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EXACT S-MATRICES FOR ANOMALY-FREE NON-LINEAR SIGMA MODELS ON SYMMETRIC SPACES

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The existence of a conserved quantum non-local charge is used to prove the factorization equations for the two-body S-matrix pertaining to the following five classical series of non-linear σ models on symmetric spaces: SU(N) (A), SO(N) (BD), Sp(N) (C), SU(N)/SO(N) (AI) and SU($2N$)/Sp(N) (AII). For the last two cases, this proof is new. The relevant S-matrices are computed explicitly, and the bound state problem is discussed.

1. Introduction

Two-dimensional non-linear σ models, or chiral models, are known to be classically integrable whenever the fields take values in a riemannian symmetric space $\mathrm{M} = \mathrm{G}/\mathrm{H}$ [1–3]. Moreover, it is known that integrability survives in the quantized theory, through existence of a conserved quantum non-local charge, if either H is simple (when appropriate "minimal" choices for the symmetry group G and the stability group H are made) [4] or – at least in certain families of models such as the $\mathbb{C}\mathrm{P}^{N-1}$ models [5] or the grassmannian models [6] – the bosonic fields are coupled to fermionic fields in a minimal or supersymmetric way. (For a general analysis of this second possibility, see ref. [7].) If one of these conditions for the absence of an anomaly is met, the existence of a conserved quantum non-local charge leads to

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factorization of the S-matrix into two-body amplitudes, and in addition it imposes strong constraints on the latter. Depending on the concrete model, however, these constraints may or may not suffice to determine the S-matrix completely.

In the present article, we concentrate on pure (i.e. purely bosonic) chiral models defined on irreducible riemannian symmetric spaces $\mathrm{M} = \mathrm{G}/\mathrm{H}$ of the compact type. (Technically speaking, we assume, as in ref. [4], that G is compact and connected and acts almost effectively on $\mathrm{M} = \mathrm{G}/\mathrm{H}$.) These spaces fall into two types, according to whether G is simple (type I) or is the direct product $\tilde{\mathrm{G}} \times \tilde{\mathrm{G}}$ of two copies of the same simple group $\tilde{\mathrm{G}}$ (type II), both types being completely classified (see ref. [8], pp. 516 and 518). In particular, the chiral model on M is anomaly-free whenever M is either of type II, i.e., any compact connected simple Lie group $\tilde{\mathrm{G}}$, or one of the following spaces of type I:

$$
\begin{array}{ll}
\mathrm{AI}: & \mathrm{M} = \mathrm{SU}(N)/\mathrm{SO}(N), \\
\mathrm{AII}: & \mathrm{M} = \mathrm{SU}(2N)/\mathrm{Sp}(N), \\
\mathrm{AIII}: & \mathrm{M} = \mathrm{SU}(2)/\mathrm{U}(1) = \mathbb{C}\mathrm{P}^1, \\
\mathrm{BDI}: & \mathrm{M} = \mathrm{SO}(N)/\mathrm{SO}(N-1) = \mathrm{S}^{N-1}, \\
\mathrm{EI}: & \mathrm{M} = \mathrm{E}_6/\mathrm{Sp}(4), \\
\mathrm{EIV}: & \mathrm{M} = \mathrm{E}_6/\mathrm{F}_4, \\
\mathrm{EV}: & \mathrm{M} = \mathrm{E}_7/\mathrm{SU}(8), \\
\mathrm{EVIII}: & \mathrm{M} = \mathrm{E}_8/\mathrm{SO}(16), \\
\mathrm{FII}: & \mathrm{M} = \mathrm{F}_4/\mathrm{SO}(9).
\end{array}
$$

We shall disregard the type I spaces $\mathbb{C}\mathrm{P}^1 = \mathrm{S}^2$ and S^{N-1}, whose S-matrices are well known [9, 10]*, and are therefore left with the following five series of classical ($\equiv$ non-exceptional) spaces:

$$
\begin{array}{lll}
\mathrm{A} & (\text{type II}): & \mathrm{M} = \mathrm{SU}(N), \\
\mathrm{BD} & (\text{type II}): & \mathrm{M} = \mathrm{SO}(N), \\
\mathrm{C} & (\text{type II}): & \mathrm{M} = \mathrm{Sp}(N), \\
\mathrm{AI} & (\text{type I}): & \mathrm{M} = \mathrm{SU}(N)/\mathrm{SO}(N), \\
\mathrm{AII} & (\text{type I}): & \mathrm{M} = \mathrm{SU}(2N)/\mathrm{Sp}(N).
\end{array}
$$

* Ref. [10] contains a proof of the fact that the bound-state S-matrix of the $\mathbb{C}\mathrm{P}^1$ model is identical with the S-matrix of the O(3) invariant non-linear σ model on S^2 proposed in [9]: in other words, the equality $\mathbb{C}\mathrm{P}^1 = \mathrm{S}^2$ holds even at the level of S-matrix quantum field theory.

The S-matrices for the classical groups have been determined before [11, 12], and our principal objective here is to extend that analysis to the classical coset spaces appearing in the list above. For pedagogical reasons, however, we also include a discussion of the group manifold cases, with the intention of facilitating comparison with the coset space cases, as well as slightly improving the notation and clarifying several arguments, as compared to the earlier versions [11, 12].

Before embarking on our program, we would like to comment on the relation of our work to the existing literature on factorizing S-matrices. First of all, we must mention the bootstrap approach, which aims at a complete classification of all two-body S-matrices that satisfy the factorization equations, or Yang-Baxter equations (plus the additional constraints resulting from unitarity, analyticity and crossing symmetry, of course). This program seems to be almost complete by now, and we refer the reader to [9, 13–15, 17, 18] for various results in that direction. In particular, it should be pointed out that all the S-matrices we derive here have been found before in the context of the bootstrap approach; we shall have more to say on this at the end of the paper. On the other hand, it is well known that the bootstrap approach starts out from the factorization equations as an unproven assumption, and that it gives no idea as to which field-theoretical model – if any – might yield a given factorizing S-matrix. Turning the question around, it is also a priori unclear which field-theoretical models will give rise to a factorizing S-matrix and which will not. In the context of non-linear σ models, however, these deficiencies can be overcome, and there are basically two different methods for doing so.

One of these methods is the Bethe ansatz technique, which relates the bosonic chiral model to certain fermionic models (in an appropriate limit), and uses the Bethe ansatz to solve the latter [19]. For the type II spaces (groups), i.e. for the principal chiral models, this strategy works very well, and it provides a wealth of information not only on the S-matrix but also on other aspects, such as the spectrum of particles that should build the various sectors of the theory [16–18]. For the type I spaces (coset spaces), however, the approach apparently fails, and one has to resort to other ideas.

The other method is based on the exploitation of hidden symmetries, or more precisely of higher conservation laws. These come in two species, local [20, 21] and non-local [22, 23, 4], but we use only the latter since we believe that they are easier to handle and, in a sense, more powerful than the former. (For example, there is no result on the absence of anomalies for quantum local charges which is as simple and general as the criterion for the absence of anomalies for quantum non-local charges proved in ref. [4].) Unfortunately, the method yields the S-matrix in a very special sector only, namely the one between particle/antiparticle states corresponding to the original chiral fields of the model. Of course, the results obtained from the bootstrap approach and from the Bethe ansatz technique (where applicable) indicate that other particles, such as fermions and various bound states, should be included as well, but to our knowledge, the complete field-theoretical models which would give rise to all these other sectors have not been worked out in full generality. We believe this to be a highly interesting open problem that deserves further study, but it certainly lies far beyond the scope of the present paper.

2. The classical models

In order to unify and streamline the presentation as much as possible, we shall employ a manifestly gauge invariant formulation, which is based on the fact that in each of the five cases under consideration here, the target manifold in question can be represented as a (totally geodesic) submanifold of a special unitary group, defined by an appropriate constraint. More precisely*, consider the involutive automorphism σ_{I} of $\mathrm{SU}(N)$ given by

$$\sigma_{\mathrm{I}}(g) = g^* \qquad \text{for } g \in \mathrm{SU}(N), \tag{1.I}$$

and the involutive automorphism σ_{II} of $\mathrm{SU}(2N)$ given by

$$\sigma_{\mathrm{II}}(g) = -Jg^*J \qquad \text{for } g \in \mathrm{SU}(2N), \tag{1.II}$$

where J is the matrix

$$J = \begin{pmatrix} 0 & +1_N \\ -1_N & 0 \end{pmatrix} \in \mathrm{SU}(2N). \tag{2.II}$$

Both of these can be written in the common form

$$\sigma(g) = Ig^*I^{-1} \qquad \text{for } g \in \mathrm{SU}(N) \text{ resp. } \mathrm{SU}(2N), \tag{1}$$

where I is the matrix

$$I = 1_N \in \mathrm{SU}(N) \qquad \text{for the cases BD and AI}, \qquad I^{-1} = I^{\mathrm{T}} = +I = +I^*.$$

$$I = J \in \mathrm{SU}(2N) \qquad \text{for the cases C and AII}, \qquad I^{-1} = I^{\mathrm{T}} = -I = -I^*. \tag{2}$$

Then, obviously,

$$\sigma(g) = g \quad \Leftrightarrow \quad g \in \mathrm{SO}(N) \text{ resp. } \mathrm{Sp}(N), \tag{3}$$

* We follow the convention that *, T and − stand for complex conjugation, transposition and hermitian conjugation of matrices, respectively – even when the matrix entries are operators in a Hilbert space, for which "conjugate" means "adjoint".

or more specifically,

$$\sigma_{\mathrm{I}}(g)=g \quad \Leftrightarrow \quad g\in \mathrm{SO}(N) \text{ for } g\in \mathrm{SU}(N). \tag{3.I}$$

$$\sigma_{\mathrm{II}}(g)=g \quad \Leftrightarrow \quad g\in \mathrm{Sp}(N) \text{ for } g\in \mathrm{SU}(2N). \tag{3.II}$$

Similarly,

$$\sigma(g)=g^{-1} \quad \Leftrightarrow \quad g\in \mathrm{SU}(N)/\mathrm{SO}(N) \text{ resp. } \mathrm{SU}(2N)/\mathrm{Sp}(N), \tag{4}$$

or more specifically,

$$\sigma_{\mathrm{I}}(g)=g^{-1} \quad \Leftrightarrow \quad g\in \mathrm{SU}(N)/\mathrm{SO}(N) \text{ for } g\in \mathrm{SU}(N). \tag{4.I}$$

$$\sigma_{\mathrm{II}}(g)=g^{-1} \quad \Leftrightarrow \quad g\in \mathrm{SU}(2N)/\mathrm{Sp}(N) \text{ for } g\in \mathrm{SU}(2N). \tag{4.II}$$

Here, we have identified the coset spaces $\mathrm{SU}(N)/\mathrm{SO}(N)$ and $\mathrm{SU}(2N)/\mathrm{Sp}(N)$ with their images under the Cartan immersion; see ref. [8], p. 347 and ref. [2] for details. Finally, we extend our convention (2) to the remaining case by demanding

$$I=1_N\in \mathrm{SU}(N) \qquad \text{for the case } A, \qquad I^{-1}=I^{\mathrm{T}}=+I=+I^*. \tag{5}$$

The merit of introducing the matrix I is that this will allow us to give common formulae for all five cases, and when this is not possible, at least to deal simultaneously with the "subgroup cases" BD and C on the one hand and with the "quotient space cases" or "coset space cases" AI and AII on the other hand: we shall in such situations indicate the type of model to which a given formula refers by adding a letter A (for the case A) or S (for the subgroup cases) or Q (for the quotient space cases). Moreover, certain formulae will contain $\pm$ or $\mp$ signs: then the general convention is that the upper sign will refer to the cases A and/or BD and AI (where I is symmetric) while the lower sign will refer to the cases C and AII (where I is antisymmetric).

