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# NON-LOCAL CHARGES FOR NON-LINEAR SIGMA MODELS ON GRASSMANN MANIFOLDS

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We discuss the non-local charge for the grassmannian non-linear sigma models with and without fermion interactions, both in the classical and in the quantized theory. As suspected, conservation of the quantum non-local charge for the "pure" model is spoiled by anomalies, while it is restored when minimally or supersymmetrically coupled fermions are added. In the last two cases, we draw conclusions on the factorizing $S$-matrix.

## 1. Introduction

Generalized non-linear sigma models, or chiral models, defined on riemannian symmetric spaces $M = G/H$ [1], represent an important class of $(1+1)$-dimensional field theories. One of the basic reasons for their outstanding rôle is that due to their many analogies with $(3+1)$-dimensional non-abelian gauge theories [2], they have been very useful in developing and testing general ideas about the latter [3, 4]. (As examples of common properties of the two types of models, we mention, at the classical level, their geometrical nature, the non-trivial topological structure of the space of field configurations (instantons) and their conformal invariance [2], and at the quantum level, the phenomena of dynamical mass generation and of asymptotic freedom [5].) In addition, the "pure" generalized non-linear sigma models have classically an infinite number of non-local conservation laws [6]. As will be shown elsewhere [7] in full generality, this statement remains true when

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fermions belonging to a given representation of H are coupled to the bosons; this includes the cases of minimal coupling and supersymmetric coupling in the $\mathbb{C}P^{n-1}$ model [8].

The importance of the non-local conservation laws in the quantum theory lies in the fact that if they survive quantization, then particle production is suppressed and the $S$-matrix is factorizable [9]; it can then be calculated exactly [10]. In general, however, quantum fluctuations will produce anomalies in the non-local conservation laws [11, 12], and the above conclusions about particle production and the $S$-matrix cannot be drawn.

For the "pure" generalized non-linear sigma models, there is a simple criterion for the absence or (possible) presence of anomalies in the first quantum non-local charge (whose conservation is already sufficient for the aforementioned conclusions on particle production and the $S$-matrix) [13]: namely, there is no anomaly if the stability group H is simple. Otherwise, anomalies may be generated by the components of the gauge field strengths belonging to the various simple components of the stability algebra $\mathscr{h}$.

The picture changes when fermions are coupled to the bosons. As will be shown elsewhere [7], the axial anomalies coming from the fermionic sector have the same group-theoretical structure as the ones coming from the bosonic sector, and no case is known where the "pure" model has no anomalies, but the model with fermions does. The really interesting phenomenon, however, is the fact that there are cases where the two types of anomalies exactly cancel each other, leaving us with a conserved quantum non-local charge, and hence with a factorizable $S$-matrix. This was first shown for the $\mathbb{C}P^{n-1}$ model with minimally or supersymmetrically coupled fermions [8, 12].

The main aim of this paper is to extend this result on the mutual cancellation of anomalies to non-linear sigma models defined on complex Grassmann manifolds:

$$G_{\mathbb{C}}(p, q) = U(n)/U(p)\times U(q) = SU(n)/S(U(p)\times U(q)) \qquad (n = p+q) \quad (1.1)$$

or on real Grassmann manifolds

$$G_{\mathbb{R}}(p, q) = SO(n)/SO(p)\times SO(q) \qquad (n = p+q)\,, \quad (1.2)$$

with minimally or supersymmetrically coupled fermions, thus showing that the cancellation phenomenon is independent of the question whether the gauge group is abelian or non-abelian. Therefore, both the minimal model and the supersymmetric model have a conserved quantum non-local charge, and as in the case $p = 1$ [8, 9], it can be shown that this forces their $S$-matrices to factorize into two-body amplitudes. It also imposes severe constraints on these amplitudes, but in contrast to the $\mathbb{C}P^{n-1}$ case, we have not been able to find a completely explicit solution.

Throughout the paper, we shall use the following conventions: Latin indices $a, b, c, d, \ldots$ (color indices) run from 1 to $p$ (we assume $p \leq q$) and refer to the local internal symmetry under $U(p)$, while Latin indices $i, j, k, l, \ldots$ (flavor indices) run from 1 to $n$ and refer to the global internal symmetry under $U(n)$ (upper and lower indices are not distinguished). Greek indices $\kappa, \lambda, \mu, \nu, \ldots$, on the other hand, refer to two-dimensional space-time, and the corresponding conventions (including those on $\gamma$-matrices) are summarized in an appendix. Finally, all classical spinor fields are understood to be anticommuting $c$-numbers.

## 2. The classical model

In this section, we briefly discuss the formulation and the basic integrability properties of the classical non-linear sigma models on complex Grassmann manifolds (1.1). (The real case (1.2) is entirely analogous, the only difference being that all fields are real instead of complex - which for spinor fields means Majorana instead of Dirac spinors.)

We begin with the "pure" Grassmann model, which is written in terms of a complex $(n \times p)$-matrix field $z = z(x)$ subject to the constraint

$$z^+ z = 1_p \qquad \text{i.e.} \qquad \bar{z}_i^a z_i^b = \delta^{ab} . \tag{2.1}$$

In this representation, this model has a local $U(p)$ invariance, where gauge transformations act according to

$$z \to zh \qquad \text{i.e.} \qquad z_i^a \to h^{ba} z_i^b , \tag{2.2}$$

with unitary $(p \times p)$-matrix fields $h = h(x)$. This enforces the use of covariant derivatives:

$$\begin{aligned} D_\mu z &= \partial_\mu z - z A_\mu \qquad \text{i.e.} \qquad D_\mu z_i^a = \partial_\mu z_i^a - A_\mu^{ba} z_i^b , \\ D_\mu D_\nu z &= \partial_\mu D_\nu z - D_\nu z A_\mu \qquad \text{i.e.} \qquad D_\mu D_\nu z_i^a = \partial_\mu D_\nu z_i^a - A_\mu^{ba} D_\nu z_i^b , \end{aligned} \tag{2.3}$$

etc., with the gauge potential

$$A_\mu = z^+ \partial_\mu z \qquad \text{i.e.} \qquad A_\mu^{ab} = \bar{z}_i^a \partial_\mu z_i^b , \tag{2.4}$$

and the gauge field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = D_\mu z^+ D_\nu z - D_\nu z^+ D_\mu z ,$$

i.e. (2.5)

$$F_{\mu\nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + A_\mu^{ac} A_\nu^{cb} - A_\nu^{ac} A_\mu^{cb} = \overline{D_\mu z_i^a} D_\nu z_i^b - \overline{D_\nu z_i^a} D_\mu z_i^b ,$$

both of which are antihermitian $(p \times p)$-matrix fields. The lagrangian reads

$$L = g^{\mu\nu} \operatorname{tr} (D_\mu z^+ D_\nu z) = g^{\mu\nu} \overline{D_\mu z_i^a} D_\nu z_i^a . \tag{2.6}$$

This lagrangian has a global $U(n)$ invariance, where global symmetry transformations act according to

$$z \to g_0 z \qquad \text{i.e.} \qquad z_i^a \to (g_0)_{ij} z_j^a , \tag{2.7}$$

with unitary $(n \times n)$-matrices $g_0$. Consequently, the model possesses a conserved

Noether current, which is the antihermitian $(n \times n)$-matrix field given by

$$j_\mu = zD_\mu z^+ - D_\mu z z^+ \qquad \text{i.e.} \qquad (j_\mu)_{ij} = z_i^a \overline{D_\mu z_j^a} - D_\mu z_i^a \bar{z}_j^a \,. \tag{2.8}$$

The equation of motion is

$$g^{\mu\nu}(D_\mu D_\nu z + zD_\mu z^+ D_\nu z) = 0 \qquad \text{i.e.} \qquad g^{\mu\nu}(D_\mu D_\nu z_i^a + z_i^b \overline{D_\mu z_j^b} D_\nu z_j^a) = 0 \,. \tag{2.9}$$

It implies (and is in fact equivalent to) conservation of the current:

$$g^{\mu\nu}\partial_\mu j_\nu = 0 \,. \tag{2.10}$$

Moreover, we have the crucial identity

$$\partial_\mu j_\nu - \partial_\nu j_\mu + 2[j_\mu, j_\nu] = 0 \,. \tag{2.11}$$

From (2.10) and (2.11), we can check [6] that the equation of motion implies (and is in fact equivalent to) the integrability, for any value of the real parameter $\lambda$, of the following system of first-order differential equations:

$$\partial_\mu U^{(\lambda)} = U^{(\lambda)}\{(1-\cosh\lambda) j_\mu - \sinh(\lambda)\varepsilon_{\mu\nu} g^{\nu\kappa} j_\kappa\} \,, \tag{2.12}$$

where $U^{(\lambda)}$ is a U$(n)$-valued field which serves as a generating functional for the non-local charges. In particular, the first non-local charge $Q^{(1)}$, whose conservation (i.e. time independence) can also be verified directly from (2.10) and (2.11), is

$$Q^{(1)}(t) = \int \mathrm{d}y_1\, \mathrm{d}y_2\, \theta(y_1 - y_2)[j_0(t, y_1), j_0(t, y_2)] - \int \mathrm{d}y\, j_1(t, y) \,. \tag{2.13}$$

Turning to the Grassmann model with fermions, we shall distinguish two cases: the minimal model and the supersymmetric model. It can be shown that both of these have a common group-theoretical origin, the former being derived from the (dual of the) fundamental representation of U$(n)$ on $\mathbb{C}^n$ and the latter from the adjoint representation of U$(n)$ on u$(n)$; see [7] for more details. Here, we shall deal with them simultaneously and shall simply indicate the model to which a given formula refers by an index M (for "minimal") or S (for "supersymmetric"). For more information on the supersymmetric case, we refer the reader to the literature, e.g. [14, 15].

The minimal and supersymmetric models involve, apart from the $(n \times p)$-matrix scalar field $z = (z_i^a)$, a $p$-vector Dirac spinor field $\psi = (\psi^a)$ and an $(n \times p)$-matrix Dirac spinor field $\psi = (\psi_i^a)$ respectively. In the supersymmetric case, one has, in addition, the constraint

$$z^+\psi = 0 \qquad \text{i.e.} \qquad \bar{z}_i^a \psi_i^b = 0 \,. \tag{2.14$_\mathrm{S}$}$$

All fields are subject to local U$(p)$ transformations, which act according to (2.2) and

$$\psi \to \psi h \qquad \text{i.e.} \qquad \psi^a \to h^{ba}\psi^b \,. \tag{2.15$_\mathrm{M}$}$$

$$\psi \to \psi h \qquad \text{i.e.} \qquad \psi_i^a \to h^{ba}\psi_i^b \,. \tag{2.15$_\mathrm{S}$}$$

The corresponding covariant derivatives are

$$D_\mu \psi = \partial_\mu \psi - \psi A_\mu \qquad \text{i.e.} \qquad D_\mu \psi^a = \partial_\mu \psi^a - A_\mu^{ba} \psi^b\,, \tag{2.16$_\mathrm{M}$}$$

$$D_\mu \psi = \partial_\mu \psi - \psi A_\mu \qquad \text{i.e.} \qquad D_\mu \psi_i^a = \partial_\mu \psi_i^a - A_\mu^{ba} \psi_i^b\,, \tag{2.16$_\mathrm{S}$}$$

etc. The lagrangians read*

$$\begin{aligned} L &= g^{\mu\nu}\,\mathrm{tr}\,(D_\mu z^+ D_\nu z) + \tfrac{1}{2} i \bar\psi \overleftrightarrow{\not{D}} \psi + \tfrac{1}{4} L_{\mathrm{FM}} \\ &= g^{\mu\nu} \overline{D_\mu z_i^a} D_\nu z_i^a + \tfrac{1}{2} i g^{\mu\nu} \bar\psi^a \gamma_\mu \overleftrightarrow{D}_\nu \psi^a + \tfrac{1}{4} L_{\mathrm{FM}}\,, \end{aligned} \tag{2.17a$_\mathrm{M}$}$$

with

$$L_{\mathrm{FM}} = (\bar\psi^a \psi^a)(\bar\psi^b \psi^b) - (\bar\psi^a \gamma_5 \psi^a)(\bar\psi^b \gamma_5 \psi^b) = -g^{\mu\nu} (\bar\psi^a \gamma_\mu \psi^b)(\bar\psi^b \gamma_\nu \psi^a)\,, \tag{2.17b$_\mathrm{M}$}$$

and

$$\begin{aligned} L &= g^{\mu\nu}\,\mathrm{tr}\,(D_\mu z^+ D_\nu z) + \tfrac{1}{2} i \bar\psi \overleftrightarrow{\not{D}} \psi + \tfrac{1}{4} L_{\mathrm{FS}} \\ &= g^{\mu\nu} \overline{D_\mu z_i^a} D_\nu z_i^a + \tfrac{1}{2} i g^{\mu\nu} \bar\psi_i^a \gamma_\mu \overleftrightarrow{D}_\nu \psi_i^a + \tfrac{1}{4} L_{\mathrm{FS}}\,, \end{aligned} \tag{2.17a$_\mathrm{S}$}$$

with

$$\begin{aligned} L_{\mathrm{FS}} &= (\bar\psi_i^a \psi_i^b)(\bar\psi_j^b \psi_j^a) - (\bar\psi_i^a \gamma_5 \psi_i^b)(\bar\psi_j^b \gamma_5 \psi_j^a) - g^{\mu\nu} (\bar\psi_i^a \gamma_\mu \psi_i^b)(\bar\psi_j^b \gamma_\nu \psi_j^a) \\ &= -g^{\mu\nu} \{ (\bar\psi_i^a \gamma_\mu \psi_j^a)(\bar\psi_j^b \gamma_\nu \psi_i^b) + (\bar\psi_i^a \gamma_\mu \psi_i^b)(\bar\psi_j^b \gamma_\nu \psi_j^a) \}\,. \end{aligned} \tag{2.17b$_\mathrm{S}$}$$

