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TWISTED CHIRAL MODELS WITH WESS-ZUMINO TERMS, AND STRINGS

M. FORGER[1]

CERN, Geneva, Switzerland

P. ZIZZI[2]

Institut für Theoretische Physik, Universität Karlsruhe, D-7500 Karlsruhe 1, Fed. Rep. Germany

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We discuss the problem of consistent symmetric space-type constraints for principal chiral models with Wess-Zumino terms, and arrive at twisted string theories on group manifolds.

One of the most remarkable recent achievements in field theory has been the progress in understanding the rôle of conformal invariance in two-dimensional QFT [1, 2], together with the discovery of large new classes of models that exhibit such an invariance. In fact, we suspect that many of the known integrable relativistic two-dimensional models can be modified in such a way that the usual phenomenon of dynamical mass generation is suppressed, and conformal invariance sets in even at the quantum level. Consider, in particular, the class of integrable chiral models ($\equiv$ non-linear σ models), which are known to be precisely the ones defined on riemannian symmetric spaces [3–5]. What we would like to find is a corresponding class of modified chiral models, which should also be associated with riemannian symmetric spaces, and such that the field equations force the light-cone components of the currents to depend on only one of the two light-cone coordinates: $J_+ = J_+(x^+)$, $J_- = J_-(x^-)$. Then, as a result, these current components would generate two mutually commuting Kac-Moody algebras, which in turn would give rise to two mutually commuting Virasoro algebras, via the Sugawara construction [6], and thus establish conformal invariance.

In the course of his work on non-abelian bosonization in two dimensions [7], Witten has shown how this program can be carried out for compact Lie groups

[1] On leave of absence from Fakultät für Physik der Universität Freiburg, Federal Republic of Germany.
[2] Alexander von Humboldt Fellow.

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(type II spaces), and that two key steps have to be taken in order to arrive at the desired modification, namely:

(i) change the lagrangian by adding a WZ term;

(ii) change the definition of the currents.

On the other hand, attempts to generalize Witten's approach to the remaining riemannian symmetric spaces (type I spaces) have so far failed, and statements such as that the third de Rham cohomology group $H^3(M)$ vanishes for spaces M of type I, or that spaces of type I are certainly not parallelizable (together with the observation that WZ terms originate from torsion tensors on parallelizable manifolds [8]), have led to a widespread belief that such a generalization is impossible.

The main point of the present note is to show that this is not true. In fact, we shall provide an easy and elegant way out of the apparent impasse. To describe it in words, we need only recall that a symmetric space can be realized, via the so-called Cartan immersion, as a totally geodesic submanifold of the corresponding group, and that this submanifold can be described by a simple constraint. The basic idea is then that apart from (i) and (ii) above, a third key step has to be taken to arrive at the desired modification, namely:

(iii) change the constraint.

In the following, we shall work this out in more detail, and in particular, we shall see that sticking to the old constraint would be inconsistent, for a variety of reasons.

As a preparatory step, let us first discuss the most naive guess for a possible modification of the chiral model, which would be to require that the target manifold M for the chiral field $q = q(x)$ carries, in addition to its given metric tensor g, a given two-form ω, and to consider the following lagrangian:

$$L = \left(\frac{1}{2\lambda_s} \eta^{\mu\nu} g_{\alpha\beta} + \frac{1}{2\lambda_a} \varepsilon^{\mu\nu} \omega_{\alpha\beta} \right) \partial_\mu q^\alpha \, \partial_\nu q^\beta \,, \tag{1}$$

where η is the standard metric tensor and ε is the standard volume two-form on two-dimensional flat space-time*, while λ_s and λ_a are coupling constants (with $\lambda_s > 0$ to ensure stability). The corresponding equations of motion are:

$$\frac{1}{\lambda_s} \eta^{\mu\nu} \left(\partial_\mu \partial_\nu q^\alpha + \Gamma^\alpha_{\beta\gamma} \partial_\mu q^\beta \, \partial_\nu q^\gamma \right) - \frac{1}{2\lambda_a} \varepsilon^{\mu\nu} g^{\alpha\delta} (d\omega)_{\beta\gamma\delta} \partial_\mu q^\beta \, \partial_\nu q^\gamma = 0 \,. \tag{2}$$

* Our conventions are:

$$\mu, \nu = 0,1; \qquad \eta_{00} = +1, \qquad \eta_{11} = -1, \qquad \varepsilon_{01} = 1, \qquad x_\pm = x_0 \pm x_1, \qquad \partial_\pm = \tfrac{1}{2}(\partial_0 \pm \partial_1)$$

in the Minkowski case,

$$\mu, \nu = 1,2; \qquad \eta_{11} = \eta_{22} = 1, \qquad \varepsilon_{12} = i, \qquad x_\pm = x_1 \pm i x_2, \qquad \partial_\pm = \tfrac{1}{2}(\partial_1 \mp i \partial_2)$$

in the euclidean case.