With these conventions, the basic field of the model is a matrix field g which, classically, satisfies the unitarity and determinant constraints

$$g^{-1}=g^{+}, \tag{6}$$

$$\det g=1, \tag{7}$$

plus additional constraints depending on the model at hand:

$$g=\sigma(g)\equiv Ig^*I^+ \qquad \text{or} \qquad g^{\mathrm{T}}=Ig^+I^+, \tag{8.S}$$

$$g^+=\sigma(g)\equiv Ig^*I^+ \qquad \text{or} \qquad g^{\mathrm{T}}=IgI^+. \tag{8.Q}$$

In particular, (8.S) means that for the subgroup cases, the field g is invariant under charge conjugation $g \to g^{c}$, which is defined by

$$g^{c} = I g^{*} I^{+} . \tag{9}$$

The action is the standard one:

$$S = \tfrac{1}{2} \int \mathrm{d}^2 x \, \operatorname{tr}\left(g^{+} \partial^{\mu} g g^{+} \partial_{\mu} g \right) . \tag{10}$$

It admits various global internal symmetries, namely continuous ones, described by the connected Lie group

$$\mathrm{G} = \mathrm{SU}(N)_{\mathrm{L}} \times \mathrm{SU}(N)_{\mathrm{R}} , \tag{11.A}$$

$$\mathrm{G} = \mathrm{SO}(N)_{\mathrm{L}} \times \mathrm{SO}(N)_{\mathrm{R}} , \tag{11.BD}$$

$$\mathrm{G} = \mathrm{Sp}(N)_{\mathrm{L}} \times \mathrm{Sp}(N)_{\mathrm{R}} , \tag{11.C}$$

$$\mathrm{G} = \mathrm{SU}(N) , \tag{11.AI}$$

$$\mathrm{G} = \mathrm{SU}(2N) , \tag{11.AII}$$

which acts according to

$$g \to g_{\mathrm{L}} g g_{\mathrm{R}}^{+} , \tag{12.A}$$

$$g \to g_{\mathrm{L}} g g_{\mathrm{R}}^{+} , \tag{12.S}$$

$$g \to \sigma(g_0) g g_0^{+} = I g_0^{*} I^{+} g g_0^{+} , \tag{12.Q}$$

as well as discrete ones, namely charge conjugation $g \to g^{c} = I g^{*} I^{+}$ and the transformation $g \to g^{+}$, or equivalently, the transformation $g \to I g^{\mathrm{T}} I^{+}$. As usual, the continuous global symmetries lead to a Noether current J_{μ} taking values in the Lie algebra $\mathfrak{g}$ of G:

$$J_{\mu} = \left(J_{\mu}^{\mathrm{L}}, J_{\mu}^{\mathrm{R}} \right), \qquad J_{\mu}^{\mathrm{L}} = -\partial_{\mu} g g^{+} , \qquad J_{\mu}^{\mathrm{R}} = +g^{+} \partial_{\mu} g . \tag{13.A}$$

$$J_{\mu} = \left(J_{\mu}^{\mathrm{L}}, J_{\mu}^{\mathrm{R}} \right), \qquad J_{\mu}^{\mathrm{L}} = -\partial_{\mu} g g^{+} , \qquad J_{\mu}^{\mathrm{R}} = +g^{+} \partial_{\mu} g . \tag{13.S}$$

$$J_{\mu} = g^{+} \partial_{\mu} g - \sigma\left(\partial_{\mu} g g^{+} \right) . \tag{13.Q}$$

In the group manifold cases, the currents J_{μ}^{L} and J_{μ}^{R} are exchanged under the discrete symmetry $g \to g^{+}$, while in the coset space cases, the expression for J_{μ} can be simplified due to the fact that the constraints (6) and (8.Q) imply $\sigma(\partial_\mu g g^+) = -g^+ \partial_\mu g$. Therefore, it is sufficient to work with the current

$$j_\mu = g^+ \partial_\mu g, \tag{14}$$

whose conservation

$$\partial^\mu j_\mu = 0 \tag{15}$$

is completely equivalent to the equations of motion derived from (10), and which, classically, also satisfies the flatness condition

$$\partial_\mu j_\nu - \partial_\nu j_\mu + [j_\mu, j_\nu] = 0, \tag{16}$$

thus giving rise not only to the standard conserved classical charge

$$Q^{(0)} = \int \mathrm{d}y\, j_0(t, y), \tag{17}$$

but also to a conserved classical non-local charge, namely

$$Q^{(1)} = \int \mathrm{d}y_1 \mathrm{d}y_2\, \varepsilon(y_1 - y_2)\, j_0(t, y_1)\, j_0(t, y_2) + 2\int \mathrm{d}y\, j_1(t, y). \tag{18}$$

3. Quantization

For the quantum theory, we note that all non-linear relations appearing in the classical context must be handled with care, while linear relations do not cause any problems: they can simply be required to hold as operator identities. Thus the non-linear conditions (6) and (7) must be reformulated as constraints

$$\mathcal{N}[g^+ g] = \text{const}\, 1 = \mathcal{N}[g g^+], \tag{19}$$

$$\mathcal{N}[\det g] = \text{const} \tag{20}$$

on normal products, with renormalization scheme dependent constants, while the additional linear constraints (8) remain as they stand. Similarly, interpreting (14) in the sense of normal products,

$$j_\mu = \mathcal{N}[g^+ \partial_\mu g], \tag{21}$$

the conservation law (15) remains as it stands, while the flatness condition (16) is replaced by a short-distance expansion for the (matrix) commutator of two currents at nearby (spacelike separated) points

$$j_\mu(x+\varepsilon)\,j_\nu(x-\varepsilon)-j_\nu(x-\varepsilon)\,j_\mu(x+\varepsilon)\sim C^{\rho}_{\mu\nu}(\varepsilon)\,j_\rho(x)+D^{\sigma\rho}_{\mu\nu}(\varepsilon)(\partial_\sigma j_\rho)(x)\,, \quad (22)$$

where $\sim$ means equality up to terms that go to zero as $\varepsilon \to 0$ [4], and the coefficients can be determined completely [7, 22, 23] in terms of a single function of $-\varepsilon^2$. In the usual way, this gives rise not only to the standard conserved quantum charge

$$Q^{(0)}=\int \mathrm{d}y\, j_0(t,y)\,, \quad (23)$$

but also to a conserved quantum non-local charge, namely

$$Q^{(1)}=\lim_{\delta\to 0} Q^{(1)}_\delta(t)\,, \quad (24)$$

with

$$Q^{(1)}_\delta(t)=\int_{|y_1-y_2|\geq\delta}\mathrm{d}y_1\,\mathrm{d}y_2\,\varepsilon(y_1-y_2)\,j_0(t,y_1)\,j_0(t,y_2)-Z(\delta)\int\mathrm{d}y\,j_1(t,y)\,, \quad (25)$$

where $Z(\delta)$ is a function that diverges logarithmically as $\delta \to 0$ [4, 7, 22, 23]. Moreover, if T is the generator of Lorentz transformations, we have the commutation relation

$$[T, Q^{(1)}]=-\frac{N'}{2\pi}Q^{(0)}\,, \quad (26)$$

where

$$N'=N\,, \quad (27.\mathrm{A})$$

$$N'=N-2\,, \quad (27.\mathrm{BD})$$

$$N'=2N+2\,, \quad (27.\mathrm{C})$$

$$N'=N\,, \quad (27.\mathrm{AI})$$

$$N'=2N\,. \quad (27.\mathrm{AII})$$

(The proof of an analogous relation, with an appropriate constant, for the spherical models can be found in [23]; we shall not give any details here.)

In order to make contact with S-matrix theory, we have to specify the particle content of our models. One natural requirement in this context is consistency with the various symmetries, which include G-invariance (cf. eq. (11)) as well as charge conjugation invariance. This forces particles and antiparticles to be arranged in mutually conjugate representations of G, and the simplest consistent choice is of course to suppose that these representations are irreducible – which is precisely the conclusion that we shall arrive at. In fact, the most straightforward way to figure out the correct representations is to view the interacting field g as an interpolating field in the sense of LSZ, and to introduce asymptotic free fields g_{ex}, where "ex" stands for "in" or "out", such that g converges weakly to g_{in} for $t \to -\infty$ and to g_{out} for $t \to +\infty$. This seems plausible since the interacting field g is gauge invariant, so there are no long-range forces (no confinement), and the corresponding particles should therefore all be visible as asymptotic states generated by the asymptotic free fields g_{ex}. (Of course, that does not exclude the formation of bound states.) In addition, we expect all our models to exhibit dynamical mass generation, so the matrix fields g_{ex} should be massive, and all their matrix elements should have the same mass $m > 0$, since they are transformed into each other under the action of G.

