunately, however, self-dual Yang-Mills theories fail to define genuine field theories, mainly because they do not seem to admit a reasonable lagrangian formulation, and one therefore does not know how to quantize them

This somewhat unsatisfactory situation is greatly improved by supersymmetry. Namely, it is known that supersymmetric Yang-Mills theories, when written down in superspace, contain constraint equations of precisely the desired type (certain components of the supercurvature tensor multiplet are supposed to vanish): they must be imposed in order to eliminate unphysical degrees of freedom [16, 17] In four dimensions, these constraints are of purely algebraic nature if $N = 1$ or $N = 2$, but they imply equations of motion if $N = 3$ or $N = 4$ More specifically, it can be shown that for $N = 3$ or $N = 4$ supersymmetric Yang-Mills theories in four dimensions, the superspace constraints are exactly equivalent to the standard equations of motion in terms of component fields [Such an equivalence has first been claimed, but not fully proved, in ref [17], and has then usually been taken for granted in the literature–probably because the importance of the statement, as the crucial prerequisite for an interpretation in terms of integrable systems, has not generally been appreciated An explicit proof, for $N = 3$, has only been given recently [18]] As a result, these theories are integrable, in the (weak) sense of admitting a Lax representation (in superspace) [19]. Moreover, it is known that the latter can be used as a starting point for the derivation of higher (non-local) conservation laws [20, 21].

The purpose of the present paper is to show that supersymmetric Yang-Mills theories in ten dimensions [22] exhibit the same structure, in the sense of integrable systems, as $N=3$ or $N=4$ supersymmetric Yang-Mills theories in four dimensions. This is certainly not surprising, since the latter can be obtained from the former by dimensional reduction [22], but it does show that integrability is not an artifact introduced by the process of dimensional reduction but rather a structural property of the ten-dimensional theory compatible with it. A further advantage is that the constructions involved are much more transparent and, from a technical point of view, more manageable in ten dimensions than they are in four dimensions. In particular, this goes for the Lax representation (i.e., the linear system) itself, which finds a natural geometric interpretation in terms of concepts borrowed from twistor theory [23] and, due to the twistorial nature of the spectral parameter, suggests a notion of harmonic superspace that could prove to be the clue for an off-shell formulation of the theory [24, 25]. Finally, there is an independent motivation for looking at supersymmetric Yang-Mills theory directly in ten dimensions: namely, the fact that the effective field theory obtained in the low-energy limit of the type I superstring [26] or the heterotic superstring [27] is a supersymmetric coupled Einstein Yang-Mills theory [28–30]. In fact, we suspect that the results obtained here for the Yang-Mills sector can be extended to the full effective field theory, and maybe even to the superstring theory itself; we hope to come back to this problem in the future.

2. Lagrangian formulation versus superspace formulation

We begin by fixing our notations and conventions, which to a large extent coincide with those of ref. [23]. Latin indices denote vector components running from 0 to 9, while Greek indices denote chiral Majorana spinor components running from 1 to 16: more specifically, upper/lower Greek indices stand for components of chiral Majorana spinors of positive/negative chirality, so we shall have to distinguish systematically between upper and lower indices. The Clifford algebra over ten-dimensional Minkowski space is represented by (32×32) matrices, namely the ten γ-matrices γ_m satisfying the standard anticommutation relations, together with their (normalized) totally antisymmetrized products $\gamma_{m_1 \ldots m_r}$, in such a way that

$$
\gamma_{m_1 \ldots m_r} = \begin{pmatrix} 0 & \left(\left(\sigma_{m_1 \ldots m_r}\right)^{\alpha\beta}\right) \\ \left(\left(\sigma_{m_1 \ldots m_r}\right)_{\alpha\beta}\right) & 0 \end{pmatrix} \quad \text{for } r \text{ odd},
$$

$$
\gamma_{m_1 \ldots m_r} = \begin{pmatrix} \left(\left(\sigma_{m_1 \ldots m_r}\right)^{\alpha}{}_{\beta}\right) & 0 \\ 0 & \left(\left(\sigma_{m_1 \ldots m_r}\right)_{\alpha}{}^{\beta}\right) \end{pmatrix} \quad \text{for } r \text{ even} \qquad (2.1)
$$

Thus $\gamma_{11} = \gamma_0\gamma_1 \ldots \gamma_9$ is diagonal. Useful relations are

$$\sigma_m\sigma_n + \sigma_n\sigma_m = 2g_{mn}, \tag{2.2}$$

$$\sigma_m\sigma_n = \sigma_{mn} + g_{mn}, \tag{2.3}$$

$$\sigma_m\sigma_{pq} = \sigma_{mpq} + g_{mp}\sigma_q - g_{mq}\sigma_p, \tag{2.4}$$

together with the following symmetry property of the $\sigma_{m_1 \ldots m_r}$

$$\sigma^T_{m_1 \ldots m_r} = (-1)^{(r-1)/2}\sigma_{m_1 \ldots m_r} \qquad \text{for } r \text{ odd},$$

$$\sigma^T_{m_1 \ldots m_r} = (-1)^{r/2}\sigma_{m_1 \ldots m_r} \qquad \text{for } r \text{ even}. \tag{2.5}$$

(We follow the convention that all equations involving σ-matrices in which the spinor indices are suppressed should be understood to hold whenever these indices are substituted back in any meaningful way.) We also have the important cyclic identity

$$(\sigma_m)^{\alpha\beta}(\sigma^m)^{\gamma\delta} + (\sigma_m)^{\beta\gamma}(\sigma^m)^{\alpha\delta} + (\sigma_m)^{\gamma\alpha}(\sigma^m)^{\beta\delta} = 0,$$

$$(\sigma_m)_{\alpha\beta}(\sigma^m)_{\gamma\delta} + (\sigma_m)_{\beta\gamma}(\sigma^m)_{\alpha\delta} + (\sigma_m)_{\gamma\alpha}(\sigma^m)_{\beta\delta} = 0 \tag{2.6}$$

With these preliminaries out of the way, let us briefly recall the lagrangian formulation of ten-dimensional supersymmetric Yang-Mills theories [22] and then confront it with their superspace formulation Throughout, we shall work with an unspecified gauge group it is simply any compact Lie group G, with Lie algebra $\mathfrak{g}$ carrying a given Ad(G)-invariant inner product (,) For simplicity, we suppose $\mathfrak{g}$ to be given in some faithful representation by anti-hermitian matrices and (.,.) to be some negative multiple of the trace form in that representation, and in terms of an arbitrary basis of generators T_a in $\mathfrak{g}$, we write $X = X^aT_a$,

$$[T_a, T_b] = f^c_{ab}T_c, \quad (T_a, T_b) = g_{ab}, \quad f_{abc} = g_{ad}f^d_{bc}, \quad (g^{ab}) = (g_{ab})^{-1}$$

Now in the lagrangian context, we are dealing with an ordinary field theory in ten-dimensional Minkowski space, involving a $\mathfrak{g}$-valued commuting vector field A_m and a $\mathfrak{g}$-valued anticommuting chiral Majorana spinor field χ^α, which transform according to

$$A_m \to g^{-1}A_m g + g^{-1}\partial_m g, \quad \chi^\alpha \to g^{-1}\chi^\alpha g \tag{2.7}$$

under gauge transformations, and governed by the lagrangian

$$L = -\tfrac{1}{4}g_{ab}F^{a}_{mn}F^{mnb} + \tfrac{1}{2}g_{ab}\chi^{\alpha a}(\sigma^{m})_{\alpha\beta}\mathscr{D}_{m}\chi^{\beta b}, \tag{2 8}$$

where $\mathscr{D}_m = \partial_m + A_m$ is the usual covariant derivative, and $F_{mn} = [\mathscr{D}_m, \mathscr{D}_n]$ A standard variational calculation gives the equations of motion

$$\not{\mathscr{D}}\chi = 0, \qquad \text{i e} \qquad (\sigma^{m})_{\alpha\beta}\mathscr{D}_{m}\chi^{\beta} = 0, \tag{2.9}$$

$$\mathscr{D}^{m}F_{mn} = \tfrac{1}{2}(\sigma_{n})_{\alpha\beta}\{\chi^{\alpha}, \chi^{\beta}\} \tag{2 10}$$