Now if we require that the second term in the lagrangian (1) does not explicitly break any of the continuous global symmetries of the first term, and if M is riemannian symmetric, we can write $\mathrm{M} = \mathrm{G/H}$ and conclude that g and ω are both G-invariant. But this implies that ω is closed. (It is a general theorem that on symmetric spaces G/H, all G-invariant differential forms are automatically closed: this generalizes the well-known fact that all bi-invariant differential forms on Lie groups are automatically closed, and is equally easy to prove.) Therefore, the second term in the lagrangian (1) has no effect on the equations of motion. Moreover, if ω is exact, then it is even a pure divergence:

$$\omega = d\theta \Rightarrow \varepsilon^{\mu\nu}\omega_{\alpha\beta}\partial_\mu q^\alpha\, \partial_\nu q^\beta = 2\varepsilon^{\mu\nu}\,\partial_\mu\left(\theta_\alpha \partial_\nu q^\alpha\right). \tag{3}$$

Otherwise, it can be viewed as a topological density, and so although it does not change the classical theory, it will have an effect on the quantum theory: this is precisely what happens if M is hermitian symmetric, ω being the Kähler two-form. But a short look at $\mathbb{C}\mathrm{P}^{N-1}$ models or grassmannian models within the $1/N$ expansion [9, 10] reveals that, whatever the influence of the topological term may be, it is certainly not going to suppress dynamical mass generation, at least for large N. (In fact, the topological term is non-perturbative in $1/N$ and can only be computed in semi-classical instanton gas type approximations, while the dynamically generated mass gap appears in lowest order in $1/N$.) For all these reasons, eq. (1) is the wrong ansatz.

Of course, it is well-known – at least for the principal chiral models ($\equiv$ non-linear σ models on group manifolds) – that the correct term to be added to the standard chiral lagrangian is the so-called Wess-Zumino (WZ) term which, in its two-dimensional form, was first employed by Novikov [11] and by Witten [7]. As a functional of the fields, the WZ term is both non-local and multivalued, but its variation is both local and single-valued. For the sake of completeness, let us briefly review the construction: this will be important for understanding the whys and hows in the modification of the constraint, to be introduced later.

We begin by collecting our conventions and notation. Let G be a connected compact simple Lie group and $\mathfrak{g}$ be its Lie algebra. By $(\cdot,\cdot)$ we shall denote the standard scalar product on $\mathfrak{g}$: this is by definition the unique ad-invariant negative definite inner product on $\mathfrak{g}$ in which the long roots are normalized to have length $\sqrt{2}$. Explicitly, we have for the classical groups

$$\begin{aligned} (X, Y) &= \mathrm{tr}(XY) \qquad \text{for } X, Y \in \mathrm{su}(N) \text{ or } \mathrm{sp}(N), \\ (X, Y) &= \tfrac{1}{2}\mathrm{tr}(XY) \qquad \text{for } X, Y \in \mathrm{so}(N), \end{aligned} \tag{4}$$

where the trace is understood to be taken in the defining representation (which for the symplectic case is given by the inclusions $\mathbb{H}^N = \mathbb{C}^{2N}$, $\mathrm{sp}(N) \subset \mathrm{su}(2N)$). Next, let

σ denote an automorphism of G of order 2 and also (by abuse of notation) the corresponding automorphism of $\mathfrak{g}$ of order 2: then letting H be the closed subgroup of G consisting of the fixed points under σ defines an irreducible riemannian symmetric space $M = G/H$ of type I. (See ref. [12], p. 518, for a complete list.) In the following, we shall consider M as a submanifold of G, namely the connected 1-component of the submanifold*

$$M_\sigma = \{ g \in G | \sigma(g) = g^{-1} \} \tag{5}$$

of G; this can be achieved via the Cartan immersion

$$\begin{aligned} G/H &\to M_\sigma, \\ \tilde{g}H &\mapsto g = \sigma(\tilde{g})\tilde{g}^{-1} \end{aligned} \tag{6}$$

(see, for example, ref. [12], p. 347.).