To be more explicit, let us introduce the Fourier representations

$$g_{\mathrm{ex}}(t, y) = \int \mathrm{d}\mu(q)\{b_{\mathrm{ex}}(q)\exp(-iE(q)t + iqy) + d_{\mathrm{ex}}^*(q)\exp(+iE(q)t - iqy)\}, \tag{28}$$

where the integration measure is

$$\mathrm{d}\mu(q) = \frac{\mathrm{d}q}{2\pi 2E(q)} = \frac{\mathrm{d}\theta}{4\pi}, \qquad \begin{aligned} E(q) &= \sqrt{q^2 + m^2} = m\cosh\theta, \\ q &= m\sinh\theta, \end{aligned} \tag{29}$$

and the creation and annihilation operators appearing in (28) are supposed to satisfy the following canonical commutation relations:

$$\begin{aligned} \left[b_{\mathrm{ex}}^{ij}(q_1), b_{\mathrm{ex}}^{*kl}(q_2)\right] &= \left[d_{\mathrm{ex}}^{ij}(q_1), d_{\mathrm{ex}}^{*kl}(q_2)\right] = 2\pi 2E r^{ij,kl}\delta(q_1 - q_2), \\ \left[b_{\mathrm{ex}}^{ij}(q_1), d_{\mathrm{ex}}^{*kl}(q_2)\right] &= \left[d_{\mathrm{ex}}^{ij}(q_1), b_{\mathrm{ex}}^{*kl}(q_2)\right] = 2\pi 2E s^{ij,kl}\delta(q_1 - q_2), \end{aligned} \tag{30}$$

with

$$r^{ij,kl} = \delta^{ik}\delta^{jl}, \qquad s^{ij,kl} = 0, \tag{31.A}$$

$$r^{ij,kl} = \delta^{ik}\delta^{jl}, \qquad s^{ij,kl} = I^{ik}I^{jl}, \tag{31.S}$$

$$r^{ij,kl} = \delta^{ik}\delta^{jl} + I^{il}I^{jk}, \qquad s^{ij,kl} = 0. \tag{31.Q}$$

(All other commutators are supposed to vanish.) Moreover, asymptotic states are defined by

$$|\theta kl\rangle_{\mathrm{ex}} = \left(Ib_{\mathrm{ex}}^*(q(\theta))\right)^{kl}|0\rangle, \qquad |\overline{\theta kl}\rangle_{\mathrm{ex}} = \left(d_{\mathrm{ex}}^*(q(\theta))I\right)^{kl}|0\rangle. \tag{32}$$

(These are the asymptotic one-body states; asymptotic two-body states are defined correspondingly*.) This fixes the structure of the asymptotic free theory, since the commutation relations (30) and (31) contain, among other things, the rules for defining composite operators by means of Wick ordering : :. Examples of such operators are provided by the asymptotic free currents $j_{\mu,\mathrm{ex}}$: these are traceless anti-hermitian matrix fields given by

$$\tilde{j}_{\mu,\mathrm{ex}} = \tfrac{1}{2}\left(: g_{\mathrm{ex}}^{+} \partial_\mu g_{\mathrm{ex}} : - : \partial_\mu g_{\mathrm{ex}}^{+} g_{\mathrm{ex}} : \right), \tag{33}$$

and

$$j_{\mu,\mathrm{ex}} = \tilde{j}_{\mu,\mathrm{ex}} - \frac{1}{N'} \operatorname{tr} \tilde{j}_{\mu,\mathrm{ex}}, \tag{34}$$

where N' is given by (27). (For the subgroup cases, the value of N' in (34) is irrelevant since the constraint (35.S) below implies that $\tilde{j}_{\mu,\mathrm{ex}}$ is already traceless.) Notice also that the g_{ex} do not satisfy the free field analogue of the non-linear constraints (19) and (20): this was the reason why we had to make the $j_{\mu,\mathrm{ex}}$ traceless and anti-hermitian by hand. (Other consequences will emerge later on.) The g_{ex} can and will, however, be made to satisfy the free field analogue of the additional linear constraints (8): this is achieved by imposing the following simple identities between the various creation and annihilation operators:

$$d_{\mathrm{ex}}^{(*)}(q) I = I b_{\mathrm{ex}}^{(*)}(q), \tag{35.S}$$

$$b_{\mathrm{ex}}^{(*)\mathrm{T}}(q) I = I b_{\mathrm{ex}}^{(*)}(q), \qquad d_{\mathrm{ex}}^{(*)\mathrm{T}}(q) I = I d_{\mathrm{ex}}^{(*)}(q). \tag{35.Q}$$

For asymptotic states, this means that

$$|\overline{\theta k l}\rangle_{\mathrm{ex}} = |\theta k l\rangle_{\mathrm{ex}}, \tag{36.S}$$

$$|\theta k l\rangle_{\mathrm{ex}} = \pm |\theta l k\rangle_{\mathrm{ex}}, \qquad |\overline{\theta k l}\rangle_{\mathrm{ex}} = \pm |\overline{\theta l k}\rangle_{\mathrm{ex}}. \tag{36.Q}$$

Another condition is that the g_{ex} should have the same transformation laws under global symmetries as the interacting field g: this is achieved by imposing appropriate transformation laws on the various creation and annihilation operators. More precisely, (11) and (12), with g replaced by g_{ex}, lead to

$$b_{\mathrm{ex}}^{(*)}(q) \to g_{\mathrm{L}}^{(*)} b_{\mathrm{ex}}^{(*)}(q) g_{\mathrm{R}}^{(*)+}, \qquad d_{\mathrm{ex}}^{(*)}(q) \to g_{\mathrm{L}}^{(*)*} d_{\mathrm{ex}}^{(*)}(q) g_{\mathrm{R}}^{(*)\mathrm{T}}, \tag{37.A}$$

$$b_{\mathrm{ex}}^{(*)}(q) \to g_{\mathrm{L}}^{(*)} b_{\mathrm{ex}}^{(*)}(q) g_{\mathrm{R}}^{(*)+}, \qquad d_{\mathrm{ex}}^{(*)}(q) \to g_{\mathrm{L}}^{(*)*} d_{\mathrm{ex}}^{(*)}(q) g_{\mathrm{R}}^{(*)\mathrm{T}}, \tag{37.S}$$

$$b_{\mathrm{ex}}^{(*)}(q) \to I g_0^{(*)*} I^{+} b_{\mathrm{ex}}^{(*)}(q) g_0^{(*)+}, \qquad d_{\mathrm{ex}}^{(*)}(q) \to I g_0^{(*)} I^{+} d_{\mathrm{ex}}^{(*)}(q) g_0^{(*)\mathrm{T}}. \tag{37.Q}$$

* The term "body" is synonymous for "particle or antiparticle".

For asymptotic states, this means that

$$|\theta kl\rangle_{\mathrm{cx}} \to g_{\mathrm{L}}^{*km} g_{\mathrm{R}}^{ln} |\theta mn\rangle_{\mathrm{cx}}, \qquad |\overline{\theta kl}\rangle_{\mathrm{cx}} \to g_{\mathrm{L}}^{km} g_{\mathrm{R}}^{*ln} |\overline{\theta mn}\rangle_{\mathrm{cx}}, \tag{38.A}$$

$$|\theta kl\rangle_{\mathrm{cx}} \to g_{\mathrm{L}}^{km} g_{\mathrm{R}}^{ln} |\theta mn\rangle_{\mathrm{cx}}, \tag{38.S}$$

$$|\theta kl\rangle_{\mathrm{cx}} \to g_{0}^{km} g_{0}^{ln} |\theta mn\rangle_{\mathrm{cx}}, \qquad |\overline{\theta kl}\rangle_{\mathrm{ex}} \to g_{0}^{ckm} g_{0}^{cln} |\overline{\theta mn}\rangle_{\mathrm{ex}}. \tag{38.Q}$$

Similarly, under charge conjugation $g_{\mathrm{ex}} \to g_{\mathrm{cx}}^{\mathrm{c}} = I g_{\mathrm{ex}}^{*} I^{+}$,

$$b_{\mathrm{ex}}^{(*)}(q) \to I d_{\mathrm{ex}}^{(*)}(q) I^{+}, \qquad d_{\mathrm{ex}}^{(*)}(q) \to I b_{\mathrm{cx}}^{(*)}(q) I^{+}, \tag{39}$$

so

$$|\theta kl\rangle_{\mathrm{ex}} \to |\overline{\theta kl}\rangle_{\mathrm{cx}}, \qquad |\overline{\theta kl}\rangle_{\mathrm{ex}} \to |\theta kl\rangle_{\mathrm{cx}}. \tag{40}$$

Finally, under the discrete symmetry $g_{\mathrm{ex}} \to I g_{\mathrm{ex}}^{\mathrm{T}} I^{+}$,

$$b_{\mathrm{cx}}^{(*)}(q) \to I b_{\mathrm{ex}}^{(*)\mathrm{T}}(q) I^{+}, \qquad d_{\mathrm{cx}}^{(*)}(q) \to I d_{\mathrm{cx}}^{(*)\mathrm{T}}(q) I^{+}, \tag{41}$$

so

$$|\theta kl\rangle_{\mathrm{cx}} \to \pm |\theta lk\rangle_{\mathrm{ex}}, \qquad |\overline{\theta kl}\rangle_{\mathrm{ex}} \to \pm |\overline{\theta lk}\rangle_{\mathrm{cx}}. \tag{42}$$

For consistency, we have to require, of course, that the additional linear constraints (35) and the various symmetries (37), (39) and (41) are compatible with the commutation relations (30): this imposes severe restrictions on the group-theoretical structure of the coefficients r and s. It is straightforward to verify, however, that the choice made in (31) does satisfy all these restrictive conditions.

To summarize, we see that the asymptotic one-particle states $|\theta kl\rangle_{\mathrm{cx}}$ transform according to the following irreducible representations of G:

$N_{\mathrm{L}}^{*} \otimes N_{\mathrm{R}}$	of	$\mathrm{SU}(N)_{\mathrm{L}} \times \mathrm{SU}(N)_{\mathrm{R}}$	(A-series)
$N_{\mathrm{L}} \otimes N_{\mathrm{R}}$	of	$\mathrm{SO}(N)_{\mathrm{L}} \times \mathrm{SO}(N)_{\mathrm{R}}$	(BD-series)
$N_{\mathrm{L}} \otimes N_{\mathrm{R}}$	of	$\mathrm{Sp}(N)_{\mathrm{L}} \times \mathrm{Sp}(N)_{\mathrm{R}}$	(C-series)
$N \otimes_{\mathrm{s}} N$	of	$\mathrm{SU}(N)$	(AI-series)
$2N \otimes_{\mathrm{a}} 2N$	of	$\mathrm{SU}(2N)$	(AII-series)

(Here, s stands for "symmetric" and a stands for "antisymmetric".) Moreover, these states are neutral (particles = antiparticles) in the subgroup cases and charged (particles ≠ antiparticles) in the remaining cases.