Both lagrangians are invariant under global $\mathrm{U}(n)$ transformations, which act according to (2.7) and

$$\psi \to \psi \qquad \text{i.e.} \qquad \psi^a \to \psi^a\,. \tag{2.18$_\mathrm{M}$}$$

$$\psi \to g_0 \psi \qquad \text{i.e.} \qquad \psi_i^a \to (g_0)_{ij} \psi_j^a\,. \tag{2.18$_\mathrm{S}$}$$

Consequently, the model possesses a conserved Noether current, which is the antihermitian $(n \times n)$-matrix field

$$J_\mu = j_\mu + j_\mu^{\mathrm{M}}\,, \tag{2.19}$$

given by (2.8) and the matter field contribution

$$(j_\mu^{\mathrm{M}})_{ij} = -i z_i^a \bar\psi^a \gamma_\mu \psi^b \bar z_j^b\,; \tag{2.20$_\mathrm{M}$}$$

$$(j_\mu^{\mathrm{M}})_{ij} = i \bar\psi_j^a \gamma_\mu \psi_i^a - i z_i^a \bar\psi_k^a \gamma_\mu \psi_k^b \bar z_j^b\,. \tag{2.20$_\mathrm{S}$}$$

The equations of motion split into a bosonic equation:

$$g^{\mu\nu} ( D_\mu D_\nu z_i^a + z_i^b \overline{D_\mu z_j^b} D_\nu z_j^a + i D_\mu z_i^b \bar\psi^b \gamma_\nu \psi^a ) = 0\,, \tag{2.21$_\mathrm{M}$}$$

$$g^{\mu\nu} ( D_\mu D_\nu z_i^a + z_i^b \overline{D_\mu z_j^b} D_\nu z_j^a + i \bar\psi_j^b \gamma_\mu \psi_i^b D_\nu z_j^a + i D_\mu z_i^b \bar\psi_j^b \gamma_\nu \psi_j^a ) = 0\,, \tag{2.21$_\mathrm{S}$}$$

* Here, we use one of the Fierz identities for Dirac spinors (based on anticommuting $c$-numbers); see the appendix.

and a fermionic equation:

$$\not{D}\psi^a = -\tfrac{1}{2}ig^{\mu\nu}(\bar{\psi}^b\gamma_\mu\psi^a)\gamma_\nu\psi^b\,; \tag{2.22$_\mathrm{M}$}$$

$$\begin{aligned}\not{D}\psi_i^a - z_i^b\bar{z}_j^b\not{D}\psi_j^a &= \tfrac{1}{2}i\{(\bar{\psi}_j^b\psi_j^a)\psi_i^b - (\bar{\psi}_j^b\gamma_5\psi_j^a)\gamma_5\psi_i^b - g^{\mu\nu}(\bar{\psi}_j^b\gamma_\mu\psi_j^a)\gamma_\nu\psi_i^b\} \\ &= -\tfrac{1}{2}ig^{\mu\nu}\{(\bar{\psi}_j^b\gamma_\mu\psi_i^b)\gamma_\nu\psi_j^a + (\bar{\psi}_j^b\gamma_\mu\psi_j^a)\gamma_\nu\psi_i^b\}\,. \end{aligned}\tag{2.22$_\mathrm{S}$}$$

As a consequence, the two pieces $j_\mu$ and $j_\mu^{\mathrm{M}}$ of the current $J_\mu$ satisfy

$$g^{\mu\nu}(\partial_\mu j_\nu - [j_\mu, j_\nu^{\mathrm{M}}]) = 0\,, \tag{2.23}$$

$$g^{\mu\nu}(\partial_\mu j_\nu^{\mathrm{M}} + [j_\mu, j_\nu^{\mathrm{M}}]) = 0\,, \tag{2.24}$$

as well as

$$\partial_\mu j_\nu^{\mathrm{M}} - \partial_\nu j_\mu^{\mathrm{M}} + [j_\mu, j_\nu^{\mathrm{M}}] - [j_\nu, j_\mu^{\mathrm{M}}] + [j_\mu^{\mathrm{M}}, j_\nu^{\mathrm{M}}] = 0\,. \tag{2.25}$$

(In fact, a rather lengthy calculation shows that for both models, (2.23) follows from the bosonic equation (2.21), while (2.24) and (2.25) follow from the fermionic equation (2.22); we refer the reader to [7] for the proof in a more general situation.) Of course, (2.23) and (2.24) imply conservation of the current:

$$g^{\mu\nu}\partial_\mu J_\nu = 0\,. \tag{2.26}$$

Moreover, combining (2.11) and (2.25), we get

$$\partial_\mu(J_\nu + j_\nu^{\mathrm{M}}) - \partial_\nu(J_\mu + j_\mu^{\mathrm{M}}) + 2[J_\mu, J_\nu] = 0\,. \tag{2.27}$$

From (2.23)-(2.27), we can check [16] that the equations of motion imply the integrability, for any value of the real parameter $\lambda$, of the following system of first-order differential equations:

$$\begin{aligned}\partial_\mu U^{(\lambda)} = U^{(\lambda)}\{&(1-\cosh\lambda)j_\mu - \sinh(\lambda)\varepsilon_{\mu\nu}g^{\nu\kappa}j_\kappa \\ &+ \tfrac{1}{2}(1-\cosh(2\lambda))j_\mu^{\mathrm{M}} - \tfrac{1}{2}\sinh(2\lambda)\varepsilon_{\mu\nu}g^{\nu\kappa}j_\kappa^{\mathrm{M}}\}\,,\end{aligned} \tag{2.28}$$

where $U^{(\lambda)}$ is a U($n$)-valued field which serves as a generating functional for the non-local charges. In particular, the first non-local charge $Q^{(1)}$, whose conservation (i.e. time independence) can also be verified directly from (2.26) and (2.27), is

$$Q^{(1)}(t) = \int \mathrm{d}y_1\,\mathrm{d}y_2\,\theta(y_1-y_2)[J_0(t,y_1), J_0(t,y_2)] - \int \mathrm{d}y(J_1(t,y) + j_1^{\mathrm{M}}(t,y))\,. \tag{2.29}$$

To conclude this section, we want to clarify why the model with the lagrangian (2.17$_\mathrm{M}$) is called "minimal". To this end, we consider the minimal lagrangian

$$\begin{aligned}L &= g^{\mu\nu}\,\mathrm{tr}\,(\tilde{D}_\mu z^\dagger\,\tilde{D}_\nu z) + \tfrac{1}{2}i\bar{\psi}\,\tilde{\not{D}}\psi \\ &= g^{\mu\nu}\overline{\tilde{D}_\mu z_i^a}\,\tilde{D}_\nu z_i^a + \tfrac{1}{2}ig^{\mu\nu}\bar{\psi}^a\gamma_\mu\tilde{D}_\nu\psi^a\,,\end{aligned} \tag{2.30$_\mathrm{M}$}$$

where the covariant derivatives

$$\tilde{D}_\mu z = \partial_\mu z - z\tilde{A}_\mu \quad \text{i.e.} \quad \tilde{D}_\mu z_i^a = \partial_\mu z_i^a - \tilde{A}_\mu^{ba} z_i^b\,,$$
$$\tilde{D}_\mu \psi = \partial_\mu \psi - \psi\tilde{A}_\mu \quad \text{i.e.} \quad \tilde{D}_\mu \psi^a = \partial_\mu \psi^a - \tilde{A}_\mu^{ba} \psi^b\,, \tag{2.31$_\mathrm{M}$}$$

refer to a new, a priori independent gauge potential $\tilde{A}_\mu$. (This procedure is motivated by the quantum theory, where - at least in the functional integral approach and within the $1/n$ expansion (see sect. 3) - the gauge potential acquires the status of an independent field.) Now the lagrangian (2.30$_\mathrm{M}$) contains no derivatives of $\tilde{A}_\mu$, so that variation with respect to $\tilde{A}_\mu$ leads to an algebraic equation of motion:

$$\tilde{A}_\mu^{ab} = A_\mu^{ab} - \tfrac{1}{2} i\bar{\psi}^a \gamma_\mu \psi^b\,, \tag{2.32$_\mathrm{M}$}$$

and with this equation, (2.30$_\mathrm{M}$) reduces to (2.17$_\mathrm{M}$).

Of course, the supersymmetric model can be handled in the same spirit. However, part of the above motivation is lost because we have to start from a minimal lagrangian with an additional chiral Gross-Neveu-type interaction term, namely

$$\begin{aligned} L &= g^{\mu\nu}\,\mathrm{tr}\,(\tilde{D}_\mu z^+ \tilde{D}_\nu z) + \tfrac{1}{2} i\bar{\psi}\,\tilde{\tilde{\not{D}}}\psi + \tfrac{1}{4}\tilde{L}_{\mathrm{FS}} \\ &= g^{\mu\nu}\overline{\tilde{D}_\mu z_i^a}\,\tilde{D}_\nu z_i^a + \tfrac{1}{2} i g^{\mu\nu}\bar{\psi}_i^a \gamma_\mu \tilde{\tilde{D}}_\nu \psi_i^a + \tfrac{1}{4}\tilde{L}_{\mathrm{FS}}\,, \end{aligned} \tag{2.30a$_\mathrm{S}$}$$

with

$$\tilde{L}_{\mathrm{FS}} = (\bar{\psi}_i^a \psi_i^b)(\bar{\psi}_j^b \psi_j^a) - (\bar{\psi}_i^a \gamma_5 \psi_i^b)(\bar{\psi}_j^b \gamma_5 \psi_j^a) = -g^{\mu\nu}(\bar{\psi}_i^a \gamma_\mu \psi_j^a)(\bar{\psi}_j^b \gamma_\nu \psi_i^b)\,, \tag{2.30b$_\mathrm{S}$}$$

where as before, the covariant derivatives

$$\tilde{D}_\mu z = \partial_\mu z - z\tilde{A}_\mu \quad \text{i.e.} \quad \tilde{D}_\mu z_i^a = \partial_\mu z_i^a - \tilde{A}_\mu^{ba} z_i^b\,,$$
$$\tilde{D}_\mu \psi = \partial_\mu \psi - \psi\tilde{A}_\mu \quad \text{i.e.} \quad \tilde{D}_\mu \psi_i^a = \partial_\mu \psi_i^a - \tilde{A}_\mu^{ba} \psi_i^b\,, \tag{2.31$_\mathrm{S}$}$$

refer to a new, a priori independent gauge potential $\tilde{A}_\mu$. Once again, $\tilde{A}_\mu$ satisfies an algebraic equation of motion:

$$\tilde{A}_\mu^{ab} = A_\mu^{ab} - \tfrac{1}{2} i\bar{\psi}_i^a \gamma_\mu \psi_i^b\,, \tag{2.32$_\mathrm{S}$}$$

and with this equation, (2.30$_\mathrm{S}$) reduces to (2.17$_\mathrm{S}$).