With these preliminaries out of the way, we turn to the chiral model on G, which is a theory of G-valued fields on two-dimensional flat space-time; our presentation essentially follows that of Witten [7]. The definition of the WZ term requires, however, that we extend any such field configuration $g = g(x)$ to a field configuration $\bar{g} = \bar{g}(s, x)$ which depends on an additional parameter s, $0 \leqslant s \leqslant 1$, in such a way that it is identically 1 (or some constant) at $s = 0$ and equal to the original field configuration at $s = 1$:

$$\bar{g}|_{s=0} = 1, \qquad \bar{g}|_{s=1} = g. \tag{7}$$

This supposes, of course, that such an extension always exists, but under standard boundary conditions this is indeed the case. (In the euclidean case, for example, where the standard boundary condition on smooth maps g: $\mathbb{R}^2 \to G$ is that they can be extended to smooth maps g: $S^2 \to G$, where S^2 is the two-dimensional unit sphere, the existence of extensions to smooth maps $\bar{g}$: $B^3 \to G$, where B^3 is the three-dimensional unit ball, S^2 is its boundary and the variable s above is the radial coordinate, is guaranteed by the fact that $\pi_2(G) = \{0\}$.) Now if we denote the partial derivative with respect to s by a dot, the lagrangian reads

$$L = L_{CH} + L_{WZ}, \tag{8}$$

with

$$L_{CH} = -\frac{1}{2\lambda}\eta^{\mu\nu}\left(g^{-1}\partial_\mu g, g^{-1}\partial_\nu g\right), \tag{9}$$

$$L_{WZ} = \frac{n}{8\pi}\varepsilon^{\mu\nu}\int_0^1 ds\left(\bar{g}^{-1}\dot{\bar{g}}, \left[\bar{g}^{-1}\partial_\mu\bar{g}, \bar{g}^{-1}\partial_\nu\bar{g}\right]\right), \tag{10}$$

* This is a submanifold in the sense that each of its connected components is, but different connected components may have different dimensions.

where λ and n are coupling constants (with $\lambda > 0$ to ensure stability). Note that the WZ term has here been normalized in such a way that in the euclidean case, with standard boundary conditions, computing $S_{WZ} = \int d^2x\, L_{WZ}$ for two different extensions $\bar{g}_1$ and $\bar{g}_2$ of the same field configuration g will give results that differ at most by some integer multiple of $2\pi i n$, so n itself should be an integer in order to guarantee that $\exp(-S_{WZ})$ is single-valued. The equations of motion following from L read

$$\left(\eta^{\mu\nu} - \frac{n\lambda}{4\pi}\varepsilon^{\mu\nu}\right)\left(\partial_\mu \partial_\nu g - \partial_\mu g g^{-1} \partial_\nu g\right) = 0. \tag{11}$$

Moreover, L has a global symmetry under left translations $g \to g_L g$, $\bar{g} \to g_L \bar{g}$ and under right translations $g \to g g_R$, $\bar{g} \to \bar{g} g_R$, with g_L, g_R independent of s and x, which leads to two Noether currents

$$\tilde{j}_\mu^L = -\partial_\mu g g^{-1} - \frac{n\lambda}{4\pi}\varepsilon_{\mu\nu}\int_0^1 ds \left[\dot{\bar{g}}\bar{g}^{-1}, \partial^\nu \bar{g}\bar{g}^{-1}\right],$$

$$\tilde{j}_\mu^R = +g^{-1}\partial_\mu g + \frac{n\lambda}{4\pi}\varepsilon_{\mu\nu}\int_0^1 ds \left[\bar{g}^{-1}\dot{\bar{g}}, \bar{g}^{-1}\partial^\nu \bar{g}\right]. \tag{12}$$

These can be rewritten as the sums of standard currents and pure curls: namely,

$$\tilde{j}_\mu^L = j_\mu^L + \varepsilon_{\mu\nu}\partial^\nu \phi^L,$$

$$\tilde{j}_\mu^R = j_\mu^R + \varepsilon_{\mu\nu}\partial^\nu \phi^R, \tag{13}$$

with

$$j_\mu^L = -\partial_\mu g g^{-1} - \frac{n\lambda}{4\pi}\varepsilon_{\mu\nu}\partial^\nu g g^{-1},$$

$$j_\mu^R = +g^{-1}\partial_\mu g - \frac{n\lambda}{4\pi}\varepsilon_{\mu\nu} g^{-1}\partial^\nu g, \tag{14}$$

and

$$\phi^L = \frac{n\lambda}{4\pi}\int_0^1 ds\, \dot{\bar{g}}\bar{g}^{-1},$$

$$\phi^R = \frac{n\lambda}{4\pi}\int_0^1 ds\, \bar{g}^{-1}\dot{\bar{g}}. \tag{15}$$