4. Ansatz for the *S*-matrix

Symmetry considerations similar to the ones discussed so far can also be used to write down the general ansatz for the two-body *S*-matrix. First of all, energy-momentum conservation implies the absence of inelastic scattering in the two-body sector (since all particles/antiparticles have equal mass m). Next, charge conjugation invariance means that

$$
\begin{aligned}
{}_{\text{out}}\langle \theta_1' k_1' l_1', \theta_2' k_2' l_2' | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{\text{in}} &= {}_{\text{out}}\langle \bar{\theta}_1' \bar{k}_1' \bar{l}_1', \bar{\theta}_2' \bar{k}_2' \bar{l}_2' | \bar{\theta}_1 \bar{k}_1 \bar{l}_1, \bar{\theta}_2 \bar{k}_2 \bar{l}_2 \rangle_{\text{in}}, \\
{}_{\text{out}}\langle \overline{\theta_1' k_1' l_1'}, \theta_2' k_2' l_2' | \overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2 \rangle_{\text{in}} &= {}_{\text{out}}\langle \theta_1' k_1' l_1', \overline{\theta_2' k_2' l_2'} | \theta_1 k_1 l_1, \overline{\theta_2 k_2 l_2} \rangle_{\text{in}}.
\end{aligned} \tag{43}
$$

In addition, the particle-particle = antiparticle-antiparticle scattering amplitudes are symmetric under the exchanges $1 \leftrightarrow 2$ and $1' \leftrightarrow 2'$. The main restriction, however, comes from G-invariance, which implies that among all possible contractions of indices, only certain combinations are allowed. More concretely, this works as follows.

For the case A, indices k_1, k_2, $\bar{k}_1$, $\bar{k}_2$, k_1', k_2', $\bar{k}_1'$, $\bar{k}_2'$ transform under $\mathrm{SU}(N)_{\mathrm{L}}$ while indices l_1, l_2, $\bar{l}_1$, $\bar{l}_2$, l_1', l_2', $\bar{l}_1'$, $\bar{l}_2'$ transform under $\mathrm{SU}(N)_{\mathrm{R}}$, so they cannot be contracted with each other. Moreover, the representations N and N^* of $\mathrm{SU}(N)$ are inequivalent. This leads to

$$
\begin{aligned}
&{}_{\text{out}}\langle \theta_1' k_1' l_1', \theta_2' k_2' l_2' | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{\text{in}} \\
&\quad = +16\pi^2 \delta(\theta_1' - \theta_1)\, \delta(\theta_2' - \theta_2) \\
&\qquad \times \Big\{ +u_1(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} + u_2(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_1} \delta^{l_2' l_2} \\
&\qquad\qquad + u_3(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_2} \delta^{l_2' l_1} + u_4(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_2} \delta^{l_2' l_1} \Big\} \\
&\qquad + 16\pi^2 \delta(\theta_1' - \theta_2)\, \delta(\theta_2' - \theta_1) \\
&\qquad\quad \times \Big\{ +u_4(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} + u_3(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_1} \delta^{l_2' l_2} \\
&\qquad\qquad + u_2(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_2} \delta^{l_2' l_1} + u_1(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_2} \delta^{l_2' l_1} \Big\},
\end{aligned} \tag{44.A}
$$

$$
\begin{aligned}
&{}_{\text{out}}\langle \overline{\theta_1' k_1' l_1'}, \theta_2' k_2' l_2' | \overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2 \rangle_{\text{in}} \\
&\quad = +16\pi^2 \delta(\theta_1' - \theta_1)\, \delta(\theta_2' - \theta_2) \\
&\qquad \times \Big\{ +t_1(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} + t_2(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_1} \delta^{l_2' l_2} \\
&\qquad\qquad + t_3(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_2} \delta^{l_2' l_1} + t_4(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_2} \delta^{l_2' l_1} \Big\} \\
&\qquad + 16\pi^2 \delta(\theta_1' - \theta_2)\, \delta(\theta_2' - \theta_1) \\
&\qquad\quad \times \Big\{ +r_4(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} + r_3(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_1} \delta^{l_2' l_2} \\
&\qquad\qquad + r_2(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_2} \delta^{l_2' l_1} + r_1(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_2} \delta^{l_2' l_1} \Big\}
\end{aligned} \tag{45.A}
$$

with $\theta = |\theta_1 - \theta_2|$; invariance under the discrete symmetry (41) and (42) is then guaranteed by the additional conditions

$$u_2 = u_3, \qquad t_2 = t_3, \qquad r_2 = r_3. \tag{46.A}$$

For the subgroup cases BD resp. C, indices k_1, k_2, k_1', k_2' transform under $\mathrm{SO}(N)_{\mathrm{L}}$ resp. $\mathrm{Sp}(N)_{\mathrm{L}}$ while indices l_1, l_2, l_1', l_2' transform under $\mathrm{SO}(N)_{\mathrm{R}}$ resp. $\mathrm{Sp}(N)_{\mathrm{R}}$, so they cannot be contracted with each other. However, the representations N and N^* of $\mathrm{SO}(N)$ resp. $2N$ and $2N^*$ of $\mathrm{Sp}(N)$ are equivalent, with the matrix I as the intertwining operator. This leads to

$$\begin{aligned}
&{}_{\mathrm{out}}\langle \theta_1' k_1' l_1', \theta_2' k_2' l_2' | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{\mathrm{in}} \\
&\quad = +16\pi^2 \delta(\theta_1' - \theta_1)\, \delta(\theta_2' - \theta_2) \\
&\qquad \times \Big\{ + u_1(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} + u_2(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_1} \delta^{l_2' l_2} \\
&\qquad\quad + u_3(\theta) I^{k_1' k_2'} I^{k_1 k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} + u_4(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_2} \delta^{l_2' l_1} \\
&\qquad\quad + u_5(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_2} \delta^{l_2' l_1} + u_6(\theta) I^{k_1' k_2'} I^{k_1 k_2} \delta^{l_1' l_2} \delta^{l_2' l_1} \\
&\qquad\quad + u_7(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} I^{l_1' l_2'} I^{l_1 l_2} \\
&\qquad\quad + u_8(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} I^{l_1' l_2'} I^{l_1 l_2} \\
&\qquad\quad + u_9(\theta) I^{k_1' k_2'} I^{k_1 k_2} I^{l_1' l_2'} I^{l_1 l_2} \Big\} \\
&\qquad + 16\pi^2 \delta(\theta_1' - \theta_2)\, \delta(\theta_2' - \theta_1) \\
&\qquad \times \Big\{ + u_5(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} + u_4(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_1} \delta^{l_2' l_2} \\
&\qquad\quad \pm u_6(\theta) I^{k_1' k_2'} I^{k_1 k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} + u_2(\theta) \delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_2} \delta^{l_2' l_1} \\
&\qquad\quad + u_1(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_2} \delta^{l_2' l_1} \pm u_3(\theta) I^{k_1' k_2'} I^{k_1 k_2} \delta^{l_1' l_2} \delta^{l_2' l_1} \\
&\qquad\quad \pm u_8(\theta)\, \delta^{k_1' k_1} \delta^{k_2' k_2} I^{l_1' l_2'} I^{l_1 l_2} \pm u_7(\theta)\, \delta^{k_1' k_2} \delta^{k_2' k_1} I^{l_1' l_2'} I^{l_1 l_2} \\
&\qquad\quad + u_9(\theta) I^{k_1' k_2'} I^{k_1 k_2} I^{l_1' l_2'} I^{l_1 l_2} \Big\},
\end{aligned} \tag{44.S}$$

with $\theta = |\theta_1 - \theta_2|$; invariance under the discrete symmetry (41) and (42) is then guaranteed by the additional conditions

$$u_2 = u_4, \qquad u_3 = u_7, \qquad u_6 = u_8. \tag{46.S}$$

For the quotient space cases AI resp. AII, we must take into account the symmetry resp. antisymmetry under exchanges $k_1 \leftrightarrow l_1$, $k_2 \leftrightarrow l_2$, $\bar{k}_1 \leftrightarrow \bar{l}_1$, $\bar{k}_2 \leftrightarrow \bar{l}_2$,

$k_1' \leftrightarrow l_1'$, $k_2' \leftrightarrow l_2'$, $\bar{k}_1' \leftrightarrow \bar{l}_1'$, $\bar{k}_2' \leftrightarrow \bar{l}_2'$. Moreover, the representations N' and N'^* of $\mathrm{SU}(N')$, with $N' = N$ resp. $2N$, are inequivalent. This leads to

$$
\begin{aligned}
&{}_{\text{out}}\langle \theta_1' k_1' l_1', \theta_2' k_2' l_2' | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{\text{in}} \\
&\quad = +16\pi^2 \delta(\theta_1' - \theta_1)\, \delta(\theta_2' - \theta_2) \\
&\qquad \times \Big\{ + u_1(\theta) \Big(\delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} \pm k_1' \leftrightarrow l_1', k_2' \leftrightarrow l_2' \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2 \Big) \\
&\qquad\quad + u_2(\theta) \Big(\delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_1} \delta^{l_2' l_2} \pm k_1' \leftrightarrow l_1', k_2' \leftrightarrow l_2' \text{ and } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2 \Big) \\
&\qquad\quad + u_3(\theta) \Big(\delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_2} \delta^{l_2' l_1} \pm k_1' \leftrightarrow l_1', k_2' \leftrightarrow l_2' \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2 \Big) \Big\} \\
&\qquad + 16\pi^2 \delta(\theta_1' - \theta_2)\, \delta(\theta_2' - \theta_1) \\
&\qquad \times \Big\{ + u_3(\theta) \Big(\delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} \pm k_1' \leftrightarrow l_1', k_2' \leftrightarrow l_2' \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2 \Big) \\
&\qquad\quad + u_2(\theta) \Big(\delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_1} \delta^{l_2' l_2} \pm k_1' \leftrightarrow l_1', k_2' \leftrightarrow l_2' \text{ and } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2 \Big) \\
&\qquad\quad + u_1(\theta) \Big(\delta^{k_1' k_2} \delta^{k_2' k_1} \delta^{l_1' l_2} \delta^{l_2' l_1} \pm k_1' \leftrightarrow l_1', k_2' \leftrightarrow l_2' \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2 \Big) \Big\} .
\end{aligned}
\tag{44.Q}
$$