The variation of the fields A_m and χ^α under an infinitesimal supersymmetry transformation, parametrized by an anticommuting chiral Majorana spinor ε^α, is

$$\delta_{\varepsilon}A_{m} = (\sigma_{m})_{\alpha\beta}\varepsilon^{\alpha}\chi^{\beta}, \tag{2 11}$$

$$\delta_{\varepsilon}\chi^{\alpha} = \tfrac{1}{2}(\sigma^{mn})^{\alpha}{}_{\beta}\varepsilon^{\beta}F_{mn}. \tag{2.12}$$

As usual, one can show that on solutions of the field equations (2 9) and (2 10), the variation of the lagrangian (2 8) becomes a pure divergence, and the algebra closes

Next, let us recall that ten-dimensional flat Minkowski superspace is parametrized by even co-ordinates x^m and odd co-ordinates θ^α, and hence any superfield ϕ has a θ-expansion as follows

$$\phi(x,\theta) = \mathring{\phi}(x) + \sum_{r=1}^{16}\phi^{(r)}(x,\theta), \quad \phi^{(r)}(x,\theta) = \frac{1}{r!}\theta^{\alpha_1}\ldots\theta^{\alpha_r}\phi_{\alpha_1\ldots\alpha_r}(x) \tag{2 13}$$

[We shall often call $\phi^{(r)}(x,\theta)$ the term of order r in the θ-expansion of $\phi(x,\theta)$, and we also recall that superfields $\phi(x,\theta)$ are called even/odd if the component fields of even order are commuting/anticommuting objects and the component fields of odd order are anticommuting/commuting objects] Moreover, the superderivatives D_α are defined by

$$D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} - (\sigma^{m})_{\alpha\beta}\theta^{\beta}\partial_{m}, \tag{2.14}$$

and satisfy

$$\{D_{\alpha}, D_{\beta}\} = -2(\sigma^{m})_{\alpha\beta}\partial_{m} \tag{2 15}$$

Finally, we introduce the following change of notation: the symbols A_m, F_{mn}, χ^α and g employed so far, and in (2 7)–(2.12) in particular, are to be replaced, throughout the rest of this paper, by the symbols $\mathring{A}_m$, $\mathring{F}_{mn}$, $\mathring{\chi}^\alpha$ and $\mathring{g}$, respectively, where the superscript $^\circ$ indicates, as in (2 13), that we are dealing with ordinary fields (depending on x) instead of superfields (depending on x and θ)

Now we are ready to exhibit the superspace formulation of ten-dimensional supersymmetric Yang-Mills theories, based on a $\mathfrak{g}$-valued superconnection one-form A in superspace, representing the gauge potentials, which gives rise to a $\mathfrak{g}$-valued supercurvature two-form F in superspace, representing the gauge fields, and to a supercovariant superderivative $\mathscr{D}$, acting on $\mathfrak{g}$-valued superfields As usual, we expand the differential forms A, F on superspace in terms of component superfields A_m, A_α, F_{mn}, $F_{\alpha m}$, $F_{\alpha\beta}$; then A_m, F_{mn} and $F_{\alpha\beta}$ are even while A_α and $F_{\alpha m}$ are odd, and

$$\mathscr{D}_m = \partial_m + [A_m, .\} , \tag{2 16}$$

$$\mathscr{D}_\alpha = D_\alpha + [A_\alpha, .\} , \tag{2 17}$$

and

$$F_{mn} = [\mathscr{D}_m, \mathscr{D}_u] = \partial_m A_n - \partial_n A_m + [A_m, A_n] , \tag{2.18}$$

$$F_{\alpha m} = [\mathscr{D}_\alpha, \mathscr{D}_m] = D_\alpha A_m - \partial_m A_\alpha + [A_\alpha, A_m] , \tag{2 19}$$

$$F_{\alpha\beta} = \{\mathscr{D}_\alpha, \mathscr{D}_\beta\} + 2(\sigma^m)_{\alpha\beta}\mathscr{D}_m = D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\} + 2(\sigma^m)_{\alpha\beta} A_m \tag{2 20}$$

For later reference, we also list the Bianchi identities·

$$\mathscr{D}_m F_{np} + \mathscr{D}_n F_{pm} + \mathscr{D}_p F_{mn} = 0 , \tag{2.21}$$

$$\mathscr{D}_\alpha F_{mn} - \mathscr{D}_m F_{\alpha n} + \mathscr{D}_n F_{\alpha m} = 0 , \tag{2 22}$$

$$\mathscr{D}_\alpha F_{\beta m} + \mathscr{D}_\beta F_{\alpha m} + \mathscr{D}_m F_{\alpha\beta} - 2(\sigma^n)_{\alpha\beta} F_{mn} = 0 , \tag{2.23}$$

$$\mathscr{D}_\alpha F_{\beta\gamma} + \mathscr{D}_\beta F_{\gamma\alpha} + \mathscr{D}_\gamma F_{\alpha\beta} - 2\big((\sigma^m)_{\beta\gamma} F_{\alpha m} + (\sigma^m)_{\gamma\alpha} F_{\beta m} + (\sigma^m)_{\alpha\beta} F_{\gamma m}\big) = 0 \tag{2 24}$$

Supergauge transformations are represented by G-valued even superfields g and act according to

$$A_m \to g^{-1} A_m g + g^{-1} \partial_m g , \qquad A_\alpha \to g^{-1} A_\alpha g + g^{-1} D_\alpha g , \tag{2 25}$$

$$F_{mn} \to g^{-1} F_{mn} g , \qquad F_{\alpha m} \to g^{-1} F_{\alpha m} g , \qquad F_{\alpha\beta} \to g^{-1} F_{\alpha\beta} g \tag{2.26}$$

Finally, as mentioned in the introduction, the necessity for eliminating unphysical degrees of freedom requires imposing some constraints on F The correct choice, apparently first proposed in ref. [31], and used more extensively in ref [23], is extremely simple

$$F_{\alpha\beta} = 0 . \tag{2 27}$$

[This form of the constraint is in fact suggested by the observation that $F_{\alpha\beta}$ contains, as its zero order component, a gauge covariant field of dimension (length)$^{-1}$, and that no such field is present in the lagrangian formulation] Moreover, it can be shown that under dimensional reduction, (2 27) provides exactly the usual constraint equations of four-dimensional $N = 4$ supersymmetric Yang-Mills theories [16, 17]

3. Equivalence of the constraint with the field equations

3 1 DERIVATION OF THE FIELD EQUATIONS FROM THE CONSTRAINT

Our objective in this subsection is to show that the constraint (2 27) is, at least partially, of a dynamical nature, and more specifically, that it implies precisely the field equations (2 9) and (2.10). [This part of the equivalence proof has also been carried out in ref [23] and is included mainly for the sake of completeness.]

First of all, combining (2 27) with the Bianchi identity (2 24), we infer the existence of a spinorial superfield χ^α such that

$$F_{\alpha m} = (\sigma_m)_{\alpha\beta}\chi^\beta \tag{3 1}$$

The other two Bianchi identities involving spinorial covariant derivatives, namely (2.22) and (2 23), can now be used to express the spinorial covariant derivatives $\mathscr{D}_\alpha F_{mn}$ and $\mathscr{D}_\alpha \chi^\beta$ of the superfields F_{mn} and χ^β in terms of these superfields themselves and their vector covariant derivatives For F_{mn}, the result is immediate from (2.22) and (3.1)

$$\mathscr{D}_\alpha F_{mn} = (\sigma_n)_{\alpha\gamma}\mathscr{D}_m\chi^\gamma - (\sigma_m)_{\alpha\gamma}\mathscr{D}_n\chi^\gamma \tag{3 2}$$

For χ^β, some algebra gives:

$$\mathscr{D}_\alpha\chi^\beta = -\tfrac{1}{2}(\sigma^{mn})_\alpha{}^\beta F_{mn} \tag{3.3}$$