In particular, the conservation law

$$\partial^\mu j_\mu^L = 0,$$

$$\partial^\mu j_\mu^R = 0, \tag{16}$$

for any of these two currents is not only a consequence of the equations of motion, but is in fact completely equivalent to them. In addition, in view of the obvious identities

$$-\partial_\mu(\partial_\nu g g^{-1}) + \partial_\nu(\partial_\mu g g^{-1}) + [\partial_\mu g g^{-1}, \partial_\nu g g^{-1}] = 0,$$

$$+\partial_\mu(g^{-1}\partial_\nu g) - \partial_\nu(g^{-1}\partial_\mu g) + [g^{-1}\partial_\mu g, g^{-1}\partial_\nu g] = 0, \tag{17}$$

the conservation laws (16) lead to the curl conditions

$$\partial_\mu j_\nu^{\mathrm{L}} - \partial_\nu j_\mu^{\mathrm{L}} + [j_\mu^{\mathrm{L}}, j_\nu^{\mathrm{L}}] = 0,$$

$$\partial_\mu j_\nu^{\mathrm{R}} - \partial_\nu j_\mu^{\mathrm{R}} + [j_\mu^{\mathrm{R}}, j_\nu^{\mathrm{R}}] = 0. \tag{18}$$

This implies that the model is classically integrable [3, 13, 14].

At the quantum level, the model is still integrable, in the sense of admitting conserved quantum non-local charges [14, 15], and one can show that it exhibits asymptotic freedom in the UV regime [16, 17]. While these properties hold for any value of n, physics in the IR regime depends crucially on whether $n = 0$ or $n \neq 0$. Namely, if $n = 0$, i.e. for the standard chiral model, the β-function has no zeroes, which implies that the theory flows to strong coupling, while if $n \neq 0$, the β-function has a non-trivial zero at $\lambda = 4\pi/|n|$, and the theory becomes conformally invariant at this critical point [7].

The particular rôle of the critical coupling $\lambda = 4\pi/|n|$, for $n \neq 0$, can be clearly seen even in the classical theory. Indeed, in this case, the lagrangian can be rewritten in the form

$$L = +\frac{n}{4\pi}\int_0^1 \mathrm{d}s\, (\bar{g}^{-1}\dot{\bar{g}}, \partial_-(\bar{g}^{-1}\partial_+\bar{g})) - \partial^\mu G_\mu \qquad \text{if } n > 0,$$

$$L = -\frac{n}{4\pi}\int_0^1 \mathrm{d}s\, (\bar{g}^{-1}\dot{\bar{g}}, \partial_+(\bar{g}^{-1}\partial_-\bar{g})) + \partial^\mu G_\mu \qquad \text{if } n < 0, \tag{19}$$

with

$$G_\mu = \frac{n}{4\pi}\int_0^1 \mathrm{d}s\, (\bar{g}^{-1}\dot{\bar{g}}, \bar{g}^{-1}\partial_\mu\bar{g}), \tag{20}$$

while the equations of motion become

$$\partial_-\partial_+ g - \partial_- g g^{-1}\partial_+ g = 0 \qquad \text{if } n > 0,$$

$$\partial_+\partial_- g - \partial_+ g g^{-1}\partial_- g = 0 \qquad \text{if } n < 0, \tag{21}$$

and the currents read

$$j_+^{\mathrm{L}} = 0, \qquad j_-^{\mathrm{L}} = -2\partial_- g g^{-1}; \qquad j_+^{\mathrm{R}} = +2g^{-1}\partial_+ g, \qquad j_-^{\mathrm{R}} = 0 \qquad \text{if } n > 0,$$
$$j_+^{\mathrm{L}} = -2\partial_+ g g^{-1}, \qquad j_-^{\mathrm{L}} = 0; \qquad j_+^{\mathrm{R}} = 0, \qquad j_-^{\mathrm{R}} = +2g^{-1}\partial_- g \qquad \text{if } n < 0. \tag{22}$$

Now we define a new current J_μ according to

$$J_+ = +g^{-1}\partial_+ g, \qquad J_- = -\partial_- g g^{-1} \qquad \text{if } n > 0,$$
$$J_+ = -\partial_+ g g^{-1}, \qquad J_- = +g^{-1}\partial_- g \qquad \text{if } n < 0. \tag{23}$$