$$
\begin{aligned}
&{}_{\text{out}}\langle \overline{\theta_1' k_1' l_1'}, \theta_2' k_2' l_2' | \overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2 \rangle_{\text{in}} \\
&\quad = +16\pi^2 \delta(\theta_1' - \theta_1)\, \delta(\theta_2' - \theta_2) \\
&\qquad \times \Big\{ + t_1(\theta) \Big(\delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} \pm k_1' \leftrightarrow l_1', k_2' \leftrightarrow l_2' \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2 \Big) \\
&\qquad\quad + t_2(\theta) \Big(I^{k_1' k_2'} I^{k_1 k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} \pm k_1' \leftrightarrow l_1', k_2' \leftrightarrow l_2' \text{ and } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2 \Big) \\
&\qquad\quad + t_3(\theta) \Big(I^{k_1' k_2'} I^{k_1 k_2} I^{l_1' l_2'} I^{l_1 l_2} \pm k_1' \leftrightarrow l_1', k_1 \leftrightarrow l_1 \text{ or } k_2' \leftrightarrow l_2', k_2 \leftrightarrow l_2 \Big) \Big\} \\
&\qquad + 16\pi^2 \delta(\theta_1' - \theta_2)\, \delta(\theta_2' - \theta_1) \\
&\qquad \times \Big\{ + r_3(\theta) \Big(\delta^{k_1' k_1} \delta^{k_2' k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} \pm k_1' \leftrightarrow l_1', k_2' \leftrightarrow l_2' \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2 \Big) \\
&\qquad\quad + r_2(\theta) \Big(I^{k_1' k_2'} I^{k_1 k_2} \delta^{l_1' l_1} \delta^{l_2' l_2} \pm k_1' \leftrightarrow l_1', k_2' \leftrightarrow l_2' \text{ and } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2 \Big) \\
&\qquad\quad + r_1(\theta) \Big(I^{k_1' k_2'} I^{k_1 k_2} I^{l_1' l_2'} I^{l_1 l_2} \pm k_1' \leftrightarrow l_1', k_1 \leftrightarrow l_1 \text{ or } k_2' \leftrightarrow l_2', k_2 \leftrightarrow l_2 \Big) \Big\} ,
\end{aligned}
\tag{45.Q}
$$

with $\theta = |\theta_1 - \theta_2|$; invariance under the discrete symmetry (41) and (42) is then automatic.

5. The quantum non-local charge and the factorization equations

Up to this point, we have simply elaborated the consequences of the global symmetries of the two-body S-matrix. Clearly, the next step is to take into account the various other properties of these scattering amplitudes, and to derive the restrictions they impose on the (as yet undetermined) functions u_i, t_i and r_i appearing in (44), (45). As it turns out, the most stringent constraints result from the fact that the S-matrix must commute with the conserved quantum non-local charge, and it is these constraints that we are going to analyze next.

To proceed with that derivation, we must first compute the action of this charge on asymptotic one-body and two-body states, or equivalently, its expression in terms of asymptotic creation and annihilation operators. This was first carried out for the spherical models in ref. [22], but the strategy used there can equally well be applied in all the cases under consideration here: it essentially amounts to repeating the relevant definitions with interacting fields replaced by asymptotic free fields. (This implies, of course, that the normal ordering prescription and, hence, all renormalization constants for the interacting theory must also be replaced by their free field counterparts.) More specifically, apart from the standard conserved asymptotic quantum charges

$$Q_{\mathrm{ex}}^{(0)} = \int \mathrm{d}y\, j_{0,\mathrm{ex}}(t, y), \tag{47}$$

we define asymptotic quantum non-local charges

$$Q_{\mathrm{ex}}^{(1)}(t) = \lim_{\delta \to 0} Q_{\mathrm{ex},\delta}^{(1)}(t), \tag{48}$$

with

$$\begin{aligned} Q_{\mathrm{ex},\delta}^{(1)}(t) = &\int_{|y_1 - y_2| \geq \delta} \mathrm{d}y_1\, \mathrm{d}y_2\, \varepsilon(y_1 - y_2)\, j_{0,\mathrm{ex}}(t, y_1)\, j_{0,\mathrm{ex}}(t, y_2) \\ &- Z_{\mathrm{ex}}(\delta) \int \mathrm{d}y\, j_{1,\mathrm{ex}}(t, y), \end{aligned} \tag{49}$$

where

$$Z_{ex}(\delta) = -\frac{N'}{4\pi}\left(\ln\left(\tfrac{1}{2}m\delta\right) + \gamma - 1\right), \tag{50}$$

γ is Euler's constant, and N' is given by (27). Notice that since the g_{ex} do not satisfy the free field analogue of the non-linear constraints (19) and (20), and the $j_{\mu,ex}$ therefore do not satisfy the free field analogue of the short-distance expansion (22), the limits in (48) are still explicitly time-dependent. We may, however, take the limits

$$Q_{in}^{(1)} = \lim_{t\to -\infty} Q_{in}^{(1)}(t), \qquad Q_{out}^{(1)} = \lim_{t\to +\infty} Q_{out}^{(1)}(t), \tag{51}$$

and after a rather tedious calculation along the lines of ref. [22], we arrive at the following expressions in terms of asymptotic creation and annihilation operators:

$$\begin{aligned} Q_{ex}^{(0)} = -\tfrac{1}{2}i\int \mathrm{d}\mu(q)\, :&\left(b_{ex}^{+}(q)b_{ex}(q) - d_{ex}^{\mathrm{T}}(q)d_{ex}^{*}(q)\right) \\ &- \frac{1}{N'}\operatorname{tr}\left(b_{ex}^{+}(q)b_{ex}(q) - d_{ex}^{\mathrm{T}}(q)d_{ex}^{*}(q)\right):, \end{aligned} \tag{52}$$

$$Q_{in}^{(1)} = \tfrac{1}{4}N'(+A_{in} + B_{in}), \qquad Q_{out}^{(1)} = \tfrac{1}{4}N'(-A_{out} + B_{out}), \tag{53}$$

$$\begin{aligned} A_{ex} = \frac{1}{N'}\int &\mathrm{d}\mu(q_1)\,\mathrm{d}\mu(q_2)\,\varepsilon(q_1 - q_2) \\ &\times :\left(b_{ex}^{+}(q_1)b_{ex}(q_1) - d_{ex}^{\mathrm{T}}(q_1)d_{ex}^{*}(q_1)\right)\left(b_{ex}^{+}(q_2)b_{ex}(q_2) - d_{ex}^{\mathrm{T}}(q_2)d_{ex}^{*}(q_2)\right):, \end{aligned} \tag{54}$$

$$\begin{aligned} B_{ex} = \frac{i}{\pi}\int \mathrm{d}\mu(q)\ln\frac{E(q)+q}{m}\, :&\left(b_{ex}^{+}(q)b_{ex}(q) - d_{ex}^{\mathrm{T}}(q)d_{ex}^{*}(q)\right) \\ &- \frac{1}{N'}\operatorname{tr}\left(b_{ex}^{+}(q)b_{ex}(q) - d_{ex}^{\mathrm{T}}(q)d_{ex}^{*}(q)\right):, \end{aligned} \tag{55}$$

where N' is given by (27). (For the subgroup cases, the value of N' in (52) and (55) is irrelevant since the constraint (35.S) forces the trace terms to vanish. It does appear, however, in the relative normalizations of the terms A_{ex} and B_{ex}.) This in turn leads to the following explicit formulae in terms of asymptotic states*

$$A^{ij}_{\mathrm{ex}}|\theta k l\rangle_{\mathrm{ex}} = 0\,,$$
$$A^{ij}_{\mathrm{ex}}|\overline{\theta k l}\rangle_{\mathrm{ex}} = 0\,; \tag{56}$$

$$B^{ij}_{\mathrm{ex}}|\theta k l\rangle_{\mathrm{ex}} = +\frac{i\theta}{\pi}\left(\delta^{jl}|\theta k i\rangle_{\mathrm{ex}} - \frac{1}{N}\delta^{ij}|\theta k l\rangle_{\mathrm{ex}}\right),$$
$$B^{ij}_{\mathrm{ex}}|\overline{\theta k l}\rangle_{\mathrm{ex}} = -\frac{i\theta}{\pi}\left(\delta^{il}|\overline{\theta k j}\rangle_{\mathrm{ex}} - \frac{1}{N}\delta^{ij}|\overline{\theta k l}\rangle_{\mathrm{ex}}\right); \tag{57.A}$$

$$B^{ij}_{\mathrm{ex}}|\theta k l\rangle_{\mathrm{ex}} = +\frac{i\theta}{\pi}\left(\delta^{jl}|\theta k i\rangle_{\mathrm{ex}} - I^{il}I^{jn}|\theta k n\rangle_{\mathrm{ex}}\right), \tag{57.S}$$

$$B^{ij}_{\mathrm{ex}}|\theta k l\rangle_{\mathrm{ex}} = +\frac{i\theta}{\pi}\left(\delta^{jl}|\theta k i\rangle_{\mathrm{ex}} \pm \delta^{jk}|\theta l i\rangle_{\mathrm{ex}} - \frac{2}{N'}\delta^{ij}|\theta k l\rangle_{\mathrm{ex}}\right),$$
$$B^{ij}_{\mathrm{ex}}|\overline{\theta k l}\rangle_{\mathrm{ex}} = -\frac{i\theta}{\pi}\left(I^{il}I^{jn}|\overline{\theta k n}\rangle_{\mathrm{ex}} \pm I^{ik}I^{jn}|\overline{\theta l n}\rangle_{\mathrm{ex}} - \frac{2}{N'}\delta^{ij}|\overline{\theta k l}\rangle_{\mathrm{ex}}\right); \tag{57.Q}$$

$$\begin{aligned} &A^{ij}_{\mathrm{ex}}|\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{\mathrm{ex}} \\ &\quad = \frac{1}{N}\varepsilon(\theta_1-\theta_2)\left(+\delta^{in_1}\delta^{jl_2}\delta^{l_1n_2} - \delta^{in_2}\delta^{jl_1}\delta^{l_2n_1}\right)|\theta_1 k_1 n_1, \theta_2 k_2 n_2\rangle_{\mathrm{ex}}\,, \end{aligned}$$

$$\begin{aligned} &A^{ij}_{\mathrm{ex}}|\overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2\rangle_{\mathrm{ex}} \\ &\quad = \frac{1}{N}\varepsilon(\theta_1-\theta_2)\left(+\delta^{in_2}\delta^{jn_1}\delta^{l_2l_1} - \delta^{il_1}\delta^{jl_2}\delta^{n_2n_1}\right)|\overline{\theta_1 k_1 n_1}, \theta_2 k_2 n_2\rangle_{\mathrm{ex}}\,; \end{aligned} \tag{58.A}$$

$$\begin{aligned} &A^{ij}_{\mathrm{ex}}|\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{\mathrm{ex}} \\ &\quad = \frac{1}{N'}\varepsilon(\theta_1-\theta_2)\big(+\delta^{in_1}\delta^{jl_2}\delta^{l_1n_2} - \delta^{in_2}\delta^{jl_1}\delta^{l_2n_1} \\ &\qquad\qquad - \delta^{in_1}I^{jn_2}I^{l_1l_2} + \delta^{in_2}I^{jn_1}I^{l_2l_1} \\ &\qquad\qquad - I^{il_1}\delta^{jl_2}I^{n_2n_1} + I^{il_2}\delta^{jl_1}I^{n_1n_2} \\ &\qquad\qquad + I^{il_1}I^{jn_2}\delta^{l_2n_1} - I^{il_2}I^{jn_1}\delta^{l_1n_2}\big)|\theta_1 k_1 n_1, \theta_2 k_2 n_2\rangle_{\mathrm{ex}}\,; \end{aligned} \tag{58.S}$$