In terms of the covariant derivatives $\tilde{D}_\mu$, the Noether current $J_\mu$ (cf. (2.19), (2.20)) takes the simple form

$$(J_\mu)_{ij} = z_i^a \overline{\tilde{D}_\mu z_j^a} - \tilde{D}_\mu z_i^a \bar{z}_j^a\,; \tag{2.33$_\mathrm{M}$}$$

$$(J_\mu)_{ij} = z_i^a \overline{\tilde{D}_\mu z_j^a} - \tilde{D}_\mu z_i^a \bar{z}_j^a + i\bar{\psi}_j^a \gamma_\mu \psi_i^a\,. \tag{2.33$_\mathrm{S}$}$$

In sect. 3, we shall find it convenient to slightly modify the minimal model by allowing the fermionic sector to come in $n$ identical copies. (Otherwise, we would not be able to set up a consistent $1/n$-expansion.) We leave it to the reader as an exercise to adapt the preceding analysis to this situation.

## 3. The quantum model and the $1/n$ expansion

The $1/n$ expansion for the $G_{\mathbb{C}}(p, n-p)$ model is, in many respects, a straightforward generalization of the $1/n$ expansion for the $\mathbb{C}P^{n-1}$ model, which has been formulated in some detail in [14, 17]; see also ref. [18]. For simplicity, we work in euclidean space; the final results can then be transferred to Minkowski space by the standard Wick rotation techniques. Throughout, the action $S = \int d^2x\, L$ (as given in sect. 2) and the fields $z, \bar{z}, \psi, \bar{\psi}$ are rescaled according to $S \to (n/2f)S$, $z \to (n/2f)^{1/2} z$, $\bar{z} \to (n/2f)^{1/2}\bar{z}$, $\psi \to (n/2f)^{1/2}\psi$, $\bar{\psi} \to (n/2f)^{1/2}\bar{\psi}$. (Note that the minimal model and the supersymmetric model now both involve an $(n \times p)$-matrix scalar field $z = (z_i^a)$ and an $(n \times p)$-matrix Dirac spinor field $\psi = (\psi_i^a)$, and differ only in the structure of the fermionic self-interaction term $L_{\mathrm{F}}$.)

### 3.1. THE PURE MODEL

For the pure Grassmann model, the generating functional of the euclidean Green functions is

$$Z(J, \bar{J}) = N^{-1} \int \mathscr{D}\bar{z}\, \mathscr{D}z \prod_{x,a,b} \delta\left( \bar{z}_i^a(x) z_i^b(x) - \frac{n}{2f}\delta^{ab} \right) \times \exp\left[ -S + \int d^2x \{ \bar{J}_i^a z_i^a + \bar{z}_i^a J_i^a \} \right], \tag{3.1}$$

with

$$\begin{aligned} S &= \int d^2x (\partial_\mu \bar{z}_i^a + A_\mu^{ab} \bar{z}_i^b)(\partial_\mu z_i^a - A_\mu^{ca} z_i^c) \\ &= \int d^2x \left\{ \partial_\mu \bar{z}_i^a \partial_\mu z_i^a + \frac{n}{2f} A_\mu^{ab} A_\mu^{ba} \right\}, \end{aligned} \tag{3.2}$$

$$A_\mu^{ab} = \frac{f}{n} \bar{z}_i^a \overleftrightarrow{\partial}_\mu z_i^b \tag{3.3}$$

(compare (2.3), (2.4), (2.6)). The functional integral in (3.1) is rewritten by introducing hermitian $(p \times p)$-matrix fields $\alpha$ and $\lambda$ which act as Lagrange multipliers for the (rescaled) constraint (2.1) and for the quartic $A_\mu^2$ interaction terms, respectively:

$$\prod_{x,a,b} \delta\left( \bar{z}_i^a(x) z_i^b(x) - \frac{n}{2f}\delta^{ab} \right) \sim \int \left[ \prod_{x,a,b} d\alpha^{ab}(x) \right] \exp\left[ \int d^2x \left\{ \frac{i}{\sqrt{n}} \alpha^{ba} \left( \bar{z}_i^a z_i^b - \frac{n}{2f}\delta^{ab} \right) \right\} \right];$$

$$\exp\left[ \int d^2x \left\{ -\frac{n}{2f} A_\mu^{ab} A_\mu^{ba} \right\} \right] \sim \int \left[ \prod_{x,\mu,a,b} d\lambda_\mu^{ab}(x) \right] \exp\left[ \int d^2x \left\{ -\frac{1}{2f} \lambda_\mu^{ab} \lambda_\mu^{ba} + \frac{i\sqrt{n}}{f} A_\mu^{ab} \lambda_\mu^{ba} \right\} \right]. \tag{3.4}$$

(Here, $\sim$ means equality up to normalization constants.) Indeed, inserting (3.4) into (3.1), we see that the functional integral over $\bar{z}, z$ becomes gaussian and can be carried out. The result is

$$Z(J,\bar{J}) = N^{-1}\int \mathscr{D}\alpha\,\mathscr{D}\lambda\,\exp[-S_{\text{eff}} + S_{\text{source}}] \tag{3.5}$$

with

$$S_{\text{eff}} = n\,\text{tr}\ln(\Delta_B) + \frac{i\sqrt{n}}{2f}\int d^2x\,\text{tr}\,\alpha \tag{3.6}$$

and

$$S_{\text{source}} = \bar{J}\cdot\Delta_{\text{B}}^{-1}J\,. \tag{3.7}$$

Here, $\Delta_{\text{B}}$ denotes the $(p\times p)$-matrix differential operator given by

$$\Delta_{\text{B}}z = D_\mu^* D_\mu z + m^2 z - \frac{i}{\sqrt{n}}z\alpha\,, \tag{3.8}$$

with $D_\mu^*$ the adjoint of $D_\mu$ ($D_\mu^* = -D_\mu$ under appropriate boundary conditions), $m>0$ an arbitrary - and so far irrelevant - constant (obtained from a redefinition of the normalization constant $N$ in (3.1)), and

$$D_\mu z = \partial_\mu z + \frac{i}{\sqrt{n}}z\lambda_\mu\,. \tag{3.9}$$

Thus $D_\mu$ is the covariant derivative with respect to a new gauge potential, namely $-in^{-1/2}\lambda_\mu$ (compare (2.3)). Its appearance reflects a new local $U(p)$ invariance of the effective theory, with gauge transformations acting according to

$$\begin{aligned} &z\to zh\,, \qquad J\to Jh\,,\\ &\alpha\to h^{-1}\alpha h\,, \qquad \lambda_\mu\to h^{-1}\lambda_\mu h - i\sqrt{n}\,h^{-1}\partial_\mu h\,. \end{aligned} \tag{3.10}$$

The next step is to expand the logarithm in (3.6), which leads to an expansion of the effective action in powers of $1/\sqrt{n}$:

$$S_{\text{eff}} = \sum_{\nu=1}^{\infty} n^{1-\nu/2}S^{(\nu)} + \text{const}\,. \tag{3.11}$$

The first nontrivial term is computed to be*

$$S^{(1)} = i\,\text{tr}\,\tilde{\alpha}(0)\left(\frac{1}{2f} - \int\frac{d^2p}{4\pi^2}\frac{1}{p^2+m^2}\right). \tag{3.12}$$

Regularizing the logarithmically divergent integral in (3.12) with, e.g., a Pauli-Villars cutoff $\Lambda$, we see that in order to cancel the infinite contribution to $S^{(1)}$, the bare

* Our convention on Fourier transformations, denoted by $\tilde{}$, is the same as in refs. [14, 17].

coupling $f = f(\Lambda)$ must depend on the cutoff $\Lambda$ according to

$$\frac{2\pi}{f(\Lambda)} = \ln\frac{\Lambda^2}{\mu^2} + \frac{2\pi}{f_{\mathrm{ren}}(\mu)}. \tag{3.13}$$

Moreover, in order to cancel the finite contribution as well, and hence to satisfy the saddle-point condition $S^{(1)} = 0$, the renormalized coupling $f_{\mathrm{ren}} = f_{\mathrm{ren}}(\mu)$ must depend on the normalization point $\mu$ according to

$$\frac{2\pi}{f_{\mathrm{ren}}(\mu)} = \ln\frac{\mu^2}{m^2}. \tag{3.14}$$

As in the $\mathbb{C}\mathrm{P}^{n-1}$ case, the model therefore exhibits asymptotic freedom ($f(\Lambda) \to 0$ as $\Lambda \to \infty$) and dynamical mass generation (all coupling constants can be eliminated in favor of the, now physical, mass parameter $m$). The second nontrivial term is precisely the term quadratic in the fields $\alpha$, $\lambda$ and can be written

$$S^{(2)} = \tfrac{1}{2}\int \mathrm{d}^2x\, \mathrm{d}^2y \{\alpha^{ab}(x)\Gamma^{(\alpha)}(x-y)\alpha^{ba}(y) + \lambda_\mu^{ab}(x)\Gamma_{\mu\nu}^{(\lambda)}(x-y)\lambda_\nu^{ba}(y)\}, \tag{3.15}$$

where

$$\tilde{\Gamma}^{(\alpha)}(p) = A(p), \qquad \tilde{\Gamma}_{\mu\nu}^{(\lambda)}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)F(p), \tag{3.16}$$

$$A(p) = \frac{1}{2\pi}\frac{1}{\sqrt{p^2(p^2+4m^2)}}\ln\frac{\sqrt{p^2+4m^2}+\sqrt{p^2}}{\sqrt{p^2+4m^2}-\sqrt{p^2}},$$

$$F(p) = (p^2+4m^2)A(p) - \frac{1}{\pi}. \tag{3.17}$$

For later use, we note the behavior of the functions $A$ and $F$ at small momenta and at large momenta (to lowest order):

$$A(p) \approx \frac{1}{4\pi m^2}, \qquad F(p) \approx \frac{1}{12\pi}\frac{p^2}{m^2} \qquad \text{for} \qquad p^2 \ll m^2; \tag{3.18}$$

$$A(p) \approx \frac{1}{2\pi p^2}\ln\frac{p^2}{m^2}, \qquad F(p) \approx \frac{1}{2\pi}\ln\frac{p^2}{m^2} \qquad \text{for} \qquad p^2 \gg m^2. \tag{3.19}$$

We also want to mention that the calculation of $\Gamma_{\mu\nu}^{(\lambda)}$ will, in general, lead to the result given in (3.16) only up to a term of the form $a\delta_{\mu\nu}$, where $a$ is a constant whose concrete value depends on the specific scheme employed in the regularization of the logarithmically divergent integrals appearing in this calculation. Later, however, we shall use a gauge in which the gauge field propagator is transverse, and this will force $a = 0$.

So far, we have been able to manage without having to fix the gauge, but this is of course a necessary prerequisite for the derivation of Feynman rules. As long as the gauge condition is linear, the gauge fixing + Faddeev-Popov procedure amounts to a redefinition of the generating functional $Z$ by introducing a $(p \times p)$-matrix field $\zeta$*, say, for the Faddeev-Popov ghosts and inserting a factor $\int \mathscr{D}\bar{\zeta}\mathscr{D}\zeta \exp[-S_{gf} - S_{FP}]$ on the r.h.s. of (3.5) ([19], pp. 579-582). In addition, we want to maintain euclidean covariance. Both requirements are met by choosing a covariant linear gauge condition $\partial'_\mu \lambda_\mu = 0$, where $\partial'_\mu$ is any two-vector of pseudo-differential operators; more explicitly, we assume $\partial'_\mu$ to be given by

$$(\partial'_\mu f)(x) = \int \frac{d^2p}{4\pi^2} e^{+ip\cdot x} ip_\mu L(p)\tilde{f}(p), \tag{3.20}$$

with $L$ an even function of momentum which is left unspecified for the moment. The gauge-fixing and Faddeev-Popov contributions to the total action are then

$$S_{gf} = \frac{1}{2\gamma} \int d^2x\, \partial'_\mu \lambda_\mu^{ab} \partial'_\nu \lambda_\nu^{ba}, \tag{3.21}$$

$\gamma$ being some non-negative constant, and

$$S_{FP} = \bar{\zeta} \cdot \mathscr{M}\zeta = \int d^2x\, \bar{\zeta}^{ab} \mathscr{M}^{ab,cd} \zeta^{cd}. \tag{3.22}$$

$\mathscr{M}$ denotes the Faddeev-Popov operator in this gauge, which is a $(p^2 \times p^2)$-matrix differential operator given by

$$\mathscr{M}\zeta = -\partial'_\mu D_\mu \zeta, \qquad D_\mu \zeta = \partial_\mu \zeta + \frac{i}{\sqrt{n}}[\lambda_\mu, \zeta]. \tag{3.23}$$

Moreover, the gauge-fixing term (3.21) can be combined with $S^{(2)}$ (cf. (3.11), (3.15)) and be absorbed into a redefinition of $\Gamma^{(\lambda)}$, which becomes $\Gamma^{(\lambda)}_{new} = \Gamma^{(\lambda)}_{old} + \Gamma^{(\lambda)}_{gf}$ with

$$(\tilde{\Gamma}^{(\lambda)}_{gf})_{\mu\nu}(p) = \gamma^{-1} p_\mu p_\nu L(p)^2. \tag{3.24}$$

In the usual treatment of gauge theories in four dimensions, one would of course take $\partial'_\mu = \partial_\mu$, i.e. $L = 1$, but here, in two dimensions, the corresponding gauge-fixing term in the action would lead to a breakdown of renormalizability. This defect can be cured by modifying the behavior at large momenta, and a particularly convenient choice is

$$L(p)^2 = \frac{1}{p^2} F(p), \tag{3.25}$$

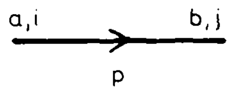
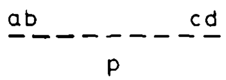
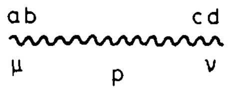
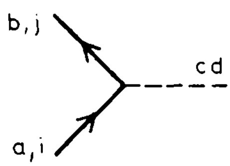
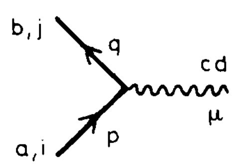
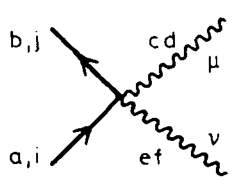
which leads to

$$\tilde{\Gamma}^{(\lambda)}_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) F(p) + \gamma^{-1} \frac{p_\mu p_\nu}{p^2} F(p). \tag{3.26}$$

* The matrix elements of this field are anticommuting $c$-numbers.