This will bring the equations of motion into the form

$$\partial_- J_+ = 0, \qquad \partial_+ J_- = 0, \tag{24}$$

with the desired general solution: $J_+ = J_+(x^+)$, $J_- = J_-(x^-)$. Note finally that these equations can be solved in closed form for the field g as well, the general solution being

$$g(x) = g_-(x^-)g_+(x^+)^{-1} \qquad \text{if } n > 0,$$
$$g(x) = g_+(x^+)g_-(x^-)^{-1} \qquad \text{if } n < 0, \tag{25}$$

and that the lagrangian possesses a huge additional symmetry, of a partly local nature, namely

$$g(x) \to g_{\mathrm{L}}(x^-)g(x)g_{\mathrm{R}}(x^+)^{-1},$$
$$\bar{g}(s,x) \to g_{\mathrm{L}}(x^-)\bar{g}(s,x)g_{\mathrm{R}}(x^+)^{-1} \qquad \text{if } n > 0,$$
$$g(x) \to g_{\mathrm{L}}(x^+)g(x)g_{\mathrm{R}}(x^-)^{-1},$$
$$\bar{g}(s,x) \to g_{\mathrm{L}}(x^+)\bar{g}(s,x)g_{\mathrm{R}}(x^-)^{-1} \qquad \text{if } n < 0. \tag{26}$$

This symmetry, which acts transitively on the space of solutions (25), is the origin of the two mutually commuting Kac-Moody current algebras that appear in the model.

Now we consider what kind of constraint, in terms of the automorphism σ, can be imposed on the G-valued fields of the model. The standard choice is to assume that the field configurations g take values in the submanifold M_σ of G, as defined by (5), i.e. that

$$\sigma(g(x)) = g(x)^{-1}. \tag{27}$$

An even stronger requirement would be that the extended field configurations $\bar{g}$ take values in the submanifold M_σ of G as well, i.e. that

$$\sigma(\bar{g}(s,x)) = \bar{g}(s,x)^{-1}. \tag{28}$$

As we shall see now, both constraints are unacceptable as soon as $n \neq 0$.

To begin with, we note that there will in general be a topological obstruction to the existence of an M_σ-valued extension $\bar{g}$ of an M_σ-valued field configuration g, even under standard boundary conditions. (In the euclidean case, for example, where the standard boundary condition on smooth maps g: $\mathbb{R}^2 \to M_\sigma$ is that they can be extended to smooth maps g: $S^2 \to M_\sigma$, the existence of extensions to smooth maps $\bar{g}$: $B^3 \to M_\sigma$ may fail due to the fact that, in general, $\pi_2(M_\sigma) \neq \{0\}$.) But even when we restrict to null homotopic field configurations, we are immediately faced with a much more serious problem, namely the fact that (28) forces the WZ term to vanish identically:

$$\sigma(\bar{g}) = \bar{g}^{-1} \Rightarrow \sigma(\bar{g}^{-1}\dot{\bar{g}}) = -\dot{\bar{g}}\bar{g}^{-1}, \qquad \sigma(\bar{g}^{-1}\partial_\mu\bar{g}) = -\partial_\mu\bar{g}\bar{g}^{-1}$$

$$\Rightarrow \left(\bar{g}^{-1}\dot{\bar{g}}, \left[\bar{g}^{-1}\partial_\mu\bar{g}, \bar{g}^{-1}\partial_\nu\bar{g}\right]\right) = -\left(\bar{g}^{-1}\dot{\bar{g}}, \left[\bar{g}^{-1}\partial_\mu\bar{g}, \bar{g}^{-1}\partial_\nu\bar{g}\right]\right).$$

This forces us to give up the constraint (28), but we could still try to maintain at least the constraint (27). In the case where $M = G/H$ is hermitian symmetric, this leads to an explicit identification of the WZ term with the topological term, at $\theta = \pi$; see ref. [18] for special cases. (Briefly, one can represent any field configuration $g = g(x)$ satisfying (27) by a gauge-dependent G-valued field $\tilde{g} = \tilde{g}(x)$, at least locally, according to

$$g(x) = \sigma(\tilde{g}(x))\tilde{g}(x)^{-1},$$

and use the generator $J \in \mathfrak{g}$ of the invariant complex structure, which satisfies $\mathrm{ad}(J) = 0$ on $\mathfrak{z}$ and $\mathrm{ad}(J)^2 = -1$ on $\mathfrak{m}$, to define an explicit extension $\bar{g} = \bar{g}(s,x)$, at least locally, according to [4]

$$\bar{g}(s,x) = \exp(\pi J)\tilde{g}(x)\exp(-s\pi J)\tilde{g}(x)^{-1}.$$

As it turns out, the s-integration in (10) can then be performed explicitly, and the result is

$$L_{\mathrm{WZ}} = \tfrac{1}{4}n\left(J, \varepsilon^{\mu\nu}F_{\mu\nu}\right),$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right] = -\left[\tilde{g}^{-1}D_\mu\tilde{g}, \tilde{g}^{-1}D_\nu\tilde{g}\right],$$