* The formulae for $Q^{(0)}_{\mathrm{ex}}$ can be obtained from those for B_{ex} by the simple substitution $i\theta/\pi \to \frac{1}{2}i$.

$$A^{ij}_{\rm ex}|\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{\rm ex}$$

$$\begin{aligned}
&= \frac{1}{N'}\varepsilon(\theta_1-\theta_2)\Big\{ +(\delta^{in_1}\delta^{jl_2}\delta^{l_1n_2} - \delta^{in_2}\delta^{jl_1}\delta^{l_2n_1})|\theta_1 k_1 n_1, \theta_2 k_2 n_2\rangle_{\rm ex} \\
&\qquad \pm(\delta^{in_1}\delta^{jl_2}\delta^{k_1n_2} - \delta^{in_2}\delta^{jk_1}\delta^{l_2n_1})|\theta_1 l_1 n_1, \theta_2 k_2 n_2\rangle_{\rm ex} \\
&\qquad \pm(\delta^{in_1}\delta^{jk_2}\delta^{l_1n_2} - \delta^{in_2}\delta^{jl_1}\delta^{k_2n_1})|\theta_1 k_1 n_1, \theta_2 l_2 n_2\rangle_{\rm ex} \\
&\qquad +(\delta^{in_1}\delta^{jk_2}\delta^{k_1n_2} - \delta^{in_2}\delta^{jk_1}\delta^{k_2n_1})|\theta_1 l_1 n_1, \theta_2 l_2 n_2\rangle_{\rm ex}\Big\},
\end{aligned}$$

$$A^{ij}_{\rm ex}|\overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2\rangle_{\rm ex}$$

$$\begin{aligned}
&= \frac{1}{N'}\varepsilon(\theta_1-\theta_2)\Big\{ +(\delta^{in_2}I^{jn_1}I^{l_2l_1} - I^{il_1}\delta^{jl_2}I^{n_2n_1})|\overline{\theta_1 k_1 n_1}, \theta_2 k_2 n_2\rangle_{\rm ex} \\
&\qquad \pm(\delta^{in_2}I^{jn_1}I^{l_2k_1} - I^{ik_1}\delta^{jl_2}I^{n_2n_1})|\overline{\theta_1 l_1 n_1}, \theta_2 k_2 n_2\rangle_{\rm ex} \\
&\qquad \pm(\delta^{in_2}I^{jn_1}I^{k_2l_1} - I^{il_1}\delta^{jk_2}I^{n_2n_1})|\overline{\theta_1 k_1 n_1}, \theta_2 l_2 n_2\rangle_{\rm ex} \\
&\qquad +(\delta^{in_2}I^{jn_1}I^{k_2k_1} - I^{ik_1}\delta^{jk_2}I^{n_2n_1})|\overline{\theta_1 l_1 n_1}, \theta_2 l_2 n_2\rangle_{\rm ex}\Big\}; \qquad (58.\mathrm{Q})
\end{aligned}$$

$$\begin{aligned}
B^{ij}_{\rm ex}|\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{\rm ex} &= +\frac{i\theta_1}{\pi}\left(\delta^{jl_1}|\theta_1 k_1 i, \theta_2 k_2 l_2\rangle_{\rm ex} - \frac{1}{N}\delta^{ij}|\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{\rm ex}\right) \\
&\quad +\frac{i\theta_2}{\pi}\left(\delta^{jl_2}|\theta_1 k_1 l_1, \theta_2 k_2 i\rangle_{\rm ex} - \frac{1}{N}\delta^{ij}|\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{\rm ex}\right),
\end{aligned}$$

$$\begin{aligned}
B^{ij}_{\rm ex}|\overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2\rangle_{\rm ex} &= -\frac{i\theta_1}{\pi}\left(\delta^{il_1}|\overline{\theta_1 k_1 j}, \theta_2 k_2 l_2\rangle_{\rm ex} - \frac{1}{N}\delta^{ij}|\overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2\rangle_{\rm ex}\right) \\
&\quad +\frac{i\theta_2}{\pi}\left(\delta^{jl_2}|\overline{\theta_1 k_1 l_1}, \theta_2 k_2 i\rangle_{\rm ex} - \frac{1}{N}\delta^{ij}|\overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2\rangle_{\rm ex}\right);
\end{aligned}$$

(59.A)

$$\begin{aligned}
B^{ij}_{\rm ex}|\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{\rm ex} &= +\frac{i\theta_1}{\pi}\left(\delta^{jl_1}|\theta_1 k_1 i, \theta_2 k_2 l_2\rangle_{\rm ex} - I^{il_1}I^{jn_1}|\theta_1 k_1 n_1, \theta_2 k_2 l_2\rangle_{\rm ex}\right) \\
&\quad +\frac{i\theta_2}{\pi}\left(\delta^{jl_2}|\theta_1 k_1 l_1, \theta_2 k_2 i\rangle_{\rm ex} - I^{il_2}I^{jn_2}|\theta_1 k_1 l_1, \theta_2 k_2 n_2\rangle_{\rm ex}\right);
\end{aligned}$$

(59.S)

$$B_{\mathrm{ex}}^{ij}|\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{\mathrm{ex}} = +\frac{i\theta_1}{\pi}\Big(\delta^{jl_1}|\theta_1 k_1 i, \theta_2 k_2 l_2\rangle_{\mathrm{ex}} \pm \delta^{jk_1}|\theta_1 l_1 i, \theta_2 k_2 l_2\rangle_{\mathrm{ex}} - \frac{2}{N'}\delta^{ij}|\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{\mathrm{ex}}\Big)$$
$$+\frac{i\theta_2}{\pi}\Big(\delta^{jl_2}|\theta_1 k_1 l_1, \theta_2 k_2 i\rangle_{\mathrm{ex}} \pm \delta^{jk_2}|\theta_1 k_1 l_1, \theta_2 l_2 i\rangle_{\mathrm{ex}} - \frac{2}{N'}\delta^{ij}|\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{\mathrm{ex}}\Big),$$

$$B_{\mathrm{ex}}^{ij}|\overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2\rangle_{\mathrm{ex}} = -\frac{i\theta_1}{\pi}\Big(I^{il_1}I^{jn_1}|\overline{\theta_1 k_1 n_1}, \theta_2 k_2 l_2\rangle_{\mathrm{ex}} \pm I^{ik_1}I^{jn_1}|\overline{\theta_1 l_1 n_1}, \theta_2 k_2 l_2\rangle_{\mathrm{ex}} - \frac{2}{N'}\delta^{ij}|\overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2\rangle_{\mathrm{ex}}\Big)$$
$$+\frac{i\theta_2}{\pi}\Big(\delta^{jl_2}|\overline{\theta_1 k_1 l_1}, \theta_2 k_2 i\rangle_{\mathrm{ex}} \pm \delta^{jk_2}|\overline{\theta_1 k_1 l_1}, \theta_2 l_2 i\rangle_{\mathrm{ex}} - \frac{2}{N'}\delta^{ij}|\overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2\rangle_{\mathrm{ex}}\Big). \quad (59.\mathrm{Q})$$

On the other hand, the standard quantum charge $Q^{(0)}$ and the quantum non-local charge $Q^{(1)}$ of the interacting theory are conserved, and since the interacting field g converges at large times to the asymptotic free fields g_{ex}, we conclude, as in ref. [22], that

$$Q_{\mathrm{in}}^{(0)} = Q^{(0)} = Q_{\mathrm{out}}^{(0)}, \quad (60)$$

$$Q_{\mathrm{in}}^{(1)} = Q^{(1)} = Q_{\mathrm{out}}^{(1)}. \quad (61)$$

For the standard charge, which generates isospin symmetry, this is a legitimate procedure, but for the non-local charge, eq. (61) is a non-trivial relation which does not follow from any of the arguments presented so far. This was first pointed out in ref. [24], where it was also argued that the condition for (61) to hold is that the action of $Q^{(1)}$ on asymptotic incoming and outgoing one-body states coincides with that of $Q_{\mathrm{in}}^{(1)}$ and $Q_{\mathrm{out}}^{(1)}$, respectively, as given by (56) and (57). But on such states, the action of $Q^{(1)}$ can be computed from the action of $Q^{(0)}$, which is known due to (60), by making use of the commutation relation (26), and the result is (61) – just as for the spherical models where this was shown in [23].