TABLE 1

Feynman rules for the Grassmann models: pure model, $D^{(\lambda)}(p)=F(p)^{-1}$; fermionic models, $D^{(\lambda)}(p)=(F(p)+1/\pi)^{-1}$

| | Line | Propagator |
|---|---|---|
| $\bar{z}/z$ line: |  | $\delta^{ab}\delta_{ij}(p^2+m^2)^{-1}$ |
| $\alpha$ line: |  | $\delta^{ad}\delta^{bc}A(p)^{-1}$ |
| $\lambda$ line: |  | $\delta^{ad}\delta^{bc}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)D^{(\lambda)}(p)$ |
| | Vertex | Vertex factor |
| $\bar{z}z\alpha$ vertex: |  | $\frac{i}{\sqrt{n}}\delta^{ad}\delta^{bc}\delta_{ij}$ |
| $\bar{z}z\lambda$ vertex: |  | $-\frac{1}{\sqrt{n}}\delta^{ad}\delta^{bc}\delta_{ij}(p_\mu+q_\mu)$ |
| $\bar{z}z\lambda\lambda$ vertex: |  | $-\frac{1}{n}(\delta^{ad}\delta^{be}\delta^{fc}+\delta^{af}\delta^{bc}\delta^{de})\delta_{ij}\delta_{\mu\nu}$ |

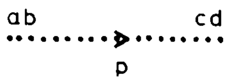
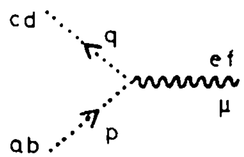
Inversion yields the propagators

$$D^{(\alpha)}(p)=A(p)^{-1}\,,\qquad D^{(\lambda)}_{\mu\nu}(p)=\left(\delta_{\mu\nu}-(1-\gamma)\frac{p_\mu p_\nu}{p^2}\right)F(p)^{-1}\,. \tag{3.27}$$

We shall usually work in the Landau gauge $\gamma=0$, which gives a transverse $\lambda$-propagator.

TABLE 2

Feynman rules for the Faddeev-Popov ghosts in the Grassmann models: pure model, $L(p)^2=(1/p^2)F(p)$; fermionic models, $L(p)^2=(1/p^2)(F(p)+1/\pi)$

| | Line | Propagator |
|---|---|---|
| $\bar{\zeta}/\zeta$ line: |   | $\delta^{ad}\delta^{bc}(p^2L(p))^{-1}$ |
| | Vertex | Vertex factor |
| $\bar{\zeta}\zeta\lambda$ vertex: |   | $\frac{1}{\sqrt{n}}(\delta^{ad}\delta^{be}\delta^{fc}-\delta^{af}\delta^{bc}\delta^{de})p_\mu L(p)$ |

Now any Green function for the pure Grassmann model has a formal expansion in terms of Feynman diagrams; the corresponding Feynman rules are collected in table 1*. When specialized to the $\mathbb{C}P^{n-1}$ model, these Feynman rules coincide with those given in ref. [17] (except for a combinatorial factor 2 in the $\bar{z}z\lambda\lambda$ vertex; see [19], p. 285, for a comment). We also refer to [17] for a list of forbidden diagrams: these are all one-loop diagrams with only one external leg (tadpoles) or with two identical external legs. Finally, we have included in table 2 the Feynman rules pertaining to the Faddeev-Popov ghosts, which are also slightly unusual due to our choice of the Faddeev-Popov operator $\mathcal{M}$ (cf. (3.20), (3.23), (3.25)). Note, however, that as far as (sub)diagrams without external ghost lines are concerned, we could just as well have used the standard Feynman rules for the ghosts (where $L=1$); in fact, it is obvious from table 2 that the $L$-factors cancel inside the ghost loops.

As a systematic device to remove UV divergences in the Green functions of our model, we shall use the framework of BPHZ renormalization [19, 20], with all subtractions of Taylor terms in the integrands performed around zero external momenta. This supposes, of course, that we use an infrared cutoff for the massless fields appearing in the theory, and the question of possible IR divergences in the UV renormalized Green functions remains to be investigated separately.

One of the first steps in the BPHZ program is to determine the superficial degree of divergence, $\delta(\Gamma)$, of a given proper (sub)diagram $\Gamma$, thus fixing the necessary number of subtractions to be performed. This degree can be obtained by using the

* We follow the usual convention of orienting lines corresponding to charged fields $A$ "from the $\bar{A}$ to the $A$" and letting their momenta flow in the direction of the resulting arrows.

asymptotic behavior of the propagators at large momenta (cf. (3.19)) and the momentum factors for the vertices, together with a few combinatorial rules (compare [21]):

$$\delta(\Gamma) = 2 - 2E_\alpha - E_\lambda - E_\zeta . \tag{3.28}$$

($E_A$ denotes the number of external lines of type A. Note that $\bar{z}/z$ lines (and $\bar{\zeta}/\zeta$ lines) can only form uninterrupted strings or internal loops, so that $E_{\bar{z}} = E_z$ (and $E_{\bar{\zeta}} = E_\zeta$).) Now according to (3.28), $\delta(\Gamma)$ does not depend on the number of external $\bar{z}/z$ lines, which at first glance seems a catastrophe because it appears to imply that the model has an infinite number of UV-divergent Green functions, hence also of possible counterterms (containing higher and higher polynomials in $\bar{z}$, $z$ and their derivatives), and is therefore not renormalizable. Fortunately, however, this problem does not arise due to cancellations of divergences between different diagrams - cancellations which were first discovered by Aref'eva [22] (for the model on spheres) and by Aref'eva and Azakov [21] (for the model on complex projective spaces).

Briefly, this cancellation mechanism works as follows. Let $\Gamma$ be a (connected) Feynman (sub)diagram with $r$ external $\bar{z}$ lines and $r$ external $z$ lines, and maybe other external lines as well, the former being arranged to form $r$ pairs of external $\bar{z}/z$ lines in such a way that the contribution from $\Gamma$ is proportional to $\delta_{i_1 j_1} \cdots \delta_{i_r j_r}$. (See fig. 1a.) Then if for some $l$, $1 \leq l \leq r$, the $l$th pair of external $\bar{z}/z$ lines of $\Gamma$ does not join at the nearest $\bar{z}z\alpha$ vertex, one can define a new (connected) Feynman (sub)diagram $\Gamma_l$ by applying the following attaching procedure to that pair: join the $\bar{z}$ line with the $z$ line and connect the resulting new $\bar{z}/z$ loop via a new internal $\alpha$-line to a new pair of external $\bar{z}/z$ lines. (See fig. 1b.) (The condition on $\Gamma$ is motivated by the fact that the attaching procedure must not be repeated on one and the same pair of external $\bar{z}/z$ lines because this would lead to a forbidden diagram.) Now if $\Gamma$ is superficially divergent and proper, it appears as a renormalization part $\gamma_l$ in $\Gamma_l$, and the Taylor terms to be subtracted in the process of renormalizing the contributions from $\Gamma$ and from $\gamma_l \subset \Gamma_l$ are both proportional to monomials $M(\partial) I_\Gamma|_0$ in partial derivatives of $I_\Gamma$, the unsubtracted integrand for $\Gamma$, with respect to the

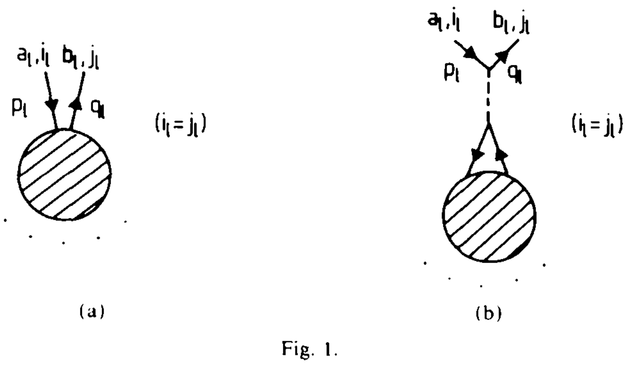



Fig. 1.

external momenta $p_1, q_1, \ldots, p_r, q_r, \ldots$, evaluated at zero. Hence due to the basic identity

$$D^{(\alpha)}(p)\int \frac{d^2k}{4\pi^2}\frac{1}{[(k+\frac{1}{2}p)^2+m^2][(k-\frac{1}{2}p)^2+m^2]}=1\,, \tag{3.29}$$

the Taylor terms proportional to $M(\partial)I_{\Gamma}|_0$ cancel as soon as $M(\partial)$ does not involve any differentiation with respect to the external momenta $p_l$ or $q_l$.

A complete treatment of the cancellation mechanism would of course require a systematic application of this attaching procedure to all sorts of combinations of pairs of external $\bar{z}/z$ lines, taking into account all possible renormalization parts according to Zimmermann's strategy, as expressed in his forest formula [19, 20]. Without going into any details, we just state the result that UV divergences of diagrams with more than two pairs of external $\bar{z}/z$ lines cancel among themselves.

The rest of the renormalization program, including the definition of normal products and the derivation of short-distance expansions, can now be carried through as usual [20]. For the Green functions, one is led to a wave function renormalization and a mass renormalization for the parton propagator, i.e. to the addition of counterterms to the lagrangian which are proportional to $\overline{D_\mu z_i^a}D_\mu z_i^a$ and to $\bar{z}_i^a z_i^a$, respectively. (Note that one may use the standard strategy of inserting (one or two) external $\lambda$ lines, with zero external momenta, in all possible ways (compare [19], pp. 336/337) in order to show that UV divergences of diagrams with one or two pairs of external $\bar{z}/z$ lines conspire to yield gauge-invariant counterterms. In other words, renormalization preserves gauge invariance, as expressed through the Ward–Takahashi identities.) For composite operators, one uses (3.29) to derive constraints on normal products, the most important of these being the quantum counterparts of the classical constraint (2.1) and of the classical expression (2.4) of the gauge potential as a composite field [21, 22], namely

$$\mathcal{N}[\bar{z}_i^a z_i^b] = \text{const}\;\delta^{ab}\,, \tag{3.30}$$

and

$$\mathcal{N}[(+i\bar{z}_i^a\partial_\mu z_i^b)^{\mathrm{T}}] = \text{const}\,\frac{1}{\sqrt{n}}\lambda_\mu^{ab} = \mathcal{N}[(-i\partial_\mu\bar{z}_i^a z_i^b)^{\mathrm{T}}]\,, \tag{3.31}$$

respectively, where $R^{\mathrm{T}}$ denotes the transverse part of $R$, i.e. $R_\mu^{\mathrm{T}} = (\delta_{\mu\nu} - \partial_\mu\partial_\nu/\partial^2)R_\nu$. The constant in both of these equations should be the same in order to guarantee that

$$\mathcal{N}[(+\bar{z}_i^a D_\mu z_i^b)^{\mathrm{T}}] = 0 = \mathcal{N}[(-\overline{D_\mu z_i^a}z_i^b)^{\mathrm{T}}]\,. \tag{3.32}$$

Its value depends on the normalization conditions, and we often choose these in such a way that it becomes zero.