$$A_\mu = \left(\tilde{g}^{-1}\partial_\mu\tilde{g}\right)_{\mathfrak{z}}, \qquad \tilde{g}^{-1}D_\mu\tilde{g} = \left(\tilde{g}^{-1}\partial_\mu\tilde{g}\right)_{\mathfrak{m}},$$

as in refs. [3–5].) However, and this is our strongest argument, (27) is compatible with the equations of motion only when $n = 0$. Indeed, for $n = 0$, the equations of motion read

$$\eta^{\mu\nu}\partial_\mu\left(g^{-1}\partial_\nu g\right) = 0 \qquad \text{or} \qquad \eta^{\mu\nu}\partial_\mu\left(\partial_\nu g g^{-1}\right) = 0$$

or

$$\eta^{\mu\nu}\left(\partial_\mu\partial_\nu g - \partial_\mu g g^{-1}\partial_\nu g\right) = 0$$

and imply that $\sigma(g)$ and g^{-1} satisfy the same equations of motion, while for $n \neq 0$, the equations of motion (21), or (24) with (23), imply that $\sigma(g)$ and g^{-1} satisfy different equations of motion, namely:

$$\begin{aligned} &\partial_-\left(\sigma(g)^{-1}\partial_+\sigma(g)\right) = \sigma\left(\partial_-\left(g^{-1}\partial_+ g\right)\right) = 0 \\ &\qquad \partial_+\left(g\partial_- g^{-1}\right) = \partial_+\left(-\partial_- g g^{-1}\right) = 0 \qquad \text{if } n > 0, \\ &\partial_+\left(\sigma(g)^{-1}\partial_-\sigma(g)\right) = \sigma\left(\partial_+\left(g^{-1}\partial_- g\right)\right) = 0 \\ &\qquad \partial_-\left(g\partial_+ g^{-1}\right) = \partial_-\left(-\partial_+ g g^{-1}\right) = 0 \qquad \text{if } n < 0. \end{aligned}$$

Of course, one may argue that this observation is irrelevant, since the correct method for incorporating (27) as an additional constraint into the theory is not to check its consistency with the given equations of motion but rather to modify the action, and hence also the equations of motion, by introducing an additional Lagrange multiplier term. (This can be done in such a way that in the path integral, the additional term – at least in a certain limit – leads to an additional δ-function factor for the submanifold M_σ of G; see ref. [18] for special cases.) It is easy to see, however, that such a Lagrange multiplier term violates the symmetry (26) which, as we recall, is the origin of the very special algebraic structure of the model, and in particular of its conformal invariance even at the quantum level.

In view of this situation, we propose to modify the constraints. Two possibilities which come to mind result from

(i) inserting an additional parity transformation P:

(ii) replacing the group inversion by the space-time inversion PT:

$$\sigma(g(t, y)) = g(t, -y)^{-1}, \tag{29i}$$

$$\sigma(\bar{g}(s, t, y)) = \bar{g}(s, t, -y)^{-1}, \tag{30i}$$

$$\sigma(g(t, y)) = g(-t, -y), \tag{29ii}$$

$$\sigma(\bar{g}(s, t, y)) = \bar{g}(s, -t, -y). \tag{30ii}$$

The problems encountered previously then disappear. Namely, (30) does not force the WZ term to vanish, and (29) is compatible with the equations of motion, as can be seen most easily by noting that their general solution (25) satisfies (29) if and only if

$$\sigma(g_+(x^+)) = g_-(x^+)g_0, \qquad \sigma(g_-(x^-)) = g_+(x^-)g_0, \tag{31i}$$

$$\sigma(g_+(x^+)) = g_+(-x^+)g_0, \qquad \sigma(g_-(x^-)) = g_-(-x^-)g_0, \tag{31ii}$$

where $g_0 \in \mathrm{G}$ is a constant which, for consistency, must satisfy

$$\sigma(g_0) = g_0^{-1}. \tag{32}$$

However, both of these constraints give rise to new difficulties. Constraint (i) couples left and right movers and is therefore not capable of generating two *mutually commuting* Virasoro algebras: it destroys conformal invariance. Constraint (ii) couples fields at (large) positive times and fields at (large) negative times: it destroys causality.