The desired additional constraints on the scattering amplitudes now follow by combining (44), (45), (46) and (53), (58), (59) with the relations

$$
\begin{aligned}
&{}_{\mathrm{out}}\langle \theta_1' k_1' l_1', \theta_2' k_2' l_2' | Q_{\mathrm{in}} | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{\mathrm{in}} = {}_{\mathrm{out}}\langle \theta_1' k_1' l_1', \theta_2' k_2' l_2' | Q_{\mathrm{out}} | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{\mathrm{in}}, \\
&{}_{\mathrm{out}}\langle \overline{\theta_1' k_1' l_1'}, \theta_2' k_2' l_2' | Q_{\mathrm{in}} | \overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2 \rangle_{\mathrm{in}} = {}_{\mathrm{out}}\langle \overline{\theta_1' k_1' l_1'}, \theta_2' k_2' l_2' | Q_{\mathrm{out}} | \overline{\theta_1 k_1 l_1}, \theta_2 k_2 l_2 \rangle_{\mathrm{in}},
\end{aligned}
\tag{62}
$$

plus the fact that $Q_{\mathrm{cx}}^{+} = -Q_{\mathrm{cx}}$. After tedious calculation, this gives

$$
\begin{aligned}
&u_2(\theta) = u_3(\theta) = -\frac{2\pi i}{N\theta} u_1(\theta), \qquad u_4(\theta) = -\frac{4\pi^2}{N^2\theta^2} u_1(\theta), \\
&t_2(\theta) = t_3(\theta) = -\frac{2\pi i}{N(i\pi-\theta)} t_1(\theta), \qquad t_4(\theta) = -\frac{4\pi^2}{N^2(i\pi-\theta)^2} t_1(\theta), \\
&r_1(\theta) = r_2(\theta) = r_3(\theta) = r_4(\theta) = 0.
\end{aligned}
\tag{63.A}
$$

$$
\begin{aligned}
&u_2(\theta) = u_4(\theta) = -\frac{2\pi i}{N'\theta} u_1(\theta), \\
&u_3(\theta) = u_7(\theta) = -\frac{2\pi i}{N'(i\pi-\theta)} u_1(\theta), \\
&u_5(\theta) = -\frac{4\pi^2}{N'^2\theta^2} u_1(\theta), \\
&u_6(\theta) = u_8(\theta) = -\frac{4\pi^2}{N'^2\theta(i\pi-\theta)} u_1(\theta), \\
&u_9(\theta) = -\frac{4\pi^2}{N'^2(i\pi-\theta)^2} u_1(\theta).
\end{aligned}
\tag{63.S}
$$

$$
\begin{aligned}
&u_2(\theta) = \mp\left(1 \pm \frac{N'\theta}{2\pi i}\right)^{-1} u_1(\theta), \qquad u_3(\theta) = -\frac{4\pi i}{N'\theta} u_2(\theta), \\
&t_2(\theta) = \mp\left(1 \pm \frac{N'(i\pi-\theta)}{2\pi i}\right)^{-1} t_1(\theta), \qquad t_3(\theta) = -\frac{4\pi i}{N'(i\pi-\theta)} t_2(\theta), \\
&r_1(\theta) = r_2(\theta) = r_3(\theta) = 0.
\end{aligned}
\tag{63.Q}
$$

We repeat that the constraints (63) are precisely the conditions which guarantee that the two-body S-matrix commutes with the conserved quantum non-local charge. On the other hand, one can verify that they force the two-body S-matrix to satisfy the factorization equations, or Yang-Baxter equations. This means that for the models under consideration here, the latter acquire a field-theoretic underpinning.

6. The minimal S-matrix

From this point onward, the further procedure is more or less standard: we just have to make use of unitarity, analyticity and crossing symmetry.

First of all, unitarity leads to

$$|u_1(\theta)|^2 = \left(\frac{\theta^2}{\theta^2 + (2\pi/N)^2}\right)^2, \qquad |t_1(\theta)|^2 = 1, \tag{64.A}$$

$$|u_1(\theta)|^2 = \left(\frac{\theta^2}{\theta^2 + (2\pi/N')^2}\right)^2, \tag{64.S}$$

$$|u_1(\theta)|^2 = \frac{\theta^2}{\theta^2 + (4\pi/N')^2}, \qquad |t_1(\theta)|^2 = 1, \tag{64.Q}$$

all other unitarity conditions then being satisfied automatically as a consequence of (63). Next, we note that eqs. (63) and (64) are a priori valid only in the physical region, i.e. for θ real and positive. Analyticity tells us, however, that the amplitudes u_i and t_i can all be continued to meromorphic functions in the complex θ-plane, or rather in the so-called physical strip of that plane defined by the condition that the imaginary part of θ should be restricted to lie between 0 and π, in such a way that

$$u_i(\theta)^* = u_i(-\theta^*), \qquad t_1(\theta)^* = t_i(-\theta^*). \tag{65}$$

To justify this assumption, we note that the usual Mandelstam variable s (total energy in the center-of-mass frame) is related to the variable θ (rapidity difference) by

$$s = 2m^2(1 + \cosh\theta) = 4m^2\cosh^2\tfrac{1}{2}\theta. \tag{66}$$

In fact, this formula establishes a homeomorphic mapping from the closed strip $0 \leqslant \operatorname{Im}\theta \leqslant \pi$ in the complex θ-plane onto the cut complex s-plane, which is biholomorphic on the open strip $0 < \operatorname{Im}\theta < \pi$ and maps

(a) the boundary $\operatorname{Im}\theta = 0$ to the branch cut $\operatorname{Re} s \geqslant 4m^2$, $\operatorname{Im} s = 0$ imposed by s-channel unitarity,

(b) the boundary $\operatorname{Im}\theta = \pi$ to the branch cut $\operatorname{Re} s \leqslant 0$, $\operatorname{Im} s = 0$ imposed by t-channel unitarity (at $u = 0$) or u-channel unitarity (at $t = 0$),

(c) the piece $\operatorname{Re}\theta = 0$, $0 < \operatorname{Im}\theta < \pi$ of the imaginary θ-axis lying in the strip to the piece $0 < \operatorname{Re} s < 4m^2$, $\operatorname{Im} s = 0$ of the real s-axis lying between the branch points.

Thus (65) is just the condition of hermitian analyticity, which requires the imaginary part of the scattering amplitude to equal its discontinuity over the branch cuts and to vanish on the interval between the two branch points (cf. ref. [25], p. 17). Moreover, crossing symmetry gives rise to one additional condition, namely

$$t_1(\theta) = u_1(i\pi - \theta), \tag{67.A}$$

$$u_1(\theta) = u_1(i\pi - \theta), \tag{67.S}$$

$$t_1(\theta) = u_1(i\pi - \theta), \tag{67.Q}$$

all other crossing relations then being satisfied automatically as a consequence of (63).

To summarize, we see that the entire two-body S-matrix can be computed from a single function $u_1(\theta)$, which is meromorphic on the open strip $0 < \operatorname{Im}\theta < \pi$, with its zeros and poles located on the imaginary axis*, and which extends continuously to the closed strip $0 \leqslant \operatorname{Im}\theta \leqslant \pi$. In addition, this function must satisfy the hermitian analyticity condition

$$u_1(\theta)^* = u_1(-\theta^*), \tag{68}$$

as well as the unitarity and crossing conditions

$$|u_1(\theta)|^2 = \left(\frac{\theta^2}{\theta^2 + (2\pi/N)^2}\right)^2, \qquad |u_1(i\pi - \theta)|^2 = 1 \qquad (\theta \text{ real}), \tag{69.A}$$

$$|u_1(\theta)|^2 = \left(\frac{\theta^2}{\theta^2 + (2\pi/N')^2}\right)^2 = |u_1(i\pi - \theta)|^2 \qquad (\theta \text{ real}), \tag{69.S}$$

$$|u_1(\theta)|^2 = \frac{\theta^2}{\theta^2 + (4\pi/N')^2}, \qquad |u_1(i\pi - \theta)|^2 = 1 \qquad (\theta \text{ real}), \tag{69.Q}$$

* This is an additional restriction made for convenience.

together with the aforementioned additional constraint for the subgroup cases:

$$u_1(\theta) = u_1(i\pi - \theta). \tag{70}$$

As usual, it is convenient to split the general solution $u_1(\theta)$ into a minimal solution $u_1^{\min}(\theta)$ and a CDD factor $P(\theta)$ [26]:

$$u_1(\theta) = u_1^{\min}(\theta) P(\theta). \tag{71}$$

In general, the term "minimal" refers to the number of zeros and poles on the physical sheet, but for the models under consideration here, it turns out that the minimal solution $u_1^{\min}(\theta)$ has no zeros and no poles whatsoever in the open strip $0 < \operatorname{Im}\theta < \pi$. Hence all zeros and all poles are collected in the CDD factor $P(\theta)$, which must satisfy the analogues of (68), (69) and (70), viz.

$$P(\theta)^* = P(-\theta^*), \tag{72}$$

$$|P(\theta)|^2 = 1, \qquad |P(i\pi - \theta)|^2 = 1 \qquad (\theta \text{ real}), \tag{73}$$

and for the subgroup cases,

$$P(\theta) = P(i\pi - \theta). \tag{74}$$

We note finally that under very mild restrictions on the growth at $\operatorname{Re}\theta \to \pm\infty$, $u_1^{\min}(\theta)$ is unique (up to a constant phase), while $u_1(\theta)$ and $P(\theta)$ are determined uniquely (up to a constant phase) by the positions and orders of their zeros and poles: this follows from combining a theorem of Phragmen and Lindelöf (cf. ref. [27], pp. 128/129) with the fact that a holomorphic function of modulus 1 is constant.

Explicit formulae are easily derived by imitating procedures found in the literature [9, 13–15, 17, 18]. Thus the unique minimal solutions $u_1^{\min}(\theta)$ are given by

$$u_1^{\min}(\theta) = \left(\frac{\Gamma(1-\theta/2\pi i)\Gamma(1/N+\theta/2\pi i)}{\Gamma(\theta/2\pi i)\Gamma(1+1/N-\theta/2\pi i)} \right)^2, \tag{75.A}$$

$$u_1^{\min}(\theta) = \left(\frac{\Gamma(\frac{1}{2}+\theta/2\pi i)\Gamma(1/N'+\theta/2\pi i)\Gamma(1-\theta/2\pi i)\Gamma(\frac{1}{2}+1/N'-\theta/2\pi i)}{\Gamma(\theta/2\pi i)\Gamma(\frac{1}{2}+1/N'+\theta/2\pi i)\Gamma(\frac{1}{2}-\theta/2\pi i)\Gamma(1+1/N'-\theta/2\pi i)} \right)^2 \tag{75.S}$$

$$u_1^{\min}(\theta) = \frac{\Gamma(1-\theta/2\pi i)\Gamma(2/N'+\theta/2\pi i)}{\Gamma(\theta/2\pi i)\Gamma(1+2/N'-\theta/2\pi i)}. \tag{75.Q}$$

Note that the solution (75.A) is the square of the class II solution of ref. [13], and the solution (75.S) is the square of the solution of ref. [9]; they have of course been derived before [11, 12, 15, 17]. In addition, the solution (75.Q) coincides with the solution $R^{(2^{+},2^{-})}(u, \eta)$ of ref. [14], with $u = \theta$ and $\eta = -2\pi i/N'$. On the other hand, let $0 < \lambda < \pi$, and define

$$P_{\lambda}(\theta) = \frac{\sinh\frac{1}{2}(\theta + i\lambda)}{\sinh\frac{1}{2}(\theta - i\lambda)} = \frac{\tanh\frac{1}{2}\theta + i\tan\frac{1}{2}\lambda}{\tanh\frac{1}{2}\theta - i\tan\frac{1}{2}\lambda}. \tag{76}$$

$$P_{\lambda,\pi-\lambda}(\theta) = \frac{\tanh\frac{1}{2}(\theta + i\lambda)}{\tanh\frac{1}{2}(\theta - i\lambda)} = \frac{\sinh\theta + i\sin\lambda}{\sinh\theta - i\sin\lambda}. \tag{77}$$

It is easily seen that in the open strip $0 < \operatorname{Im}\theta < \pi$, $P_{\lambda}(\theta)$ and $P_{\lambda,\pi-\lambda}(\theta)$ both solve (72), (73) and have no zeros, the difference being that $P_{\lambda}(\theta)$ has a simple pole at $i\lambda$, while $P_{\lambda,\pi-\lambda}(\theta)$ also solves (74) and has two simple poles at $i\lambda$ and $i(\pi - \lambda)$; in fact,

$$P_{\lambda,\pi-\lambda}(\theta) = P_{\lambda}(\theta)\,P_{\pi-\lambda}(\theta). \tag{78}$$

The signs in (76) and (77) are chosen so as to be consistent with the condition [28] that the residue of the CDD factor $P(\theta)$ (multiplied by i) must be negative at the physical poles. Thus for $P(\theta) = P_{\lambda}(\theta)$, the pole at $\theta = i\lambda$ is physical, while for $P(\theta) = P_{\lambda,\pi-\lambda}(\theta)$, the pole at $\theta = i\lambda$ is physical whereas that at $\theta = i(\pi - \lambda)$ is not: it corresponds to a physical pole in the other channel. Finally, for the subgroup cases resp. for the remaining cases, the most general CDD factor $P(\theta)$ is a product of certain $P_{\lambda,\pi-\lambda}(\theta)$'s resp. $P_{\lambda}(\theta)$'s and their inverses.