With these tools at our disposal, we can easily derive the field equations for the superfields χ^α and F_{mn} To this end, we simply combine the relation

$$\{\mathscr{D}_\alpha, \mathscr{D}_\beta\} = -2(\sigma^m)_{\alpha\beta}\mathscr{D}_m \qquad \text{or} \qquad \mathscr{D}_m = -\tfrac{1}{32}(\sigma_m)^{\alpha\beta}\{\mathscr{D}_\alpha, \mathscr{D}_\beta\}, \tag{3 4}$$

which follows from (2 20) and (2 27), with (3.2) and (3 3), and obtain

$$\not{\mathscr{D}}\chi = 0, \qquad \text{i.e.} \qquad (\sigma^m)_{\alpha\beta}\mathscr{D}_m\chi^\beta = 0, \tag{3 5}$$

$$\mathscr{D}^m F_{mn} = \tfrac{1}{2}(\sigma_n)_{\alpha\beta}\{\chi^\alpha, \chi^\beta\} \tag{3 6}$$

Hence the zero-order components $\mathring{A}_m$, $\mathring{F}_{mn}$ and $\mathring{\chi}^\alpha$ of the superfields A_m, F_{mn} and χ^α must of course be restricted to satisfy

$$\mathring{\not{\mathscr{D}}}\mathring{\chi} = 0, \qquad \text{i.e} \qquad (\sigma^m)_{\alpha\beta}\mathring{\mathscr{D}}_m\mathring{\chi}^\beta = 0, \tag{3 7}$$

$$\mathring{\mathscr{D}}^m\mathring{F}_{mn} = \tfrac{1}{2}(\sigma_n)_{\alpha\beta}\{\mathring{\chi}^\alpha, \mathring{\chi}^\beta\} \tag{3.8}$$

3 2 GAUGE CONDITION

Having shown that the constraint implies the field equations, we are left with the (more difficult) task of providing a proof for the converse statement This proof will proceed along the lines laid out in ref [18], but will be technically simpler and therefore more transparent★ To see what is involved, notice first that the supergauge freedom in the superspace formulation is of course much bigger than the ordinary gauge freedom in the lagrangian formulation Moreover, the θ-expansions for the superfields A_m, F_{mn} and χ^α contain a plethora of higher-order components which should somehow be related to their zero-order components $\mathring{A}_m$, $\mathring{F}_{mn}$ and $\mathring{\chi}^\alpha$. We must therefore

(a) restrict the supergauge freedom to the ordinary gauge freedom,

(b) give a device allowing us to promote the ordinary fields $\mathring{A}_m$, $\mathring{F}_{mn}$ and $\mathring{\chi}^\alpha$ to genuine superfields A_m, F_{mn} and χ^α

Both of these aims can be achieved by going to a special gauge which, for reasons to become clear later, will be called the recursion gauge (transverse gauge in ref [18]) and, due to the fact that it explicitly breaks supersymmetry and (as shown below) satisfies property (a) above, can be viewed as an analogue of the Wess-Zumino gauge in four-dimensional supersymmetric Yang-Mills theories [33] It reads

$$\theta^\alpha A_\alpha = 0 \tag{3 9}$$

Moreover, we introduce the following recursion operator (transverse or Euler operator in ref [18])·

$$\mathscr{R} = \theta^\alpha \frac{\partial}{\partial \theta^\alpha} = \theta^\alpha D_\alpha , \tag{3 10}$$

which acts on superfields by simply multiplying the term of order r in their θ-expansion by r, i e.,

$$\phi(x,\theta) = \mathring{\phi}(x) + \sum_{r=1}^{16} \phi^{(r)}(x,\theta) \Rightarrow (\mathscr{R}\phi)(x,\theta) = \sum_{r=1}^{16} r\phi^{(r)}(x,\theta) , \tag{3 11}$$

cf. (2.13). An equivalent way of defining the recursion gauge is then

$$\mathscr{R} = \theta^\alpha \mathscr{D}_\alpha \tag{3.12}$$

Next, we argue that the recursion gauge does satisfy property (a) above

(i) It may always be imposed. In fact, starting with an arbitrary set of potentials A_m, A_α, and gauge transforming them, via g, to a new set of potentials A'_m, A'_α, we

★ A similar proof has been given independently in ref [32], and it has there been extended, by dimensional reduction, to cover the $N=2$ theories in six dimensions as well as the $N=4$ theories in four dimensions see that the latter will satisfy the recursion gauge condition $\theta^\alpha A'_\alpha = 0$ if and only if

$$\mathscr{R} g = -\theta^\alpha A_\alpha g . \tag{3 13}$$

But this equation can always be solved Namely, looking at the θ-expansions for the superfields g and A_α, we see that, due to the presence of an explicit θ on the right-hand side of (3 13), the zero-order component $\mathring{g}$ of g is left unspecified, and the higher-order components of g are uniquely determined from the zero-order component $\mathring{g}$ of g and the components of A_α [More precisely, the argument is recursive. the term of order r in the θ-expansion for g is determined from the terms of order $\leqslant r-1$ in the θ-expansions for g and A_α, if $r \geqslant 1$]

(ii) It leaves precisely the freedom of performing ordinary gauge transformations In fact, proceeding as above, we conclude that the residual super-gauge transformations are described by superfields g satisfying $\mathscr{R} g = 0$, which means $g = \mathring{g}$

In order to analyze whether the recursion gauge also solves problem (b) above, we note that in this gauge

$$\mathscr{R} A_m = \theta^\alpha F_{\alpha m} , \tag{3 14}$$

$$(1+\mathscr{R}) A_\alpha = \theta^\beta F_{\alpha\beta} - 2(\sigma^m)_{\alpha\beta}\theta^\beta A_m , \tag{3 15}$$

This follows easily by evaluating the commutators $[\mathscr{R}, \mathscr{D}_m]$ and $[\mathscr{R}, \mathscr{D}_\alpha]$ in two different ways, namely by using $\mathscr{R} = \theta^\alpha \mathscr{D}_\alpha$ on the one hand and $\mathscr{R} = \theta^\alpha D_\alpha$ on the other hand. In particular, taking the constraint (2 27) and its consequences (3 1), (3.2) and (3.3) into account, we arrive at

$$\mathscr{R} A_m = (\sigma_m)_{\alpha\beta}\theta^\alpha \chi^\beta , \tag{3 16}$$

$$(1+\mathscr{R}) A_\alpha = -2(\sigma^m)_{\alpha\beta}\theta^\beta A_m , \tag{3.17}$$

$$\mathscr{R} F_{mn} = (\sigma_n)_{\alpha\beta}\theta^\alpha \mathscr{D}_m \chi^\beta - (\sigma_m)_{\alpha\beta}\theta^\alpha \mathscr{D}_n \chi^\beta , \tag{3 18}$$

$$\mathscr{R} \chi^\alpha = \tfrac{1}{2}(\sigma^{mn})^\alpha{}_\beta \theta^\beta F_{mn} \tag{3 19}$$

This shows that the recursion gauge does solve problem (b) above if the constraint (2.27) is imposed, since looking at the θ-expansions for the superfields A_m, F_{mn} and χ^α, we see that, due to the presence of an explicit θ on the right-hand side of (3 16), (3.18) and (3 19), the higher-order components of A_m, F_{mn} and χ^α are uniquely determined from their zero-order components $\mathring{A}_m$, $\mathring{F}_{mn}$ and $\mathring{\chi}^\alpha$ (plus covariant derivatives of the form $\mathring{\mathscr{D}}_{m_1} \cdot\ \mathring{\mathscr{D}}_{m_p}\mathring{F}_{mn}$ and $\mathring{\mathscr{D}}_{m_1}\ \ \mathring{\mathscr{D}}_{m_p}\mathring{\chi}^\alpha$), moreover, A_α is uniquely determined from A_m and has vanishing zero-order component $\mathring{A}_\alpha = 0$ However, our purpose in the following is to derive the constraint (2.27) and not to impose it a priori, so the argument will be revisited [If we were to drop the constraint (2 27), we would have to introduce a large number of additional fields in order to arrive at a system of recursion relations that closes.]