To find a way out of this impasse, let us briefly discuss the impact of boundary conditions in the principal chiral model. For the sake of definiteness, we work in Minkowski space. The standard boundary condition is

$$g(t, -\infty) = g(t, +\infty), \tag{33}$$

$$\bar{g}(s, t, -\infty) = \bar{g}(s, t, +\infty). \tag{34}$$

This can be viewed as the infinite volume limit of the same model defined in a finite box of length L, with standard boundary or periodicity condition

$$g(t, y) = g(t, y + nL), \qquad n \in \mathbb{Z}, \tag{35}$$

$$\bar{g}(s, t, y) = \bar{g}(s, t, y + nL), \qquad n \in \mathbb{Z}. \tag{36}$$

We now use the automorphism σ to change the boundary conditions. The twisted boundary condition is

$$\sigma(g(t, -\infty)) = g(t, +\infty), \tag{37}$$

$$\sigma(\bar{g}(s, t, -\infty)) = \bar{g}(s, t, +\infty). \tag{38}$$

This can again be viewed as the infinite volume limit of the same model defined in a finite box of length L, with twisted boundary or periodicity condition

$$\sigma^n(g(t, y)) = g(t, y + nL), \qquad n \in \mathbb{Z}, \tag{39}$$

$$\sigma^n(g(s, t, y)) = g(s, t, y + nL), \qquad n \in \mathbb{Z}. \tag{40}$$

Again, these conditions do not force the WZ term to vanish, and they are compatible with the equations of motion, as can be seen most easily by noting that their general solution (25) satisfies eqs. (37) resp. (39) if and only if

$$\sigma\big(g_+(-\infty)\big) = g_+(+\infty)g_0 , \qquad \sigma\big(g_-(+\infty)\big) = g_-(-\infty)g_0 , \tag{41}$$

resp.

$$\sigma^n\big(g_+(x^+)\big) = g_+(x^+ + nL)g_0^n , \quad \sigma^n\big(g_-(x^-)\big) = g_-(x^- - nL)g_0^n , \quad n \in \mathbb{Z} , \tag{42}$$

where $g_0 \in \mathrm{G}$ is a constant which, for consistency, must satisfy (32). Moreover, these conditions do not couple left and right movers, or fields at different times. For the current J_μ, as defined in (23), they lead to

$$\sigma\big(J_+(-\infty)\big) = J_+(+\infty) , \qquad \sigma\big(J_-(+\infty)\big) = J_-(-\infty) , \tag{43}$$

resp.

$$\sigma^n\big(J_+(x^+)\big) = J_+(x^+ + nL) , \qquad \sigma^n\big(J_-(x^-)\big) = J_-(x^- - nL) , \qquad n \in \mathbb{Z} . \tag{44}$$

The twisted boundary conditions (37), (38) and (39), (40) that we have introduced can be viewed as substitutes for the standard constraint (27) in the standard principal chiral model (without the WZ term). We avoid calling them "constraints" since they do not impose restrictions on the values of the fields at each space-time point. In particular, we must abandon the concept of symmetric-space-*valued* fields: the only remnant of the symmetric space structure is the explicit appearance of the automorphism σ.

For the model in a finite box of length L, with twisted boundary conditions (39), (40), a further generalization is possible because we can now assume the automorphism σ to have arbitrary finite order N. (Of course, it is only for $N = 2$ that such automorphisms are related to symmetric spaces.) Noting that for the current J_μ, the twisted periodicity conditions (44), with period L, imply standard periodicity conditions

$$J_+(x^+) = J_+(x^+ + nNL) , \qquad J_-(x^-) = J_-(x^- - nNL) , \qquad n \in \mathbb{Z} , \tag{45}$$

with period NL, and performing Fourier series expansions

$$\begin{aligned} J_+(x^+) &= \sum_{r \in \mathbb{Z}/N} J_{+,r} \exp(+2\pi i r x^+/L) , \\ J_-(x^-) &= \sum_{r \in \mathbb{Z}/N} J_{-,r} \exp(-2\pi i r x^-/L) , \end{aligned} \tag{46}$$

$$\begin{aligned} J_{+,r} &= \frac{1}{NL}\int_0^{NL} \mathrm{d}x^+ \, J_+(x^+) \exp(-2\pi i r x^+/L) , \\ J_{-,r} &= \frac{1}{NL}\int_0^{NL} \mathrm{d}x^- \, J_-(x^-) \exp(+2\pi i r x^-/L) , \end{aligned} \tag{47}$$

we arrive at generators $J_{\pm,r}$ in $\mathscr{g}^c$, for $r \in \mathbb{Z}/N$, which, when expanded in terms of an orthonormal basis of generators T_a in $\mathscr{g}^c$ with (totally antisymmetric) structure constants if^{abc}, satisfy Poisson bracket commutation relations

$$\left[J^a_{+,r}, J^b_{+,s}\right]_{\mathrm{PB}} = if_c^{ab} J^c_{+,r+s}, \qquad \left[J^a_{-,r}, J^b_{-,s}\right]_{\mathrm{PB}} = if_c^{ab} J^c_{-,r+s},$$
$$\left[J^a_{+,r}, J^b_{-,s}\right]_{\mathrm{PB}} = 0, \tag{48}$$

plus the constraint

$$\sigma(J_{+,r}) = \exp(2\pi i r) J_{+,r}, \qquad \sigma(J_{-,r}) = \exp(2\pi i r) J_{-,r}. \tag{49}$$

This means that the Fourier coefficients of the currents form two mutually commuting twisted loop algebras which, upon quantization, will turn into two mutually commuting twisted Kac-Moody algebras.