Having taken care of the UV divergences, we are left with the problem of IR divergences; these appear in diagrams containing internal $\lambda$ lines, due to the pole of the gauge field propagator at $p^2=0$ (cf. (3.18)). In vacuum expectation values of gauge-invariant operators, however, the IR divergences should cancel. In fact, this cancellation has been shown to occur in the $\mathbb{C}P^{n-1}$ models [23], and although we did not check this explicitly, we expect the same mechanism to work in the Grassmann models as well. [Such an analogy can certainly not be ruled out by comparing the situation to that in four dimensions, where the structure of IR divergences in QCD is of course very different from that in QED: in fact, a look at the Feynman rules already reveals that the analogy between the non-abelian case and the abelian case is much closer in two-dimensional non-linear sigma models than it is in four-dimensional gauge theories.]

### 3.2. THE FERMIONIC MODELS

For the Grassmann models with fermions, the whole procedure is quite similar (although the supersymmetric case is technically considerably more complicated) and so are the results - with one crucial difference: just as in the Schwinger model (massless $QED_2$), vacuum polarization by the fermions shifts the pole in the gauge field propagator away from $p^2=0$.

Once again, we begin with the generating functional of the euclidean Green functions, which is

$$Z(J,\bar{J},\eta,\bar{\eta}) = N^{-1}\int \mathscr{D}\bar{z}\,\mathscr{D}z\,\mathscr{D}\bar{\psi}\,\mathscr{D}\psi \prod_{x,a,b}\delta\left(\bar{z}_i^a(x)z_i^b(x)-\frac{n}{2f}\delta^{ab}\right) \times\exp\left[-S+\int d^2x\{\bar{J}_i^a z_i^a+\bar{z}_i^a J_i^a+\bar{\eta}_i^a\psi_i^a+\bar{\psi}_i^a\eta_i^a\}\right], \tag{3.33$_M$}$$

with

$$S=\int d^2x\{(\partial_\mu\bar{z}_i^a+A_\mu^{ab}\bar{z}_i^b)(\partial_\mu z_i^a-A_\mu^{ca}z_i^c)+\tfrac{1}{2}i\bar{\psi}_i^a\overleftrightarrow{\not\partial}\psi_i^a-i\bar{\psi}_i^a\gamma_\mu\psi_i^b A_\mu^{ba}\}$$

$$=\int d^2x\left\{\partial_\mu\bar{z}_i^a\partial_\mu z_i^a+\tfrac{1}{2}i\bar{\psi}_i^a\overleftrightarrow{\not\partial}\psi_i^a+\frac{n}{2f}A_\mu^{ab}A_\mu^{ba}\right\}, \tag{3.34$_M$}$$

$$A_\mu^{ab}=\frac{f}{n}\bar{z}_i^a\overleftrightarrow{\partial_\mu}z_i^b+B_\mu^{ab}, \tag{3.35}$$

$$B_\mu^{ab}=-i\frac{f}{n}\bar{\psi}_i^a\gamma_\mu\psi_i^b, \tag{3.36$_M$}$$

in the minimal case, and

$$Z(J,\bar{J},\eta,\bar{\eta}) = N^{-1}\int \mathscr{D}\bar{z}\,\mathscr{D}z\,\mathscr{D}\bar{\psi}\,\mathscr{D}\psi \prod_{x,a,b}\delta\left(\bar{z}_i^a(x)z_i^b(x)-\frac{n}{2f}\delta^{ab}\right) \times\prod_{x,a,b}\delta(\bar{z}_i^b(x)\psi_i^a(x))\delta(\bar{\psi}_i^a(x)z_i^b(x)) \times\exp\left[-S+\int d^2x\{\bar{J}_i^a z_i^a+\bar{z}_i^a J_i^a+\bar{\eta}_i^a\psi_i^a+\bar{\psi}_i^a\eta_i^a\}\right], \tag{3.33$_S$}$$

with

$$S=\int d^2x\left\{(\partial_\mu\bar{z}_i^a+A_\mu^{ab}\bar{z}_i^b)(\partial_\mu z_i^a-A_\mu^{ca}z_i^c)+\tfrac{1}{2}i\bar{\psi}_i^a\overleftrightarrow{\not\partial}\psi_i^a-i\bar{\psi}_i^a\gamma_\mu\psi_i^bA_\mu^{ba}\right.$$

$$\left.+\frac{f}{2n}((\bar{\psi}_i^a\psi_i^b)(\bar{\psi}_j^b\psi_j^a)-(\bar{\psi}_i^a\gamma_5\psi_i^b)(\bar{\psi}_j^b\gamma_5\psi_j^a))\right\}$$

$$=\int d^2x\left\{\partial_\mu\bar{z}_i^a\partial_\mu z_i^a+\tfrac{1}{2}i\bar{\psi}_i^a\overleftrightarrow{\not\partial}\psi_i^a+\frac{n}{2f}A_\mu^{ab}A_\mu^{ba}-\frac{n}{2f}B^{ab}B^{ba}-\frac{n}{2f}B_5^{ab}B_5^{ba}\right\},\tag{3.34$_S$}$$

$$A_\mu^{ab}=\frac{f}{n}\bar{z}_i^a\overleftrightarrow{\partial_\mu}z_i^b+B_\mu^{ab},\tag{3.35}$$

$$B_\mu^{ab}=-i\frac{f}{n}\bar{\psi}_i^a\gamma_\mu\psi_i^b,\qquad B^{ab}=-i\frac{f}{n}\bar{\psi}_i^a\psi_i^b,\qquad B_5^{ab}=\frac{f}{n}\bar{\psi}_i^a\gamma_5\psi_i^b,\tag{3.36$_S$}$$

in the supersymmetric case (compare (2.30)-(2.32); we have simply written $A_\mu$ instead of $\tilde{A}_\mu$). The functional integral in (3.33) is rewritten by introducing hermitian $(p\times p)$-matrix fields $\alpha$ and $\lambda$ as before and, in the supersymmetric case, an additional $(p\times p)$-matrix field $c$ and additional hermitian $(p\times p)$-matrix fields $\phi$ and $\phi_5$ which act as Lagrange multipliers for the (rescaled) constraint (2.14$_S$) and for the quartic $B^2$ and $B_5^2$ interaction terms, respectively:

$$\prod_{x,a,b}\delta(\bar{z}_i^b(x)\psi_i^a(x))\delta(\bar{\psi}_i^a(x)z_i^b(x))$$

$$\sim\int\left[\prod_{x,a,b}d\bar{c}^{ab}(x)\right]\left[\prod_{x,a,b}dc^{ba}(x)\right]\exp\left[\int d^2x\left\{\frac{i}{\sqrt{n}}\bar{c}^{ab}\bar{z}_i^b\psi_i^a+\frac{i}{\sqrt{n}}\bar{\psi}_i^az_i^bc^{ba}\right\}\right];$$

$$\exp\left[\int d^2x\left\{\frac{n}{2f}B^{ab}B^{ba}\right\}\right]$$

$$\sim\int\left[\prod_{x,a,b}d\phi^{ab}(x)\right]\exp\left[\int d^2x\left\{-\frac{1}{2f}\phi^{ab}\phi^{ba}+\frac{\sqrt{n}}{f}B^{ab}\phi^{ba}\right\}\right];$$

$$\exp\left[\int d^2x\left\{\frac{n}{2f}B_5^{ab}B_5^{ba}\right\}\right]$$

$$\sim\int\left[\prod_{x,a,b}d\phi_5^{ab}(x)\right]\exp\left[\int d^2x\left\{-\frac{1}{2f}\phi_5^{ab}\phi_5^{ba}+\frac{\sqrt{n}}{f}B_5^{ab}\phi_5^{ba}\right\}\right].\tag{3.37$_S$}$$

(Here, $\sim$ means equality up to normalization constants.) Indeed, inserting (3.4) and, in the supersymmetric case, (3.37$_S$) into (3.33), we see that the functional integrals over $\bar{z}$, $z$ and over $\bar{\psi}$, $\psi$ become gaussian and can be carried out. The result is

$$Z(J,\bar{J},\eta,\bar{\eta})=N^{-1}\int\mathcal{D}\alpha\,\mathcal{D}\lambda\,\exp[-S_{\text{eff}}+S_{\text{source}}]\tag{3.38$_M$}$$

with

$$S_{\mathrm{eff}} = n \operatorname{Tr} \ln (\Delta_{\mathrm{B}}) + \frac{i\sqrt{n}}{2f} \int \mathrm{d}^2x \operatorname{tr} \alpha - n \operatorname{Tr} \ln (\Delta_{\mathrm{F}}) , \tag{3.39$_{\mathrm{M}}$}$$

$$S_{\mathrm{source}} = \bar{J} \cdot \Delta_{\mathrm{B}}^{-1} J + \bar{\eta} \cdot \Delta_{\mathrm{F}}^{-1} \eta , \tag{3.40$_{\mathrm{M}}$}$$

in the minimal case, and

$$Z(J, \bar{J}, \eta, \bar{\eta}) = N^{-1} \int \mathscr{D}\alpha\, \mathscr{D}\bar{c}\, \mathscr{D}c\, \mathscr{D}\lambda\, \mathscr{D}\phi\, \mathscr{D}\phi_5 \exp[-S_{\mathrm{eff}} + S_{\mathrm{source}}] , \tag{3.38$_{\mathrm{S}}$}$$

with

$$S_{\mathrm{eff}} = n \operatorname{Tr} \ln \left( \Delta_{\mathrm{B}} + \frac{1}{n} \bar{c} \Delta_{\mathrm{F}}^{-1} c \right) + \frac{i\sqrt{n}}{2f} \int \mathrm{d}^2x \operatorname{tr} \alpha$$

$$- n \operatorname{Tr} \ln (\Delta_{\mathrm{F}}) + \frac{1}{2f} \int \mathrm{d}^2x \operatorname{tr} (\phi^2 + \phi_5^2) , \tag{3.39$_{\mathrm{S}}$}$$

$$S_{\mathrm{source}} = \bar{J}' \cdot \left( \Delta_{\mathrm{B}} + \frac{1}{n} \bar{c} \Delta_{\mathrm{F}}^{-1} c \right)^{-1} J' + \bar{\eta} \cdot \Delta_{\mathrm{F}}^{-1} \eta , \tag{3.40$_{\mathrm{S}}$}$$

in the supersymmetric case. Here, $\Delta_{\mathrm{B}}$ is as in (3.8), while $\Delta_{\mathrm{F}}$ denotes the $(p \times p)$-matrix differential operator given by

$$\Delta_{\mathrm{F}} \psi = \tfrac{1}{2} i (\not{D} - \not{D}^*) \psi , \tag{3.41$_{\mathrm{M}}$}$$

$$\Delta_{\mathrm{F}} \psi = \tfrac{1}{2} i (\not{D} - \not{D}^*) \psi + \frac{1}{\sqrt{n}} (i \psi \phi - \gamma_5 \psi \phi_5) , \tag{3.41$_{\mathrm{S}}$}$$

with $\not{D}^*$ the adjoint of $\not{D}$ ($\not{D}^* = -\not{D}$ under appropriate boundary conditions), and

$$D_\mu \psi = \partial_\mu \psi + \frac{i}{\sqrt{n}} \psi \lambda_\mu \tag{3.42}$$

in accordance with our interpretation of $-in^{-1/2}\lambda_\mu$ as a new gauge potential (compare (2.16)); the corresponding local U($p$) transformations act according to (3.10) and

$$\psi \to \psi h , \qquad \eta \to \eta h , \tag{3.43$_{\mathrm{M}}$}$$

$$\psi \to \psi h , \qquad \eta \to \eta h ,$$

$$c \to h^{-1} c h , \qquad \phi \to h^{-1} \phi h , \qquad \phi_5 \to h^{-1} \phi_5 h . \tag{3.43$_{\mathrm{S}}$}$$

Moreover, in the supersymmetric case, $\bar{c} \Delta_{\mathrm{F}}^{-1} c$ is a $(p \times p)$-matrix integral operator with matrix elements

$$(\bar{c} \Delta_{\mathrm{F}}^{-1} c)^{ab}(x, y) = c^{ac}(x) (\Delta_{\mathrm{F}}^{-1})^{cd}(x, y) \bar{c}^{db}(y) , \tag{3.44$_{\mathrm{S}}$}$$

and $J'$, $\bar{J}'$ are abbreviations for

$$J_i'^a(x) = J_i^a(x) + \frac{i}{\sqrt{n}} \int d^2y\, \eta_i^d(y)(\Delta_F^{-1})^{dc}(y,x)\bar{c}^{ca}(x)\,,$$

$$\bar{J}_i'^a(x) = \bar{J}_i^a(x) + \frac{i}{\sqrt{n}} \int d^2y\, c^{ac}(x)(\Delta_F^{-1})^{cd}(x,y)\bar{\eta}_i^d(y)\,. \qquad (3.45_S)$$

[Here and in the following, we use the convention that if $K$ is a $(p \times p)$-matrix integral operator (e.g. $\Delta_B^{-1}$, $\Delta_F^{-1}$, $\bar{c}\Delta_F^{-1}c$ or $(\Delta_B + (1/n)\bar{c}\Delta_F^{-1}c)^{-1}$) and $K^{ab}(x,y)$ is its integral kernel, then for any $p$-vector valued function $\chi$,

$$(K\chi)^a(x) = \int d^2y\, K^{ba}(y,x)\chi^b(y)\,. \qquad (3.46)$$

The transposition of indices and arguments guarantees that covariance of the operator $K$ and of its kernel $K^{ab}(x,y)$ under gauge transformations $\chi \to \chi h$ takes a simple form: $K\chi \to (K\chi)h$, $K^{ab}(x,y) \to (h^{-1})^{ac}(x)K^{cd}(x,y)h^{db}(y)$.]