3.3 DERIVATION OF THE CONSTRAINT FROM THE FIELD EQUATIONS

Let us now start out from a $\mathfrak{g}$-valued commuting vector field $\mathring{A}_m$ and a $\mathfrak{g}$-valued anticommuting spinor field $\mathring{\chi}^\alpha$ which transform according to

$$\mathring{A}_m \to \mathring{g}^{-1}\mathring{A}_m\mathring{g} + \mathring{g}^{-1}\partial_m\mathring{g}, \qquad \mathring{\chi}^\alpha \to \mathring{g}^{-1}\mathring{\chi}^\alpha\mathring{g}$$

under gauge transformations. First of all, these fields will be promoted to superfields, namely a $\mathfrak{g}$-valued even vector superfield A_m and a $\mathfrak{g}$-valued odd spinor superfield χ^α which, once again, transform according to

$$A_m \to \mathring{g}^{-1}A_m\mathring{g} + \mathring{g}^{-1}\partial_m\mathring{g}, \qquad \chi^\alpha \to \mathring{g}^{-1}\chi^\alpha\mathring{g}$$

under gauge transformations, by imposing the recursion relations (3.16) and (3.19), together with the standard definition (2.18) of F_{mn} in terms of A_m (which is supposed a priori to hold to all orders in the θ-expansion). Namely, looking at the θ-expansions for the superfields A_m and χ^α, we see that, due to the presence of an explicit θ on the right-hand side of (3.16) and (3.19), the higher-order components of A_m and χ^α are thus indeed uniquely determined from their zero-order components $\mathring{A}_m$ and $\mathring{\chi}^\alpha$, and derivatives thereof [More precisely, the argument is, once again, recursive the term of order r in the θ-expansion for A_m is determined from the term of order $r-1$ in the θ-expansion for χ^α, and the term of order r in the θ-expansion for χ^α is determined from first derivatives of the term of order $r-1$ and – in the non-abelian case – products of the terms of order p and q, with $p+q=r-1$, in the θ-expansion for A_m, if $r \geqslant 1$.] This being established, the validity of the recursion relation (3.18) follows from that of the recursion relation (3.16), together with the standard definitions (2.16) of $\mathscr{D}_m$ and (2.18) of F_{mn} in terms of A_m. In addition, the recursion relations (3.16), (3.18) and (3.19) imply that – apart from the zero-order component $\mathring{A}_m$ of A_m – all other components of A_m, F_{mn} and χ^α are made up from (products of) the gauge covariant fields $\mathring{F}_{mn}$, $\mathring{\chi}^\alpha$ and their covariant derivatives $\mathring{\mathscr{D}}_{m_1} \ldots \mathring{\mathscr{D}}_{m_p}\mathring{F}_{mn}$, $\mathring{\mathscr{D}}_{m_1} \ldots \mathring{\mathscr{D}}_{m_p}\mathring{\chi}^\alpha$. Finally, since $1+\mathscr{R}$ is an invertible operator, eq. (3.17) can be used to define the $\mathfrak{g}$-valued odd spinor superfield A_α, and the reconstruction of the relevant superfields is completed by the standard definitions (2.17) of $\mathscr{D}_\alpha$ and (2.19) resp. (2.20) of $F_{\alpha m}$ resp. $F_{\alpha\beta}$.

Now let us assume, for the remainder of this section, that the fields $\mathring{A}_m$ and $\mathring{\chi}^\alpha$ satisfy the covariant Dirac equation (3.7) and the Yang-Mills equation (3.8). To derive the constraint (2.27), we proceed in several steps, leaving it to the reader to fill in the details of the calculations.

First of all, we claim that the superfields A_m and χ^α satisfy the covariant Dirac equation (3.5) and the Yang-Mills equation (3.6). To prove this, we simply define a

$\mathfrak{g}$-valued odd spinor superfield λ_α and a $\mathfrak{g}$-valued even vector superfield C_m as follows

$$\lambda_\alpha = (\sigma^m)_{\alpha\beta}\mathscr{D}_m\chi^\beta, \tag{3 20}$$

$$C_m = \mathscr{D}^n F_{nm} - \tfrac{1}{2}(\sigma_m)_{\alpha\beta}\{\chi^\alpha, \chi^\beta\}. \tag{3 21}$$

By assumption, $\mathring{\lambda}_\alpha = 0$ and $\mathring{C}_m = 0$ Now we use the recursion relations (3 16), (3.18) and (3.19), together with (2 3), (2 4), (2.5), the cyclic identity (2 6) and the cyclicity of σ^{pqr} in p, q, r, plus the Bianchi identity (2 21), to calculate the action of the recursion operator $\mathscr{R}$ on λ_α and C_m The result is

$$\mathscr{R}\lambda_\alpha = (\sigma^m)_{\alpha\beta}\theta^\beta C_m, \tag{3 22}$$

$$\mathscr{R}C_m = (\sigma_{mn})_\alpha{}^\beta\theta^\alpha\mathscr{D}^n\lambda_b, \tag{3 23}$$

which shows recursively that $\mathring{\lambda}_\alpha = 0$ and $\mathring{C}_m = 0$ forces $\lambda_\alpha = 0$ and $C_m = 0$ to all orders in the θ-expansion.

Second, we apply a similar strategy to prove (3 1), (3 2) and (3 3) Namely, we use the recursion relations (3.16)–(3 19), together with (2 4), (2 5), (2 15), the Bianchi identity (2.21) and, most important of all, the covariant Dirac equation (3 5), to calculate the action of the invertible operator $1+\mathscr{R}$ on the difference of the two sides of (3.1), (3.2) and (3.3). The result is

$$(1+\mathscr{R})\big(F_{\alpha m} - (\sigma_m)_{\alpha\beta}\chi^\beta\big) = -(\sigma_m)_{\beta\gamma}\theta^\beta\big(\mathscr{D}_\alpha\chi^\gamma + \tfrac{1}{2}(\sigma^{pq})_\alpha{}^\gamma F_{pq}\big), \tag{3.24}$$

$$\begin{aligned}
&(1+\mathscr{R})\big(\mathscr{D}_\alpha F_{mn} - (\sigma_n)_{\alpha\beta}\mathscr{D}_m\chi^\beta + (\sigma_m)_{\alpha\beta}\mathscr{D}_n\chi^\beta\big)\\
&\quad = -(\sigma_n)_{\beta\gamma}\theta^\beta\big(\mathscr{D}_m\big(\mathscr{D}_\alpha\chi^\gamma + \tfrac{1}{2}(\sigma^{pq})_\alpha{}^\gamma F_{pq}\big) + \{F_{\alpha m} - (\sigma_m)_{\alpha\delta}\chi^\delta, \chi^\gamma\}\big)\\
&\qquad + (\sigma_m)_{\beta\gamma}\theta^\beta\big(\mathscr{D}_n\big(\mathscr{D}_\alpha\chi^\gamma + \tfrac{1}{2}(\sigma^{pq})_\alpha{}^\gamma F_{pq}\big) + \{F_{\alpha n} - (\sigma_n)_{\alpha\delta}\chi^\delta, \chi^\gamma\}\big),
\end{aligned} \tag{3 25}$$

$$\begin{aligned}
&(1+\mathscr{R})\big(\mathscr{D}_\alpha\chi^\beta + \tfrac{1}{2}(\sigma^{mn})_\alpha{}^\beta F_{mn}\big)\\
&\quad = -\tfrac{1}{2}(\sigma^{mn})^\beta{}_\gamma\theta^\gamma\big(\mathscr{D}_\alpha F_{mn} - (\sigma_n)_{\alpha\delta}\mathscr{D}_m\chi^\delta + (\sigma_m)_{\alpha\delta}\mathscr{D}_n\chi^\delta\big),
\end{aligned} \tag{3 26}$$

which shows recursively that the covariant Dirac equation (3 5) implies (3.1), (3 2) and (3.3)

The final step is to prove (2.27) This time, we use the recursion relations (3.17) and (3.18), together with (2 5), the cyclic identity (2.6) and (2.15), to calculate the action of the invertible operator $2+\mathscr{R}$ on $F_{\alpha\beta}$ The result is

$$(2+\mathscr{R})F_{\alpha\beta} = 2(\sigma^m)_{\beta\gamma}\theta^\gamma\big(F_{\alpha m} - (\sigma_m)_{\alpha\delta}\chi^\delta\big) + 2(\sigma^m)_{\alpha\gamma}\theta^\gamma\big(F_{\beta m} - (\sigma_m)_{\beta\delta}\chi^\delta\big), \tag{3 27}$$

which shows that (3.1) implies (2 27), and we are finally done

4. Higher conservation laws in superspace

In the previous section, we have proved the equivalence of the superspace constraint (2 27) with the standard equations of motion (3 7) and (3.8) in terms of component fields. The importance of that proof lies in the fact that we are able to interpret the superspace constraint $F_{\alpha\beta}=0$ as the integrability condition for some linear system in superspace, which therefore provides a Lax representation for the equations of motion With that starting point, it is then possible to derive higher conservation laws