To conclude, we propose viewing the principal chiral model with the WZ term, defined in a box of finite length, as a string theory on the corresponding group manifold: the boundary conditions then decide whether we are dealing with an untwisted or with a twisted string theory. In particular, the untwisted version, which has recently been studied by Gepner and Witten [19], is the group manifold version of a string theory on a torus. In the same way, the twisted version that we propose here is the group manifold version of a string theory on an orbifold [20]. In both bases, one divides by a finite abelian group, which in our situation is the group $\mathbb{Z}_N$ of automorphisms of G formed by the powers of σ or, more generally, when G is semi-simple but not simple, a direct product of such groups.

What remains to be checked, then, is whether all these chiral models share the two basic invariance principles of string theory.

The first such principle is conformal invariance, which is here guaranteed to hold by what has become known in the physical literature as the Sugawara construction. More specifically, this construction is known to work in the untwisted case [6] and, for certain automorphisms σ, also in the twisted case [21]: these special automorphisms are the so-called standard ones, i.e. the ones induced by automorphisms of Dynkin diagrams, cf. ref. [12], p. 505. It is however important to include inner automorphisms, since their presence has a strong influence on the mass spectrum for string states [22, 23]. This can be done, and the general result can in fact be found in the mathematical literature: it was proved several years ago by Kac and Peterson, cf. ref. [24], pp. 173–179.

The second important principle is modular invariance, which presumably requires the appearance of whole multiplets of twisted sectors, with certain restrictions on the nature of the automorphisms and their multiplicities. We have however not analyzed this problem in more detail.

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Note added

After completion of our work, we became aware of ref. [25], where a similar picture is advocated.

References

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333
[2] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83
[3] H. Eichenherr and M. Forger, Nucl. Phys. B155 (1979) 381
[4] H. Eichenherr and M. Forger, Nucl. Phys. B164 (1980) 528
[5] H. Eichenherr and M. Forger, Commun. Math. Phys. 82 (1981) 227
[6] P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303
[7] E. Witten, Commun. Math. Phys. 92 (1984) 455
[8] E. Braaten, T. Curtright and C. Zachos, Nucl. Phys. B260 (1985) 630
[9] A. D'Adda, P. di Vecchia and M. Lüscher, Nucl. Phys. B146 (1978) 63; B152 (1979) 125
[10] E. Abdalla, M. Forger and A. Lima-Santos, Nucl. Phys. B256 (1985) 145
[11] S. Novikov, Sov. Math. Dokl. 24 (1981) 222
[12] S. Helgason, Differential geometry, Lie groups and symmetric spaces (Academic Press, New York, 1978)
[13] V.E. Zakharov and A.V. Mikhailov, Sov. Phys. JETP 47 (1978) 1017
[14] M.C.B. Abdalla, Phys. Lett. 152B (1985) 215
[15] M. Forger, unpublished
[16] A.M. Polyakov, Phys. Lett. 59B (1975) 79;
E. Brézin, S. Hikami and J. Zinn-Justin, Nucl. Phys. B165 (1980) 528
[17] A.M. Polyakov and P.B. Wiegmann, Phys. Lett. 131B (1983) 121; 141B (1984) 223
[18] B.V. Ivanov, Phys. Lett. 177B (1986) 63, 67
[19] D. Gepner and E. Witten, Nucl Phys. B278 (1986) 493
[20] L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 678; B274 (1986) 285;
L. Dixon, D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B282 (1987) 13
[21] R.I. Nepomechie, University of Washington preprint 40048-02 P6 (March 1986)
[22] C. Vafa and E. Witten, Phys. Lett. 159B (1985) 265
[23] T.R. Govindrajan, T. Jayraman, A. Mukherjee and S.R. Wadia, Mod. Phys. Lett. A1 (1986) 29
[24] V.G. Kac and D.H. Peterson, Adv. Math. 53 (1984) 125
[25] W. Ogura, Osaka University preprint OU-HET 92 (June 1986)