The next step is, once again, to expand the logarithm in (3.39), and in order to make the terms $S^{(\nu)}$ in the resulting expansion (3.11) infrared finite, we introduce a fermion mass term. In the minimal case, this is done by hand, i.e. $\not{\partial}$ is replaced by $\not{\partial} + m_F$, and the limit $m_F \to 0$ is taken in the final results. For example, calculation of $S^{(1)}$ gives, once again,

$$S^{(1)} = i \operatorname{tr} \tilde{\alpha}(0)\left(\frac{1}{2f} - \int \frac{d^2p}{4\pi^2} \frac{1}{p^2 + m^2}\right) \qquad (3.47_M)$$

(even before removing the infrared cutoff $m_F$), so the saddle-point condition $S^{(1)} = 0$ implies that the minimal model exhibits, once again, asymptotic freedom and dynamical mass generation for the bosons, while the fermions stay massless. Moreover, the quadratic term $S^{(2)}$ can, once again, be written

$$S^{(2)} = \tfrac{1}{2} \int d^2x\, d^2y \{\alpha^{ab}(x)\Gamma^{(\alpha)}(x-y)\alpha^{ba}(y) + \lambda_\mu^{ab}(x)\Gamma_{\mu\nu}^{(\lambda)}(x-y)\lambda_\nu^{ba}(y)\}\,, \qquad (3.48_M)$$

where now

$$\tilde{\Gamma}^{(\alpha)}(p) = A(p)\,, \qquad \tilde{\Gamma}_{\mu\nu}^{(\lambda)}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)(p^2 + 4m^2)A(p)\,. \qquad (3.49_M)$$

In the supersymmetric case, we anticipate the fact that the quantum field $\phi$ develops a non-vanishing vacuum expectation value, and we therefore shift the variable $\phi$ in the functional integral ($3.38_S$), i.e. $\phi^{ab}$ and $\phi_5^{ab}$ are replaced by

$$\varphi^{ab} = \phi^{ab} - \sqrt{n}\, m_F \delta^{ab}\,, \qquad \varphi_5^{ab} = \phi_5^{ab}\,. \qquad (3.46_S)$$

Indeed, with this substitution, calculation of $S^{(1)}$ gives

$$S^{(1)} = i \operatorname{tr} \tilde{\alpha}(0)\left(\frac{1}{2f} - \int \frac{d^2p}{4\pi^2} \frac{1}{p^2+m^2}\right) + 2m_F \operatorname{tr} \tilde{\varphi}(0)\left(\frac{1}{2f} - \int \frac{d^2p}{4\pi^2} \frac{1}{p^2+m_F^2}\right), \qquad (3.47_S)$$

so that the saddle-point condition $S^{(1)} = 0$ implies that the supersymmetric model exhibits asymptotic freedom, dynamical mass generation with equal masses for bosons and fermions ($m = m_F$) and spontaneous chiral symmetry breaking. Moreover, the quadratic term $S^{(2)}$ can be written

$$\begin{aligned} S^{(2)} = \tfrac{1}{2} \int d^2x\, d^2y \Big\{ & \alpha^{ab}(x) \Gamma^{(\alpha)}(x-y) \alpha^{ba}(y) + \lambda_\mu^{ab}(x) \Gamma_{\mu\nu}^{(\lambda)}(x-y) \lambda_\nu^{ba}(y) \\ & + 2\bar{c}^{ab}(x) \Gamma^{(\bar{c}c)}(x-y) c^{ba}(y) \\ & + \varphi^{ab}(x) \Gamma^{(\varphi)}(x-y) \varphi^{ba}(y) + \varphi_5^{ab}(x) \Gamma^{(\varphi_5)}(x-y) \varphi_5^{ba}(y) \\ & + \lambda_\mu^{ab}(x) \Gamma_\mu^{(\lambda\varphi_5)}(x-y) \varphi_5^{ba}(y) + \varphi_5^{ab}(x) \Gamma_\nu^{(\varphi_5\lambda)}(x-y) \lambda_\nu^{ba}(y) \Big\}, \end{aligned} \qquad (3.48_S)$$

where

$$\tilde{\Gamma}^{(\alpha)}(p) = A(p), \qquad \tilde{\Gamma}_{\mu\nu}^{(\lambda)}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) p^2 A(p),$$

$$\tilde{\Gamma}^{(\bar{c}c)}(p) = -\tfrac{1}{2}(\not{p} + 2im) A(p),$$

$$\tilde{\Gamma}^{(\varphi)}(p) = (p^2 + 4m^2) A(p), \qquad \tilde{\Gamma}^{(\varphi_5)}(p) = p^2 A(p),$$

$$\tilde{\Gamma}_\mu^{(\lambda\varphi_5)}(p) = +2m\varepsilon_{\mu\kappa} p_\kappa A(p), \qquad \tilde{\Gamma}_\nu^{(\varphi_5\lambda)}(p) = -2m\varepsilon_{\nu\kappa} p_\kappa A(p). \qquad (3.49_S)$$

The procedure of fixing the gauge and introducing Faddeev-Popov ghosts in the fermionic models is identical with that in the pure model - except that it is now more convenient to replace (3.25) by

$$L(p)^2 = \frac{p^2 + 4m^2}{p^2} A(p) = \frac{1}{p^2}\left(F(p) + \frac{1}{\pi}\right), \qquad (3.50)$$

which leads to

$$\tilde{\Gamma}_{\mu\nu}^{(\lambda)}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)(p^2 + 4m^2) A(p) + \gamma^{-1} \frac{p_\mu p_\nu}{p^2} (p^2 + 4m^2) A(p). \qquad (3.51_M)$$

$$\tilde{\Gamma}_{\mu\nu}^{(\lambda)}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) p^2 A(p) + \gamma^{-1} \frac{p_\mu p_\nu}{p^2} (p^2 + 4m^2) A(p). \qquad (3.51_S)$$

Inversion yields the propagators

$$D^{(\alpha)}(p) = A(p)^{-1}, \qquad D_{\mu\nu}^{(\lambda)}(p) = \left(\delta_{\mu\nu} - (1-\gamma)\frac{p_\mu p_\nu}{p^2}\right)\left(F(p) + \frac{1}{\pi}\right)^{-1}, \qquad (3.52_M)$$

in the minimal model, and

$$D^{(\alpha)}(p)=A(p)^{-1}, \qquad D^{(\lambda)}_{\mu\nu}(p)=\left(\delta_{\mu\nu}-(1-\gamma)\frac{p_\mu p_\nu}{p^2}\right)\left(F(p)+\frac{1}{\pi}\right)^{-1},$$

$$D^{(\bar{c}c)}(p)=-2(\not{p}-2im)\left(F(p)+\frac{1}{\pi}\right)^{-1},$$

$$D^{(\varphi)}(p)=\left(F(p)+\frac{1}{\pi}\right)^{-1}, \qquad D^{(\varphi_S)}(p)=\left(F(p)+\frac{1}{\pi}\right)^{-1},$$

$$D^{(\lambda\varphi_S)}_{\mu}(p)=-2m\frac{\varepsilon_{\mu\kappa}p_\kappa}{p^2}\left(F(p)+\frac{1}{\pi}\right)^{-1}, \qquad D^{(\varphi_S\lambda)}_{\mu}(p)=+2m\frac{\varepsilon_{\mu\kappa}p_\kappa}{p^2}\left(F(p)+\frac{1}{\pi}\right)^{-1}, \tag{3.52$_S$}$$

in the supersymmetric model, where we have used

$$\left[\begin{pmatrix} p^2\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+\gamma^{-1}(p^2+4m^2)\frac{p_\mu p_\nu}{p^2} & +2m\varepsilon_{\mu\kappa}p_\kappa \\ -2m\varepsilon_{\nu\lambda}p_\lambda & p^2 \end{pmatrix}A(p)\right]^{-1}$$

$$=\begin{pmatrix} \left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+\gamma\frac{p_\mu p_\nu}{p^2} & -2m\varepsilon_{\mu\kappa}p_\kappa/p^2 \\ +2m\varepsilon_{\nu\lambda}p_\lambda/p^2 & 1 \end{pmatrix}((p^2+4m^2)A(p))^{-1}.$$

Again, the Landau gauge $\gamma=0$ gives a transverse $\lambda$ propagator.

The strategy for deriving the Feynman rules from the generating functional $Z$ is as before, and the results are summarized in tables 1-4. Once again, these rules are to be supplemented by the prescription that certain diagrams are forbidden: see [14] for a list of such diagrams.

TABLE 3

Additional Feynman rules for the minimal model

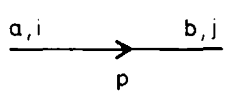
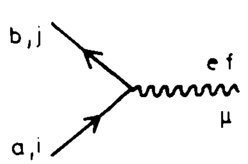
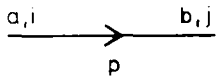
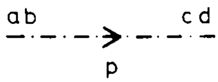
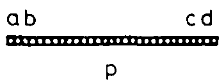
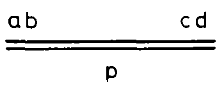
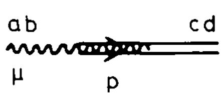
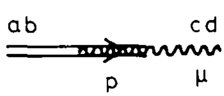
| | Line | Propagator |
|---|---|---|
| $\bar{\psi}/\psi$ line: |  | $-\delta^{ab}\delta_{ij}\frac{\not{p}}{p^2}$ |
| | **Vertex** | **Vertex factor** |
| $\bar{\psi}\psi\lambda$ vertex: |  | $\frac{1}{\sqrt{n}}\delta^{ad}\delta^{bc}\delta_{ij}\gamma_\mu$ |

TABLE 4

Additional Feynman rules for the supersymmetric model

| | Line | Propagator |
|---|---|---|
| $\bar\psi/\psi$ line: |  | $-\delta^{ab}\delta_{ij}\dfrac{\not p+im}{p^2+m^2}$ |
| $\bar c/c$ line: |  | $-2\delta^{ad}\delta^{bc}(\not p-2im)\left(F(p)+\dfrac{1}{\pi}\right)^{-1}$ |
| $\varphi$ line: |  | $\delta^{ad}\delta^{bc}\left(F(p)+\dfrac{1}{\pi}\right)^{-1}$ |
| $\varphi_5$ line: |  | $\delta^{ad}\delta^{bc}\left(F(p)+\dfrac{1}{\pi}\right)^{-1}$ |
| $\lambda\varphi_5$ line: |  | $-\delta^{ad}\delta^{bc}\dfrac{2m\varepsilon_{\mu\nu}p_\nu}{p^2}\left(F(p)+\dfrac{1}{\pi}\right)^{-1}$ |
| $\varphi_5\lambda$ line: |  | $+\delta^{ad}\delta^{bc}\dfrac{2m\varepsilon_{\mu\nu}p_\nu}{p^2}\left(F(p)+\dfrac{1}{\pi}\right)^{-1}$ |