4 1 LAX REPRESENTATION AND REFORMULATION OF THE CONSTRAINT

The constraint $F_{\alpha\beta}=0$ expresses the fact that the superconnection is flat in certain directions in superspace and that some of its components can therefore be gauged away. The corresponding supergauge transformation is represented by a G-valued even superfield R which, by its very definition, is required to satisfy a certain system of first-order linear partial differential equations – or linear system, for short – in superspace In addition, it depends on spectral parameters which appear explicitly in the defining linear system However, in contrast to the usual integrable systems in two dimensions, where the spectral parameter is simply a number, the spectral parameters here have an intrinsic geometric structure of their own, which can be expressed in terms of concepts borrowed from twistor theory [23]. (The same feature has only recently been realized to occur, in an even more direct sense, for self-dual Yang-Mills theories in four dimensions as well [14])

To describe this intrinsic geometric structure, suppose that we are given an arbitrary null vector λ in $\mathbb{R}^{1,9}$. (More precisely, it suffices to fix an arbitrary null ray $[\lambda]$ in $\mathbb{R}^{1,9}$.) Assuming that the superconnection is a pure gauge along the line in superspace generated by λ (cf. ref [23]) means that there exists some G-valued even superfield $R[\lambda]$ such that

$$\lambda^m(\sigma_m)^{\alpha\beta}A_\beta=\lambda^m(\sigma_m)^{\alpha\beta}R[\lambda]^{-1}D_\beta R[\lambda], \tag{4 1}$$

$$\lambda^m A_m=\lambda^m R[\lambda]^{-1}\partial_m R[\lambda], \tag{4 2}$$

or equivalently,

$$\lambda^m(\sigma_m)^{\alpha\beta}\mathscr{D}_\beta R[\lambda]=0, \tag{4 3}$$

$$\lambda^m\mathscr{D}_m R[\lambda]=0, \tag{4 4}$$

where by definition,

$$\mathscr{D}_\alpha R[\lambda]=D_\alpha R[\lambda]-R[\lambda]A_\alpha, \tag{4 5}$$

$$\mathscr{D}_m R[\lambda]=\partial_m R[\lambda]-R[\lambda]A_m \tag{4 6}$$

The last two equations are dictated by our convention that under supergauge transformations (2.25), the superfield $R[\lambda]$ transforms according to

$$R[\lambda] \to R[\lambda]g \tag{4 7}$$

Eqs (4 1) and (4.2) or, equivalently, (4 3) and (4 4) constitute the desired linear system, with arbitrary null vectors λ (or rather null rays $[\lambda]$) as spectral parameters, since the corresponding integrability conditions are precisely the constraint $F_{\alpha\beta} = 0$ Indeed, if that linear system admits a solution $R[\lambda]$, we can use $\lambda^2 = 0$, (2.3) and (2 20) to infer that

$$\lambda^m \lambda^n (\sigma_m)^{\alpha\beta} (\sigma_n)^{\gamma\delta} F_{\beta\delta} = 0 ,$$

and since this is supposed to hold for an arbitrary null vector λ in $\mathbb{R}^{1,9}$, (2 27) follows by polarization Conversely, assuming that (2 27) and (hence) (3 1) are valid, we can first integrate (4 2) – this is just a single equation, with lots of solutions – and then use $\lambda^2 = 0$, (2 19) and (2 20) to infer that on all such solutions $R[\lambda]$, the integrability conditions for satisfying (4 1) as well, which read

$$\left\{ \lambda^m (\sigma_m)^{\alpha\beta} \mathscr{D}_\beta , \lambda^n (\sigma_n)^{\gamma\delta} \mathscr{D}_\delta \right\} R[\lambda] = 0 ,$$

$$\left[\lambda^m (\sigma_m)^{\alpha\beta} \mathscr{D}_\beta , \lambda^n \mathscr{D}_n \right] R[\lambda] = 0 ,$$

are fulfilled. Finally, it should be observed that the solution of the linear system (4 1)–(4.4) is not unique, and that the transformation

$$R[\lambda] \to R_0[\lambda] R[\lambda] \tag{4 8}$$

will take solutions to solutions provided that

$$\lambda^m (\sigma_m)^{\alpha\beta} D_\beta R_0[\lambda] = 0 , \tag{4 9}$$

$$\lambda^m \, \partial_m R_0[\lambda] = 0 \tag{4 10}$$

Of course, the transformations (4 8) leave the superfields A_α and A_m invariant.

Next, we shall use the linear system (4 1)–(4.4), together with the constraint (2 27), to solve for the superfields A_α and A_m in terms of the superfields R To this end, we shall have to consider two arbitrary null vectors λ and μ in $\mathbb{R}^{1,9}$ which are not proportional, so that $\lambda \cdot \mu \neq 0$ Then defining

$$\sigma_\lambda = \lambda^m \sigma_m , \qquad \sigma_\mu = \mu^m \sigma_m , \tag{4.11}$$

we have, due to (2 2),

$$\sigma_\lambda^2 = 0, \qquad \sigma_\lambda \sigma_\mu + \sigma_\mu \sigma_\lambda = 2\lambda \cdot \mu , \qquad \sigma_\mu^2 = 0 \tag{4 12}$$

This implies that

$$P(\lambda,\mu)=\frac{1}{2\lambda\cdot\mu}\sigma_\lambda\sigma_\mu=\tfrac{1}{2}\left(1+\frac{\lambda^m\mu^n}{\lambda\cdot\mu}\sigma_{mn}\right),$$

$$P(\mu,\lambda)=\frac{1}{2\lambda\cdot\mu}\sigma_\mu\sigma_\lambda=\tfrac{1}{2}\left(1-\frac{\lambda^m\mu^n}{\lambda\cdot\mu}\sigma_{mn}\right), \tag{4.13}$$

are projection operators (i.e., $P^2=P$) adding up to the identity

$$P(\lambda,\mu)+P(\mu,\lambda)=1 \tag{4.14}$$

For later use, we also note that

$$P(\lambda,\mu)^{\alpha}{}_{\beta}=P(\mu,\lambda)_{\beta}{}^{\alpha},\qquad P(\mu,\lambda)^{\alpha}{}_{\beta}=P(\lambda,\mu)_{\beta}{}^{\alpha} \tag{4.15}$$

and

$$P(\lambda,\mu)\sigma^m P(\mu,\lambda)=\frac{\lambda^n\mu^m}{\lambda\cdot\mu}\sigma_n,$$

$$P(\lambda,\mu)\sigma^m+\sigma^m P(\mu,\lambda)-P(\lambda,\mu)\sigma^m P(\mu,\lambda)=\sigma^m-\frac{\lambda^m\mu^n}{\lambda\cdot\mu}\sigma_n \tag{4.16}$$

Moreover,

$$\operatorname{im}P(\lambda,\mu)\subset\operatorname{im}\sigma_\lambda\subset\ker\sigma_\lambda\subset\ker P(\mu,\lambda)=\operatorname{im}P(\lambda,\mu),$$

$$\operatorname{im}P(\mu,\lambda)\subset\operatorname{im}\sigma_\mu\subset\ker\sigma_\mu\subset\ker P(\lambda,\mu)=\operatorname{im}P(\mu,\lambda),$$

where im stands for image and ker stands for kernel, and the last equality follows from (4.14), this gives the following more precise information

$$\begin{aligned}&P(\lambda,\mu)\text{ is the projector onto }\operatorname{im}\sigma_\lambda=\ker\sigma_\lambda\text{ along }\operatorname{im}\sigma_\mu=\ker\sigma_\mu,\\&P(\mu,\lambda)\text{ is the projector onto }\operatorname{im}\sigma_\mu=\ker\sigma_\mu\text{ along }\operatorname{im}\sigma_\lambda=\ker\sigma_\lambda\end{aligned} \tag{4.17}$$