7. Bound states

From the analysis above, we conclude that the full S-matrix will be determined uniquely (up to a constant phase) once we fix the positions and orders of the zeros and poles occurring in the CDD factor $P(\theta)$. In more physical terms, this means that we must know whether the original particles and antiparticles of the model under consideration produce bound states, and if so, what are their properties - in particular, their masses. Dynamically, this is a difficult problem because it requires information on the theory off-shell, and we would like to emphasize that such information cannot be obtained by the methods discussed so far in this paper. Of course, one can use perturbation theory to analyze the question whenever a reliable perturbative scheme is available: a well-known example is provided by the $1/N$ expansion, which has the advantages of respecting the internal symmetries and of being free from infrared problems. Such an analysis has been carried out for the $O(N)$-invariant non-linear σ model on S^{N-1} [9], the N-component Gross-Neveu model [9] and the N-component chiral Gross-Neveu model [29], with the result that the factorizing S-matrix for the first model is the minimal one, while those for the second resp. third model contain CDD factors $P_{\lambda,\pi-\lambda}(\theta)$ resp. $P_{\lambda}(\theta)$ with $\lambda = 2\pi/(N-2)$ resp. $\lambda = 2\pi/N$. In physical terms, this means that the fundamental bosons of the σ model interact by repulsive forces and hence do not form bound states, while the fundamental fermions of the GN or CGN model interact by attractive forces and hence are capable of forming bound states, at least in certain channels.

For the matrix models under consideration in the present paper, however, the $1/N$ expansion does not work, since an infinite number of graphs contributes to each given order of perturbation theory. Moreover, the Bethe ansatz technique [16–19] – although capable of dealing with the problem from a different angle – is tailored to the group manifold cases and will provide no insight for the coset space cases.

In view of this situation, the best we can do is to provide some more or less heuristic arguments. One possibility is to require, for the sake of simplicity, that the two-body S-matrix should admit simple zeros and poles only, along the entire imaginary θ-axis (with the possible exception of the points $\theta = i\pi n$, $n \in \mathbb{Z}$, where (69) and (70) may enforce the appearance of double zeros and/or poles). For the group manifold cases, it turns out that this condition is sufficient to fix the CDD factor [11, 12]: the results are given in eqs. (81.A), (81.BD), (81.C) below, and they agree with those obtained from the Bethe ansatz technique [16, 17]. The other heuristic argument starts out from the observation that all the models under consideration here are subject to an additional constraint which has so far been disregarded. Namely, we can view the determinant constraint (20) imposed on the interacting field g as a hint that antiparticles are to be identified with bound states of particles if, assuming that these bound states are formed with the help of totally antisymmetric tensor products, such an identification is allowed by representation theory. This should be compared with the bootstrap condition proposed in ref. [17], which requires all particles/antiparticles, and their bound states, to belong to one of the fundamental representations of the relevant group. For the subgroup cases, the two criteria would give rise to different bound state spectra, but we are convinced that the determinant constraint alluded to above is not sufficiently restrictive to handle the groups $SO(N)$ or $Sp(N)$, and so the results of ref. [17] should be the correct ones. For the remaining cases, on the other hand, the two criteria can be expected to lead to the same bound state spectra, simply because the fundamental representations of the group $SU(N)$ are precisely the exterior powers of its defining representation. We can thus take a look at the last paragraph of sect. 3 and conclude that

antiparticles are bound states of $N-1$ particles
(A series: SU(N)),

antiparticles cannot be bound states of particles
(AI series: SU(N)/SO(N)),

antiparticles are bound states of $N-1$ particles
(AII series: SU($2N$)/Sp(N)).

But the fusion method [30] predicts that due to the factorization equations, the existence of a bound state formed of two particles, with mass $m_2 = 2m\cos\frac{1}{2}\lambda$, implies the existence of a whole spectrum of bound states formed of k particles, with mass

$$m_k = m\frac{\sin(\frac{1}{2}k\lambda)}{\sin\frac{1}{2}\lambda}. \tag{79}$$

Hence the condition $m_{N-1} = m$ will yield the position of the pole,

$$\lambda = 2\pi/N, \tag{80}$$

which should be valid for the A series and for the AII series.

To summarize the previous discussion, we arrive at the following pole factors:

$$P(\theta) = \frac{\sinh\frac{1}{2}(\theta + 2\pi i/N)}{\sinh\frac{1}{2}(\theta - 2\pi i/N)}, \qquad (\mathrm{SU}(N)), \tag{81.A}$$

$$P(\theta) = \frac{\tanh\frac{1}{2}(\theta + 2\pi i/N')}{\tanh\frac{1}{2}(\theta - 2\pi i/N')}, \qquad \begin{cases} (\mathrm{SO}(N)\colon\ N' = N-2) & (81.\mathrm{BD}) \\ (\mathrm{Sp}(N)\colon\ N' = 2N+2), & (81.\mathrm{C}) \end{cases}$$

$$P(\theta) = 1 \qquad (\mathrm{SU}(N)/\mathrm{SO}(N)), \tag{81.AI}$$

$$P(\theta) = \frac{\sinh\frac{1}{2}(\theta + 2\pi i/N)}{\sinh\frac{1}{2}(\theta - 2\pi i/N)}, \qquad (\mathrm{SU}(2N)/\mathrm{Sp}(N)), \tag{81.AII}$$

In physical terms, this means that the particles for the AI series, being symmetric rank-2 tensors under SU(N), interact repulsively and do not form bound states, while the particles for the AII series, being antisymmetric rank-2 tensors under SU($2N$), interact attractively and do form bound states: these transform as antisymmetric tensors of arbitrary even rank under SU($2N$). This suggests an unexpected connection between the SU($2N$)-invariant non-linear σ model on the coset space $SU(2N)/Sp(N)$ and the $2N$-component chiral Gross-Neveu model: the fundamental bosons in the former should be 2-particle bound states of the fundamental fermions in the latter. It would be interesting to see whether this connection between the two models can be extended off-shell.

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References

[1] H. Eichenherr and M. Forger, Nucl. Phys. B155 (1979) 381
[2] H. Eichenherr and M. Forger, Nucl. Phys. B164 (1980) 528; B282 (1987) 745
[3] H. Eichenherr and M. Forger, Comm. Math. Phys. 82 (1981) 227
[4] E. Abdalla, M. Forger and M. Gomes, Nucl. Phys. B210 [FS6] (1982) 181
[5] E. Abdalla, M.C.B. Abdalla and M. Gomes, Phys. Rev. D25 (1982) 452; D27 (1983) 825
[6] E. Abdalla, M. Forger and A. Lima-Santos, Nucl. Phys. B256 (1985) 145
[7] E. Abdalla and M. Forger, Comm. Math. Phys. 104 (1986) 123
[8] S. Helgason, Differential geometry, Lie groups and symmetric spaces (Academic Press, New York, 1978)
[9] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. B133 (1978) 525; Ann. of Phys. 120 (1979) 253
[10] M.C.B. Abdalla and A. Lima-Santos, Acta Phys. Pol. B15 (1984) 813
[11] E. Abdalla, M.C.B. Abdalla and A. Lima-Santos, Phys. Lett. 140B (1984) 71; 146B (1984) 457
[12] M.C.B. Abdalla, Phys. Lett. 164B (1985) 71
[13] B. Berg, M. Karowski, V. Kurak and P. Weisz, Nucl. Phys. B134 (1978) 125
[14] P. Kulish, N. Reshetikhin and E. Sklyanin, Lett. Math. Phys. 5 (1981) 393
[15] P. Wiegmann, Phys. Lett. 142B (1984) 173
[16] P. Wiegmann, Phys. Lett. 141B (1984) 217
[17] E. Ogievetsky, N. Reshetikhin and P. Wiegmann, preprint NORDITA 84-38; Nucl. Phys. B280 [FS18] (1987) 45
[18] E. Ogievetsky and P. Wiegmann, Phys. Lett. 168B (1986) 360
[19] A.M. Polyakov and P. Wiegmann, Phys. Lett. 131B (1983) 121
[20] A.M. Polyakov, Phys. Lett. 72B (1979) 247;
J. Lowenstein and E. Speer, Nucl. Phys. B158 (1979) 397
[21] Y.Y. Goldschmidt and E. Witten, Phys. Lett. 91B (1980) 392
[22] M. Lüscher, Nucl. Phys. B135 (1978) 1
[23] M. Lüscher, Revision of [22], unpublished
[24] D. Buchholz and J.T. Łopuszański, Lett. Math. Phys. 3 (1979) 175
[25] R. Eden, P. Landshoff, D. Olive and J. Polkinghorne, The analytic S-matrix (Cambridge University Press, Cambridge, 1966)
[26] L. Castillejo, R.H. Dalitz and F.J. Dyson, Phys. Rev. 101 (1956) 453
[27] A. Holland, Introduction to the theory of entire functions (Academic Press, New York, 1973)
[28] M. Karowski, Nucl. Phys. B153 (1979) 244
[29] R. Köberle, V. Kurak and J.A. Swieca, Phys. Rev. D20 (1979) 897
[30] B. Schroer, T.T. Truong and P. Weisz, Phys. Lett. 63B (1976) 422