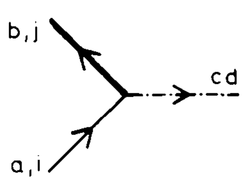
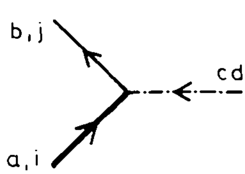
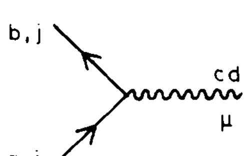
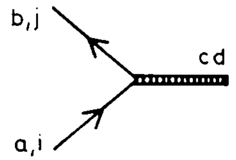
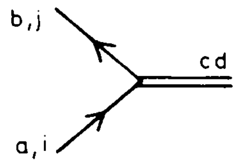
| | Vertex | Vertex factor |
|---|---|---|
| $\bar\psi zc$ vertex: |  | $\dfrac{i}{\sqrt{n}}\delta^{ad}\delta^{bc}\delta_{ij}$ |
| $\bar z\psi\bar c$ vertex: |  | $\dfrac{i}{\sqrt{n}}\delta^{ad}\delta^{bc}\delta_{ij}$ |
| $\bar\psi\psi\lambda$ vertex: |  | $\dfrac{1}{\sqrt{n}}\delta^{ad}\delta^{bc}\delta_{ij}\gamma_\mu$ |

TABLE 4 continued

| | | |
|---|---|---|
| $\bar{\psi}\psi\varphi$ vertex: |   | $-\frac{i}{\sqrt{n}}\delta^{ad}\delta^{bc}\delta_{ij}$ |
| $\bar{\psi}\psi\varphi_5$ vertex: |   | $\frac{1}{\sqrt{n}}\delta^{ad}\delta^{bc}\delta_{ij}\gamma_5$ |

As in the pure model, renormalization proceeds according to the BPHZ program [19, 20], starting out from the formula for the superficial degree of divergence, $\delta(\Gamma)$, of a given proper (sub)diagram $\Gamma$:

$$\delta(\Gamma)=2-2E_\alpha-E_\lambda-E_\zeta-E_\psi \tag{3.53$_\mathrm{M}$}$$

$$\begin{gathered}\delta(\Gamma)=2-2E_\alpha-E_\lambda-E_\zeta-E_{(\psi c)}-E_\varphi-E_{\varphi_5}-E_{\lambda\varphi_5}-E_{\varphi_5\lambda}-I_{\text{mixed}}\\ (E_{(\psi c)}=E_\psi+E_{\bar{c}}=E_{\bar{\psi}}+E_c)\end{gathered} \tag{3.53$_\mathrm{S}$}$$

($E_\mathrm{A}$ denotes the number of external lines of type A, and in the supersymmetric case, $I_{\text{mixed}}$ is the number of internal mixed ($\lambda\varphi_5$ or $\varphi_5\lambda$) lines. Note that in the minimal case, the $\bar{z}/z$ lines and $\bar{\psi}/\psi$ lines can only form uninterrupted strings or internal loops, so that $E_{\bar{z}}=E_z$ and $E_{\bar{\psi}}=E_\psi$, while in the supersymmetric case, the presence of $\bar{c}/c$ lines, $\bar{\psi}zc$ vertices and $\bar{z}\psi\bar{c}$ vertices complicates the situation; here, $E_{\bar{z}}-E_z=E_{\bar{c}}-E_c=-(E_{\bar{\psi}}-E_\psi)$.) Again, renormalizability follows after taking into account cancellations between different diagrams. In the minimal case, these are the same as in the pure model, while in the supersymmetric case, additional cancellations occur between diagrams containing pairs which are made from an external $\bar{\psi}$ line and an external $z$ line or from an external $\bar{z}$ line and an external $\psi$ line (both carrying the same flavor index): here, the attaching procedure described before must be carried out using an internal $c$ line or $\bar{c}$ line, respectively, rather than an $\alpha$ line. (See fig. 2.)

The additional cancellations are then due to the identity

$$-D^{(\bar{c}c)}(p)\int\frac{\mathrm{d}^2k}{4\pi^2}\frac{(\not{k}+\frac{1}{2}\not{p}+im)}{[(k+\frac{1}{2}p)^2+m^2][(k-\frac{1}{2}p)^2+m^2]}=1\,. \tag{3.54$_\mathrm{S}$}$$

The rest of the renormalization program proceeds as in the pure model (with standard techniques for taking care of the fermions). For composite operators, the minimal model does not present any new features, while in the supersymmetric model, one

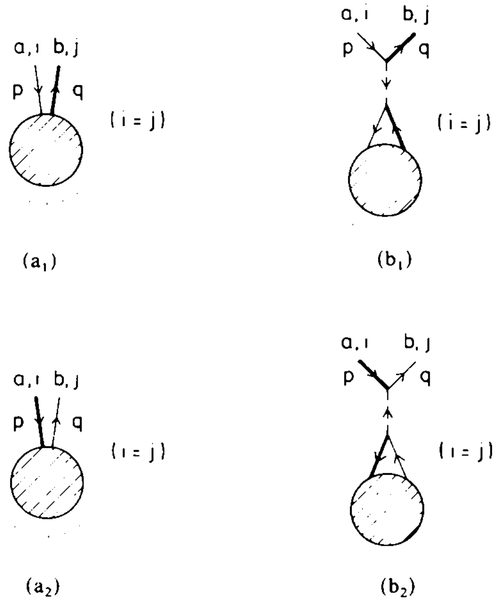



Fig. 2.

uses ($3.54_S$) to derive additional constraints on normal products, the most important of these being the quantum counterpart of the classical constraint ($2.14_S$):

$$\mathcal{N}[\bar{z}_i^a \psi_i^b] = 0 = \mathcal{N}[\bar{\psi}_i^a z_i^b]\,. \tag{$3.55_S$}$$

## 4. Calculation of the quantum non-local charge

Returning from euclidean space to Minkowski space, we shall define the quantum counterpart of the classical non-local charge as the limit

$$Q(t) = \lim_{\delta\to 0} Q^\delta(t) \tag{4.1}$$

of a cutoff charge, which reads

$$Q_{ij}^\delta(t) = \frac{1}{n}\left\{\int_{|y_1-y_2|\geq\delta} \mathrm{d}y_1\,\mathrm{d}y_2\,\varepsilon(y_1-y_2)(J_0)_{ik}(t,y_1)(J_0)_{kj}(t,y_2) - Z(\delta)\int \mathrm{d}y(J_1)_{ij}(t,y)\right\} \tag{4.2}$$

in the pure model (compare (2.13)), and

$$Q_{ij}^{\delta}(t)=\frac{1}{n}\left\{\int_{|y_1-y_2|\geq\delta} \mathrm{d}y_1\,\mathrm{d}y_2\,\varepsilon(y_1-y_2)(J_0)_{ik}(t,y_1)(J_0)_{kj}(t,y_2)\right.$$
$$\left.-Z(\delta)\int \mathrm{d}y(J_1)_{ij}(t,y)-Y'(\delta)\int \mathrm{d}y(i_1')_{ij}(t,y)-Y(\delta)\int \mathrm{d}y(i_1)_{ij}(t,y)\right\} \tag{4.3}$$

in the fermionic models (compare (2.29)). To explain our notation, we recall first that we have the constraint

$$\mathcal{N}[\bar{z}_i^a z_i^b]=c_0\delta^{ab}, \tag{4.4}$$

with $c_0$ a constant ($c_0=0$ in our normalization), while $A_\mu$ will stand for the new quantum gauge potential of sect. 3, i.e.

$$A_\mu=-\frac{i}{\sqrt{n}}\lambda_\mu\,. \tag{4.5}$$

Moreover, $j_\mu$ and $J_\mu$ continue to denote the purely bosonic current and the total conserved Noether current, respectively, and we have split the matter field contribution $j_\mu^{\mathrm{M}}=J_\mu-j_\mu$ to the latter into two pieces:

$$(i_\mu')_{ij}=-i\mathcal{N}[z_i^a\bar{\psi}_k^a\gamma_\mu\psi_k^b\bar{z}_j^b]\,,\qquad (i_\mu)_{ij}=0\,, \tag{4.6$_\mathrm{M}$}$$

$$(i_\mu')_{ij}=-i\mathcal{N}[z_i^a\bar{\psi}_k^a\gamma_\mu\psi_k^b\bar{z}_j^b]\,,\qquad (i_\mu)_{ij}=i\mathcal{N}[\bar{\psi}_j^a\gamma_\mu\psi_i^a]\,. \tag{4.6$_\mathrm{S}$}$$

(For the group-theoretical interpretation of this decomposition, we refer to [7].) Thus in the pure model

$$(J_\mu)_{ij}=(j_\mu)_{ij}=\mathcal{N}[z_i^a\overleftrightarrow{D_\mu}\bar{z}_j^a]\,, \tag{4.7}$$

while in the fermionic models

$$(j_\mu+i_\mu')_{ij}=\mathcal{N}[z_i^a\overleftrightarrow{D_\mu}\bar{z}_j^a]\,, \tag{4.8}$$

and

$$(J_\mu)_{ij}=\mathcal{N}[z_i^a\overleftrightarrow{D_\mu}\bar{z}_j^a]\,, \tag{4.9$_\mathrm{M}$}$$

$$(J_\mu)_{ij}=\mathcal{N}[z_i^a\overleftrightarrow{D_\mu}\bar{z}_j^a]+i\mathcal{N}[\bar{\psi}_j^a\gamma_\mu\psi_i^a]\,, \tag{4.9$_\mathrm{S}$}$$

where in all cases

$$\mathcal{N}[z_i^a\overleftrightarrow{D_\mu}\bar{z}_j^a]=\mathcal{N}[z_i^a\overleftrightarrow{\partial_\mu}\bar{z}_j^a+2z_i^aA_\mu^{ab}\bar{z}_j^b]\,. \tag{4.10}$$

In order to show that the coefficients $Z(\delta)$ in (4.2) and $Z(\delta)$, $Y'(\delta)$, $Y(\delta)$ in (4.3) can be chosen in such a way that the limit $Q$ in (4.1) exists, and in order to calculate its time derivative $\mathrm{d}Q/\mathrm{d}t$, we need a Wilson expansion for the (matrix) commutator $[J_\mu(x+\varepsilon), J_\nu(x)]$ of two currents at nearby (spacelike separated) points $x+\varepsilon$, $x(\varepsilon^2<0)$.

In the pure Grassmann model, this short-distance expansion is known [13] to take the form*

$$[J_\mu(x+\varepsilon), J_\nu(x)] = C^\rho_{\mu\nu}(\varepsilon)J_\rho(x) + D^{\sigma\rho}_{\mu\nu}(\varepsilon)(\partial_\sigma J_\rho)(x) + E^{(0)\sigma\rho}_{\mu\nu}(\varepsilon)\mathcal{N}[zF^{(0)}_{\sigma\rho}z^+](x) + E^{\sigma\rho}_{\mu\nu}(\varepsilon)\mathcal{N}[zF_{\sigma\rho}z^+](x)\,. \tag{4.11}$$