A particular consequence is that the entire 16-dimensional space of chiral Majorana spinors splits into the direct sum

$$\operatorname{im}\sigma_\lambda\oplus\operatorname{im}\sigma_\mu=\ker\sigma_\lambda\oplus\ker\sigma_\mu \tag{4.18}$$

of eight-dimensional subspaces. Thus we can decompose the superfield A_α according to (4 14) and (4.18) and then use (4 1)–(4 4) and (4 13) to obtain

$$A_\alpha = P(\mu,\lambda)_\alpha{}^\beta R[\lambda]^{-1} D_\beta R[\lambda] + P(\lambda,\mu)_\alpha{}^\beta R[\mu]^{-1} D_\beta R[\mu] \qquad (4\ 19)$$

On the other hand, we exploit the fact that according to (2 5), and since $F_{\alpha\beta}$ is symmetric in α and β, the constraint $F_{\alpha\beta}=0$ (136 equations) is equivalent to $(\sigma_m)^{\alpha\beta}F_{\alpha\beta}=0$ (10 equations) and $(\sigma_{mnpqr})^{\alpha\beta}F_{\alpha\beta}=0$ (126 equations)*. The first set of equations can be used to express the superfield A_m in terms of the superfield A_α

$$A_m = -\tfrac{1}{32}(\sigma_m)^{\alpha\beta}\left(D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\}\right), \qquad (4.20)$$

while the second set of equations takes the form·

$$(\sigma_{mnpqr})^{\alpha\beta}\left(D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\}\right) = 0 \qquad (4\ 21)$$

To proceed further, we introduce the gauge invariant G-valued even superfield

$$U[\lambda,\mu] = R[\lambda]R[\mu]^{-1}. \qquad (4.22)$$

Then combining (4 19) with (4.22) and performing a straightforward but somewhat tedious calculation, which uses (4 15) and (4 16), we arrive at the following equation.

$$\begin{aligned} D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\} = &-2\frac{\lambda^m\mu^n}{\lambda\cdot\mu}(\sigma_n)_{\alpha\beta}\left(R[\lambda]^{-1}\partial_m R[\lambda] - R[\mu]^{-1}\partial_m R[\mu]\right) \\ &-2(\sigma^m)_{\alpha\beta}R[\mu]^{-1}\partial_m R[\mu] \\ &+\left(P(\lambda,\mu)_\alpha{}^\gamma P(\mu,\lambda)_\beta{}^\delta + P(\lambda,\mu)_\beta{}^\gamma P(\mu,\lambda)_\alpha{}^\delta\right)R[\mu] \\ &\times D_\gamma\left(U[\lambda,\mu]^{-1}D_\delta U[\lambda,\mu]\right)R[\mu]^{-1} \end{aligned} \qquad (4.23)$$

This shows, in particular, that the set of constraints (4 21) is equivalent to the following set of constraints on $U[\lambda,\mu]$·

$$\left(P(\mu,\lambda)\sigma_{mnpqr}P(\mu,\lambda)\right)^{\alpha\beta}D_\alpha\left(U[\lambda,\mu]^{-1}D_\beta U[\lambda,\mu]\right) = 0 \qquad (4\ 24)$$

Conversely, given a gauge invariant G-valued even superfield $U[\lambda,\mu]$ satisfying the constraint (4 24), we can split it, according to (4 22), into two gauge covariant

* This counting is due to the fact that the $(\sigma_{mnpqr})^{\alpha\beta}$ are self-dual while the $(\sigma_{mnpqr})_{\alpha\beta}$ are anti-self-dual

G-valued even superfields $R[\lambda]$, $R[\mu]$ and then use (4 19) and (4 20) to define superfields A_α and A_m satisfying the constraint (2 27). Of course, such a splitting is not unique, but a change of splitting corresponds precisely to a supergauge transformation.

To summarize, we have therefore arrived at yet another formulation of the equations of motion for ten-dimensional supersymmetric Yang-Mills theories – this time in terms of a gauge invariant G-valued even superfield $U[\lambda, \mu]$ depending, in a highly non-trivial fashion, on two null rays $[\lambda] \neq [\mu]$ in $\mathbb{R}^{1,9}$ [this of course breaks invariance under the Lorentz group SO(1, 9) down to the transverse subgroup SO(8)] and satisfying a set of 126 constraint equations, which are formally reminiscent of the equations of motion for the non-linear σ model on the group G. This formulation is therefore a direct generalization of structures found previously in the case of four-dimensional self-dual Yang-Mills theories [11, 13] – except for the fact that (4.24) is of course not an equation of motion derivable from a lagrangian But on the other hand, (4.24) has already the structure of a conservation law in superspace, as we shall demonstrate in the next subsection.

The previous constructions are also closely related to the basic concepts of the harmonic superspace approach [34], which starts out by introducing extra bosonic variables (in addition to the x's and θ's) In the present case, the space of these extra variables is just the set of ordered pairs $([\lambda], [\mu])$ – simply denoted by $[\lambda, \mu]$ in what follows – consisting of null rays $[\lambda] \neq [\mu]$ in $\mathbb{R}^{1,9}$, or equivalently, the set of oriented minkowskian planes in $\mathbb{R}^{1,9}$. (The pairs have to be ordered, or equivalently, the planes have to be oriented, because $\lambda \to \mu$, $\mu \to \lambda$ forces $U \to U^{-1}$) But that is precisely the pseudo-riemannian symmetric coset space

$$S = SO(1,9)/SO(1,1) \times SO(8). \tag{4 25}$$

proposed independently in ref [24] This observation is of course just the starting point for an entire dictionary between our approach and that of ref [24], and it would be worthwhile to see that worked out in more detail.

4 2 CONSERVED CURRENTS AND EXPANSIONS

For the derivation of conservation laws, we first define a (supersymmetric) spinor current, which is a gauge invariant $\mathfrak{g}$-valued odd spinor superfield J^α, and then a (non-supersymmetric) vector current, which is a gauge invariant $\mathfrak{g}$-valued commuting vector field j_m. Both of them come with an additional, highly non-trivial dependence on two null rays $[\lambda] \neq [\mu]$ in $\mathbb{R}^{1,9}$ – i.e., they are functions on the space S introduced in (4.25) – and, once these spectral parameters are fixed, they become totally antisymmetric rank-3 tensors on the orthogonal complement to the minkowskian plane in $\mathbb{R}^{1,9}$ spanned by λ and μ The explicit definitions are

$$J^{\alpha}[\lambda, \mu, x, y, z] = \lambda^m \mu^n x^p y^q z^r \big(P(\mu, \lambda) \sigma_{mnpqr} P(\mu, \lambda) \big)^{\alpha\beta} U[\lambda, \mu]^{-1} D_\beta U[\lambda, \mu], \tag{4.26}$$

$$J_m[\lambda, \mu, x, y, z] = \int \mathrm{d}^{16}\theta (\sigma_m)_{\alpha\beta} \theta^\alpha J^\beta [\lambda, \mu, x, y, z], \tag{4 27}$$

where $x, y, z \perp \lambda, \mu$ Then the set of constraints (4 24) implies, and is in fact equivalent to, the following set of (supersymmetric) spinor conservation laws

$$D_\alpha J^\alpha = 0, \tag{4 28}$$

which in turn imply the following set of (non-supersymmetric) vector conservation laws·

$$\partial^m J_m = 0, \tag{4 29}$$

because

$$\partial^m J_m = \int \mathrm{d}^{16}\theta (\sigma^m)_{\alpha\beta} \theta^\alpha \partial_m J^\beta = \int \mathrm{d}^{16}\theta \left(\frac{\partial}{\partial \theta^\beta} - D_\beta \right) J^\beta = -\int \mathrm{d}^{16}\theta D_\beta J^\beta$$

To arrive at infinite series of conservation laws, we must expand the spinor current J^α in the spectral parameters Generally speaking, functions on the space S introduced in (4.25) should be expanded in terms of an orthonormal basis of eigenfunctions for an appropriate Lorentz invariant differential operator on S, and the coefficients in the expansion should be labelled by the spectrum of that operator, plus additional parameters to account for degeneracies. We shall, however, be less ambitious and content ourselves with expansions along curves in S generated by the action of appropriate one-parameter subgroups of the Lorentz group SO(1,9) on given points in S As far as the tensor character of the currents (i.e , their nature as totally antisymmetric rank-3 tensors) is concerned, covariance under an SO(8) (the stability group* of a single point in S) will be broken by the expansion, and what remains is covariance under an SO(7) (the joint stability group* of all points on the curve in S along which the expansion is performed) As before, this is a strategy which follows closely the one that was successfully applied in the case of four-dimensional self-dual Yang-Mills theories [13]

To be more specific, let us denote by $\{e_0, e_1, \ldots, e_9\}$ the standard orthonormal basis of $\mathbb{R}^{1,9}$ and set $e_\pm = e_0 + e_1$ Then we can write any vector x in $\mathbb{R}^{1,9}$ in the form $x = x^+ e_+ + x^- e_- + x^i e_i$ with $x^\pm = \frac{1}{2}(x^0 \pm x^1)$ and, similarly, $\sigma_\pm = \sigma_0 \pm \sigma_1$,

* More precisely, the corresponding stability group contains an additional SO(1,1) factor

$\partial_{\pm} = \partial_0 \pm \partial_1$ and $\mathscr{D}_{\pm} = \mathscr{D}_0 \pm \mathscr{D}_1$. Moreover, we decompose generators X in so(9), the stability algebra of the unit time-like vector e_0, according to

$$X = \phi X_n + X_Y \qquad \text{with } \phi \in \mathbb{R},\ n \in S^7,\ Y \in \mathrm{so}(8), \tag{4.30}$$

where

$$X_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -n^T \\ 0 & n & 0 \end{pmatrix}, \qquad X_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Y \end{pmatrix} \tag{4.31}$$

Then

$$\exp(X_Y)\exp(\phi X_n)\exp(-X_Y) = \exp\left(\phi X_{\exp(Y)_n}\right) \tag{4.32}$$

Obviously, X_n generates rotations in the plane spanned by e_1 and n, and we shall set

$$\lambda_{\pm}(n, \phi) = \exp(2\phi X_n)\ e_{\pm}\,. \tag{4.33}$$

A direct calculation gives

$$\lambda_{\pm}(n, \phi) = e_0 \pm \cos 2\phi\, e_1 \pm \sin 2\phi\, n = \cos^2\phi\, e_{\pm} + \sin^2\phi\, e_{\mp} \pm \sin 2\phi\, n\,. \tag{4.34}$$

For the following, we shall choose $n = e_2$, put $\zeta = \tan\phi$ and abbreviate $(1 + \tan^2\phi)\lambda_{\pm}(e_2, \phi)$ to $\lambda_{\pm}(\zeta)$, so that

$$\begin{aligned}
\lambda_{\pm}(\zeta) &= (1+\zeta^2)e_0 \pm (1-\zeta^2)e_1 \pm 2\zeta e_2 = e_{\pm} \pm 2\zeta e_2 + \zeta^2 e_{\mp}\,,\\
\sigma^{\zeta}_{\pm} &= (1+\zeta^2)\sigma_0 \pm (1-\zeta^2)\sigma_1 \pm 2\zeta\sigma_2 = \sigma_{\pm} \pm 2\zeta\sigma_2 + \zeta^2\sigma_{\mp}\,,\\
\partial^{\zeta}_{\pm} &= (1+\zeta^2)\partial_0 \pm (1-\zeta^2)\partial_1 \pm 2\zeta\partial_2 = \partial_{\pm} \pm 2\zeta\partial_2 + \zeta^2\partial_{\mp}\,,\\
\mathscr{D}^{\zeta}_{\pm} &= (1+\zeta^2)\mathscr{D}_0 \pm (1-\zeta^2)\mathscr{D}_1 \pm 2\zeta\mathscr{D}_2 = \mathscr{D}_{\pm} \pm 2\zeta\mathscr{D}_2 + \zeta^2\mathscr{D}_{\mp}
\end{aligned} \tag{4.35}$$

With these preliminaries out of the way, let us consider

$$R_{\pm} = R[e_{\pm}], \qquad U = R_+ R_-^{-1}, \tag{4.36}$$

and

$$R_{\pm}(\zeta) = R[\lambda_{\pm}(\zeta)], \qquad U(\zeta) = R_+(\zeta) R_-(\zeta)^{-1}, \tag{4.37}$$

together with

$$B_{\pm}(\zeta) = R_{\pm}(\zeta) R_{\pm}^{-1}, \tag{4.38}$$

which implies

$$U(\zeta) = B_+(\zeta) U B_-(\zeta)^{-1}. \tag{4.39}$$

Now the linear system for $R_{\pm}(\zeta)$ gives the following linear system for $B_{\pm}(\zeta)$

$$\left(\sigma_{\pm}^{\zeta}\right)^{\alpha\beta} B_{\pm}(\zeta)^{-1} D_{\beta} B_{\pm}(\zeta) = -\left(\sigma_{\pm}^{\zeta}\right)^{\alpha\beta} \mathscr{D}_{\beta} R_{\pm} R_{\pm}^{-1}, \tag{4.40}$$

$$B_{\pm}(\zeta)^{-1} \partial_{\pm}^{\zeta} B_{\pm}(\zeta) = -\mathscr{D}_{\pm}^{\zeta} R_{\pm} R_{\pm}^{-1}. \tag{4.41}$$

Putting

$$B_{\pm}(\zeta) = \exp(b_{\pm}(\zeta)), \tag{4 42}$$

we get

$$\frac{1-\exp(-\operatorname{ad}(b_{\pm}(\zeta)))}{\operatorname{ad}(b_{\pm}(\zeta))} \left(\sigma_{\pm}^{\zeta}\right)^{\alpha\beta} D_{\beta} b_{\pm}(\zeta) = -\left(\sigma_{\pm}^{\zeta}\right)^{\alpha\beta} \mathscr{D}_{\beta} R_{\pm} R_{\pm}^{-1}, \tag{4 43}$$

$$\frac{1-\exp(-\operatorname{ad}(b_{\pm}(\zeta)))}{\operatorname{ad}(b_{\pm}(\zeta))} \partial_{\pm}^{\zeta} b_{\pm}(\zeta) = -\mathscr{D}_{\pm}^{\zeta} R_{\pm} R_{\pm}^{-1}, \tag{4.44}$$

where for X and Y in the Lie algebra $\mathfrak{g}$, but with matrix entries that may be either commuting or anticommuting c-numbers, we have by definition

$$\frac{1-\exp(-\operatorname{ad}(X))}{\operatorname{ad}(X)} Y = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \underbrace{[X, \ldots, [X}_{k \text{ times}}, Y\} \ldots\} \tag{4 45}$$

(see ref [35], p 105) Now we expand in powers of ζ

$$b_{\pm}(\zeta) = \sum_{r=1}^{\infty} \zeta^r b_{\pm}^{(r)}. \tag{4 46}$$

Note that the expansion starts with the linear term since by (4 38), $B_{\pm}(0) = 1$ and hence by (4.42), $b_{\pm}(0) = 0$. Moreover, the advantage of expanding $b_{\pm}(\zeta)$ and not $B_{\pm}(\zeta)$ is the usual one it guarantees automatically that all coefficients $b_{\pm}^{(r)}$, and not only the first one, lie in the Lie algebra $\mathfrak{g}$ Inserting (4 46) into (4 43), (4 44) and collecting terms with the same power of ζ then gives a system of differential equations for the coefficients $b_{\pm}^{(r)}$ (in superspace) from which these can be determined recursively by successive integrations (in superspace), in particular, $b_{\pm}^{(r)}$ contains p-fold integrals of (combinations of derivatives of) $R_{\pm}$, with $p = 0, \ldots, r$, and is therefore non-local of degree r. Explicitly, the terms up to order ζ^2 are

$$(\sigma_{\pm})^{\alpha\beta} D_{\beta} b_{\pm}^{(1)} = \mp 2(\sigma_2)^{\alpha\beta} \mathscr{D}_{\beta} R_{\pm} R_{\pm}^{-1}, \tag{4.47 1}$$

$$\partial_{\pm} b_{\pm}^{(1)} = \mp 2 \mathscr{D}_2 R_{\pm} R_{\pm}^{-1}, \tag{4 48 1}$$

$$(\sigma_{\pm})^{\alpha\beta}\left(D_{\beta} b_{\pm}^{(2)} - \tfrac{1}{2}\left[b_{\pm}^{(1)}, D_{\beta} b_{\pm}^{(1)}\right]\right) \pm 2(\sigma_2)^{\alpha\beta} D_{\beta} b_{\pm}^{(1)} = -(\sigma_{\mp})^{\alpha\beta} \mathscr{D}_{\beta} R_{\pm} R_{\pm}^{-1}, \tag{4.47 2}$$

$$\left(\partial_{\pm} b_{\pm}^{(2)} - \tfrac{1}{2}\left[b_{\pm}^{(1)}, \partial_{\pm} b_{\pm}^{(1)}\right]\right) \pm 2 \partial_2 b_{\pm}^{(1)} = -\partial_{\mp} R_{\pm} R_{\pm}^{-1} \tag{4 48 2}$$

Finally, we want to exhibit expressions for the conserved currents To this end, we note first that, according to (4.34), a vector x in $\mathbb{R}^{1,9}$ will be orthogonal to $\lambda_+(n,\phi)$ and to $\lambda_-(n,\phi)$ if and only if it is a linear combination of the vector $-\sin 2\phi\, e_1 + \cos 2\phi\, n$ and the vectors orthogonal to e_0, e_1, n Therefore, using

$$\tfrac{1}{2}\big(\lambda_+(e_2,\phi) + \lambda_-(e_2,\phi)\big) = e_0,$$

$$\tfrac{1}{2}\big(\lambda_+(e_2,\phi) - \lambda_-(e_2,\phi)\big) = \cos 2\phi\, e_1 + \sin 2\phi\, e_2,$$

we see that the independent types of σ-matrix factors that appear in (4.26) are

(i) $\sigma_0(\cos 2\phi\,\sigma_1 + \sin 2\phi\,\sigma_2)\sigma_{ijk}$, with $3 \leqslant i < j < k \leqslant 9$,

(ii) $\sigma_0(\cos 2\phi\,\sigma_1 + \sin 2\phi\,\sigma_2)(-\sin 2\phi\,\sigma_1 + \cos 2\phi\,\sigma_2)\,\sigma_{jk}$, with $3 \leqslant j < k \leqslant 9$

Hence, after some algebra, we obtain conserved spinor currents $J^{\alpha}_{ijk}(\zeta)$, with $2 \leqslant i < j < k \leqslant 9$, defined by

$$J^{\alpha}_{ijk}(\zeta) = \left(\sigma^{\zeta}_{-}\sigma'_{ijk}\right)^{\alpha\beta} U(\zeta)^{-1} D_\beta U(\zeta), \tag{4 49}$$

where

$$\sigma'_{ijk} = \sigma_{0ijk} \quad \text{if } i \geqslant 3,$$
$$\sigma'_{ijk} = \sigma_{1ijk} \quad \text{if } i = 2 \tag{4.50}$$

Moreover, from (4.39),

$$U(\zeta)^{-1} D_\alpha U(\zeta) = B_-(\zeta)\big(U^{-1}B_+(\zeta)^{-1}D_\alpha B_+(\zeta)U + U^{-1}D_\alpha U$$
$$- B_-(\zeta)^{-1}D_\alpha B_-(\zeta)\big)B_-(\zeta)^{-1} \tag{4 51}$$

Again we expand in powers of ζ:

$$J^{\alpha}_{ijk}(\zeta) = \sum_{r=0}^{\infty} \zeta^r J^{\alpha(r)}_{ijk} \tag{4.52}$$

Then the $J^{\alpha(r)}_{ijk}$ can be written as functions of the $b^{(r)}_+$ and $b^{(r)}_-$. Explicitly, the expressions up to order ζ^2 are

$$J^{\alpha(0)}_{ijk} = \left(\sigma_-\sigma'_{ijk}\right)^{\alpha\beta} U^{-1}D_\beta U, \tag{4 53 0}$$

$$J^{\alpha(1)}_{ijk} = \left(\sigma_-\sigma'_{ijk}\right)^{\alpha\beta}\Big(\big[b^{(1)}_-, U^{-1}D_\beta U\big] + U^{-1}D_\beta b^{(1)}_+ U - D_\beta b^{(1)}_-\Big)$$
$$-2\left(\sigma_2\sigma'_{ijk}\right)^{\alpha\beta} U^{-1}D_\beta U, \tag{4 53 1}$$

$$J^{\alpha(2)}_{ijk} = \left(\sigma_-\sigma'_{ijk}\right)^{\alpha\beta}\Big(\big[b^{(2)}_-, U^{-1}D_\beta U\big] + \tfrac{1}{2}\big[b^{(1)}_-,\big[b^{(1)}_-, U^{-1}D_\beta U\big]\big]$$
$$+ \big[b^{(1)}_-, U^{-1}D_\beta b^{(1)}_+ U\big] - \tfrac{1}{2}\big[b^{(1)}_-, D_\beta b^{(1)}_-\big] - D_\beta b^{(2)}_-$$
$$+ U^{-1}\Big(D_\beta b^{(2)}_+ - \tfrac{1}{2}\big[b^{(1)}_+, D_\beta b^{(1)}_+\big]\Big)U\Big)$$
$$-2\left(\sigma_2\sigma'_{ijk}\right)^{\alpha\beta}\Big(\big[b^{(1)}_-, U^{-1}D_\beta U\big] + U^{-1}D_\beta b^{(1)}_+ U - D_\beta b^{(1)}_-\Big)$$
$$+\left(\sigma_+\sigma'_{ijk}\right)^{\alpha\beta} U^{-1}D_\beta U \tag{4.53 2}$$

5. Conclusions and outlook

In this paper, we have given an explicit proof that for supersymmetric Yang-Mills theories in ten dimensions, the superspace constraint is precisely equivalent to the standard equations of motion in terms of component fields As a result, these theories are integrable, in the (weak) sense of admitting a Lax representation (in superspace). Moreover, we have shown how the latter can be used as a starting point for the derivation of higher (non-local) conservation laws These integrability properties are precisely analogous to – though technically much simpler to obtain and hence more transparent than – the corresponding integrability properties of $N=3$ or $N=4$ supersymmetric Yang-Mills theories in four dimensions. One drawback is that we have so far been unable to produce a completely explicit expression – in terms of component fields, say – for even the simplest among these higher (non-local) conservation laws, and although this is not so much a matter of principle but rather a technical problem, it has hampered our understanding for their physical significance. [The technical problem is twofold: (a) integrating linear differential equations in superspace, (b) picking out the penultimate component of a superfield in its θ-expansion, cf (4 27). Of course, both problems are more likely to have manageable solutions in $d=10$ than in $d=4$.]

Despite this deficiency, it is possible, and instructive, to compare the integrability properties of higher-dimensional supersymmetric Yang-Mills theories* with those of two-dimensional non-linear σ models on symmetric spaces (cf the introduction) As we have seen, there are considerable formal analogies, mainly at the classical level. On the other hand, we expect that at the quantum level, the higher-dimensional supersymmetric Yang-Mills theories are truly interacting field theories with a non-trivial S-matrix. But then, the Coleman–Mandula theorem [36] does not allow them to admit higher local conservation laws, and there are partial – though not yet conclusive – results towards a generalized Coleman–Mandula theorem [37] which would not allow them to admit higher non-local conservation laws, either Hence it seems that upon quantization, these conservation laws are necessarily plagued by anomalies – in complete analogy with what happens for the "anomalous" two-dimensional non-linear σ models on symmetric spaces [6], such as, e g., the $\mathbb{C}P^{N-1}$ models [38] or grassmannian models [39]

Having discussed the issue of higher conservation laws, let us conclude with a few comments on another aspect of integrability that follows from the existence of a Lax representation, namely the various techniques for generating exact classical solutions. Generally speaking, this subject is much less developed in $d>2$ than it is in $d=2$, and it seems that the additional complications arising in higher dimensions are largely due to the intrinsically multi-dimensional nature of the spectral parame-

* By higher-dimensional supersymmetric Yang-Mills theories, we mean here the $N=3$ and $N=4$ theories in four dimensions, the $N=2$ theories in six dimensions, and the $(N=1)$ theories in ten dimensions

ter. A notable exception to the first part of this statement is provided by the self-dual Yang-Mills equations in $d = 4$, where the appropriate spectral parameter is a twistor [14], and the corresponding multi-dimensional spectral transform – the analogue of the Riemann – Hilbert and/or inverse scattering transform – is the twistor transform [15]

For the higher-dimensional supersymmetric Yang-Mills theories, however, the prospects for rendering the analogous twistorial techniques equally powerful are not very bright [23], and we have not investigated these questions any further

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