The coefficients in this Wilson expansion can be determined perturbatively [11] or, more elegantly, from structural properties of quantum field theory**, plus an appropriate normalization condition on the current, just as for the models on spheres [9] or complex projective spaces [12]. The result is

$$C^\rho_{\mu\nu}(\varepsilon) = \frac{n}{2\pi}\left[g_{\mu\nu}\frac{\varepsilon^\rho}{\varepsilon^2} - \frac{\delta^\rho_\mu\varepsilon_\nu + \delta^\rho_\nu\varepsilon_\mu}{\varepsilon^2} - \frac{2\varepsilon_\mu\varepsilon_\nu\varepsilon^\rho}{(\varepsilon^2)^2}\right], \tag{4.12}$$

$$D^{\sigma\rho}_{\mu\nu}(\varepsilon) = \frac{n}{2\pi}\left[g_{\mu\nu}\frac{\varepsilon^\sigma\varepsilon^\rho}{2\varepsilon^2} - \frac{\delta^\rho_\mu\varepsilon_\nu\varepsilon^\sigma}{2\varepsilon^2} - \frac{\delta^\rho_\nu\varepsilon_\mu\varepsilon^\sigma}{2\varepsilon^2} - \frac{\delta^\sigma_\mu\varepsilon_\nu\varepsilon^\rho}{2\varepsilon^2} + \frac{\delta^\sigma_\nu\varepsilon_\mu\varepsilon^\rho}{2\varepsilon^2} - (\tfrac{1}{2}\gamma + \tfrac{1}{4}\ln(-\tfrac{1}{4}m^2\varepsilon^2))\varepsilon_{\mu\nu}\varepsilon^{\sigma\rho} - \frac{\varepsilon_\mu\varepsilon_\nu\varepsilon^\sigma\varepsilon^\rho}{(\varepsilon^2)^2}\right], \tag{4.13}$$

$$E^{(0)\sigma\rho}_{\mu\nu}(\varepsilon) = E^{(0)}\varepsilon_{\mu\nu}\varepsilon^{\sigma\rho}, \qquad E^{\sigma\rho}_{\mu\nu}(\varepsilon) = E\varepsilon_{\mu\nu}\varepsilon^{\sigma\rho}, \tag{4.14}$$

where $\gamma = 0.5777\ldots$ is Euler's constant, and $E^{(0)}$, $E$ are two as yet undetermined constants. The explicit determination of these coefficients requires a perturbative analysis which can be carried out along the same lines as in the $\mathbb{C}\mathrm{P}^{n-1}$ model [12]. Briefly, one uses (4.11) (at $x=0$, say) to derive the relation

$$\begin{aligned}
&\frac{\partial}{\partial k_\lambda}\langle 0|\mathrm{T}([J_\mu(\varepsilon), J_\nu(0)] - [J_\nu(\varepsilon), J_\mu(0)])_{ij}\tilde{z}^a_l(q)\tilde{\bar{z}}^b_m(r)\tilde{A}^{cd}_\kappa(k)|0\rangle^{\mathrm{prop}}|_{q=r=k=0} \\
&= (D^{\sigma\rho}_{\mu\nu}(\varepsilon) - D^{\sigma\rho}_{\nu\mu}(\varepsilon))\frac{\partial}{\partial k_\lambda}\langle 0|\mathrm{T}(\partial_\sigma J_\rho)_{ij}(0)\tilde{z}^a_l(q)\tilde{\bar{z}}^b_m(r)\tilde{A}^{cd}_\kappa(k)|0\rangle^{\mathrm{prop}}|_{q=r=k=0} \\
&\quad - 4E^{(0)}\frac{\partial}{\partial k_\lambda}\langle 0|\mathrm{T}\mathcal{N}[(zF^{(0)}_{\mu\nu}z^+)_{ij}(0)\tilde{z}^a_l(q)\tilde{\bar{z}}^b_m(r)\tilde{A}^{cd}_\kappa(k)|0\rangle^{\mathrm{prop}}|_{q=r=k=0} \\
&\quad - 4E\frac{\partial}{\partial k_\lambda}\langle 0|\mathrm{T}\mathcal{N}[(zF_{\mu\nu}z^+)_{ij}(0)\tilde{z}^a_l(q)\tilde{\bar{z}}^b_m(r)\tilde{A}^{cd}_\kappa(k)|0\rangle^{\mathrm{prop}}|_{q=r=k=0}\,.
\end{aligned} \tag{4.15}$$

* The index $^{(0)}$ refers to contributions lying in the center u(1) of the Lie algebra u($p$), i.e. $A^{(0)}_\mu = (1/p)(\mathrm{tr}\,A_\mu)1_p$, $F^{(0)}_{\mu\nu} = (1/p)(\mathrm{tr}\,F_{\mu\nu})1_p$, etc.

** These structural properties include covariance (under Lorentz transformations, translations, parity and time reversal, internal symmetries and charge conjugation), locality, gauge invariance and current conservation; they should therefore be valid beyond perturbation theory.

(Note that the antisymmetrization in $\mu$ and $\nu$ has eliminated the $C$-terms from (4.15) due to $C^{\rho}_{\mu\nu} = C^{\rho}_{\nu\mu}$.) Now both sides of (4.15) are evaluated graphically, which is possible because there are really just a few graphs that must be computed explicitly. For example, the only contributions to any of the three terms appearing on the r.h.s. come from tree graphs. (All other graphs that might a priori contribute contain at least one loop and must be renormalized via subtractions which, as we recall, are performed around zero external momenta, so that nothing is left when we evaluate at $q = r = k = 0$.) Similarly, the only contributions to the l.h.s. come from one-loop graphs; see [12] for more details. The result of the entire calculation is

$$E^{(0)} = 0\,, \qquad E = -\frac{n}{2\pi}\,. \tag{4.16}$$

We note finally that in (4.11) we have omitted the normal product $\mathcal{N}[J_\mu, J_\nu]$ because our normalization condition forces it to vanish: indeed, going through the derivation of (2.11) once again, we find that the constraint (4.4) implies

$$\mathcal{N}[J_\mu, J_\nu] = -\tfrac{1}{2}c_0(\partial_\mu J_\nu - \partial_\nu J_\mu)\,. \tag{4.17}$$

(This also shows that if one wanted to use a normalization where $c_0 \neq 0$, one could absorb the term $\mathcal{N}[J_\mu, J_\nu]$ into a redefinition of the coefficient $D^{\sigma\rho}_{\mu\nu}$.)

Putting all this information together, we conclude that the non-local charge $Q$ is well defined if we choose

$$Z(\delta) = \frac{n}{2\pi}(\gamma - 1 + \ln(\tfrac{1}{2}m\delta))\,, \tag{4.18}$$

but that it is not conserved; rather,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{2}{\pi}\int \mathrm{d}y\, \mathcal{N}[zF_{01}z^+](t, y)\,, \tag{4.19}$$

a result valid to all orders in $1/n$.

To handle the Grassmann models with fermions, we follow the same basic strategy as in the pure case. First of all, the relevant short-distance expansion becomes

$$\begin{aligned}[J_\mu(x+\varepsilon), J_\nu(x)] = {} & C^{\rho}_{\mu\nu}(\varepsilon)J_\rho(x) + D^{\sigma\rho}_{\mu\nu}(\varepsilon)(\partial_\sigma J_\rho)(x) \\ & + \hat{C}'^{\rho}_{\mu\nu}(\varepsilon)i'_\rho(x) + \hat{D}'^{\sigma\rho}_{\mu\nu}(\varepsilon)(\partial_\sigma i'_\rho)(x) \\ & + \hat{C}^{\rho}_{\mu\nu}(\varepsilon)i_\rho(x) + \hat{D}^{\sigma\rho}_{\mu\nu}(\varepsilon)(\partial_\sigma i_\rho)(x) \\ & + E^{(0)}{}^{\sigma\rho}_{\mu\nu}(\varepsilon)\mathcal{N}[zF^{(0)}_{\sigma\rho}z^+](x) \\ & + E^{\sigma\rho}_{\mu\nu}(\varepsilon)\mathcal{N}[zF_{\sigma\rho}z^+](x) + \mathcal{N}[J_\mu, J_\nu](x)\,. \end{aligned} \tag{4.20}$$

Of course, there is a priori a large number of other operators that may appear on the r.h.s. of (4.20), such as the (matrix) commutators $\mathcal{N}[j_\rho, i'_\sigma]$, $\mathcal{N}[j_\rho, i_\sigma]$, $\mathcal{N}[i'_\rho, i'_\sigma]$ and $\mathcal{N}[i_\rho, i_\sigma]$, for example. To deal with this problem, one has to set up a complete list of such operators and analyze the behavior of the corresponding coefficients. Then combining structural properties of quantum field theory, plus an appropriate normalization condition on the current, plus Fierz identities, with general graphical arguments, one can show that all coefficients pertaining to operators not written in (4.20) vanish, that the coefficients $C^{\rho}_{\mu\nu}$, $D^{\sigma\rho}_{\mu\nu}$, $E^{(0)\sigma\rho}_{\mu\nu}$ and $E^{\sigma\rho}_{\mu\nu}$ are the same as in the pure model (cf. (4.12)-(4.14)), and that

$$\hat{C}'^{\rho}_{\mu\nu} = 0\,, \tag{4.21}$$

$$\hat{D}'^{\sigma\rho}_{\mu\nu} = 0\,, \tag{4.22}$$

$$\hat{C}^{\rho}_{\mu\nu}(\varepsilon) = -\frac{n}{\pi}\left[g_{\mu\nu}\frac{\varepsilon^{\rho}}{\varepsilon^2} - \frac{\delta^{\rho}_{\mu}\varepsilon_{\nu} + \delta^{\rho}_{\nu}\varepsilon_{\mu}}{\varepsilon^2}\right], \tag{4.23}$$

$$\begin{aligned}\hat{D}^{\sigma\rho}_{\mu\nu}(\varepsilon) = -\frac{n}{\pi}\Bigg[ & g_{\mu\nu}\frac{\varepsilon^{\sigma}\varepsilon^{\rho}}{2\varepsilon^2} + g^{\sigma\rho}\frac{\varepsilon_{\mu}\varepsilon_{\nu}}{2\varepsilon^2} - \frac{\delta^{\rho}_{\mu}\varepsilon_{\nu}\varepsilon^{\sigma}}{2\varepsilon^2} - \frac{\delta^{\rho}_{\nu}\varepsilon_{\mu}\varepsilon^{\sigma}}{2\varepsilon^2} \\ & -\left(\tfrac{1}{2}\gamma - \tfrac{1}{2} + \tfrac{1}{4}\ln\left(-\tfrac{1}{4}m^2\varepsilon^2\right)\right)\varepsilon_{\mu\nu}\varepsilon^{\sigma\rho} - \tfrac{1}{4}g_{\mu\nu}g^{\sigma\rho}\Bigg].\end{aligned} \tag{4.24}$$

This is a long and tedious analysis which has been carried out explicitly for the supersymmetric $\mathbb{C}P^{n-1}$ model [12] and can be extended to the supersymmetric $G_{\mathbb{C}}(p, n-p)$ model since the presence of color indices does not cause any problems; we shall not discuss this problem here. (Note that the corresponding investigation for the minimal model can be trivially reduced to that for the supersymmetric model.)

The essential difference between the fermionic models and the pure model, however, lies in the fact that the normal product $\mathcal{N}[J_{\mu}, J_{\nu}]$ is no longer zero: indeed, going through the derivation of (2.27) once again, but without using the fermionic equation of motion (2.22), and writing ... for all terms made up of operators which, according to the aforementioned analysis, do not contribute to the Wilson expansion (4.20), we find that the constraint (4.4) implies

$$\begin{aligned}\mathcal{N}[J_{\mu}, J_{\nu}] = & -\tfrac{1}{2}c_0(\partial_{\mu}J_{\nu} - \partial_{\nu}J_{\mu}) - \tfrac{1}{2}(\partial_{\mu}i'_{\nu} - \partial_{\nu}i'_{\mu}) + \tfrac{1}{2}c_0(\partial_{\mu}i_{\nu} - \partial_{\nu}i_{\mu}) \\ & - 2\mathcal{N}[z(D_{\mu}B_{\nu} - D_{\nu}B_{\mu})z^{+}] + \cdots\,,\end{aligned} \tag{4.25}$$

where

$$B^{ab}_{\mu} = -\tfrac{1}{2}i\mathcal{N}[\bar{\psi}^{a}_{i}\gamma_{\mu}\psi^{b}_{i}] \tag{4.26}$$

(compare (2.32)), and of course,

$$D_{\mu}B_{\nu} = \partial_{\mu}B_{\nu} + \mathcal{N}[A_{\mu}, B_{\nu}]\,. \tag{4.27}$$

Now due to the axial anomaly [24], the fourth term on the r.h.s. of (4.25) produces - possibly apart from ... terms which are due to the fermionic equation of motion (2.21) - a term proportional to $\mathcal{N}[zF_{\mu\nu}z^{+}]$, with a coefficient which can be computed graphically. Once again, the only contribution to this coefficient comes from one-loop graphs - as it must be according to the Adler-Bardeen theorem [24] - and turns out to be $-n/\pi$. On the other hand, the perturbative calculation of the coefficients $E^{(0)}$ and $E$ appearing in (4.14) suffers no modifications, and therefore (4.16) remains intact. As a result, the last two terms on the r.h.s. of (4.20) cancel, to all orders in $1/n$, except for a constant multiple of the curl of $i'_\mu$. More explicitly, the Wilson expansion (4.20) simplifies as follows:

$$[J_\mu(x+\varepsilon), J_\nu(x)] = C^\rho_{\mu\nu}(\varepsilon) J_\rho(x) + D^{\sigma\rho}_{\mu\nu}(\varepsilon)(\partial_\sigma J_\rho)(x) + \tfrac{1}{2}\varepsilon_{\mu\nu}\varepsilon^{\sigma\rho}(\partial_\sigma i'_\rho)(x) + \hat{C}^\rho_{\mu\nu}(\varepsilon) i_\rho(x) + \hat{D}^{\sigma\rho}_{\mu\nu}(\varepsilon)(\partial_\sigma i_\rho)(x) . \tag{4.28}$$

The coefficients here are given by (4.12), (4.13), (4.23), (4.24).

This information is now sufficient to conclude that the non-local charge $Q$ is well defined if we choose $Z(\delta)$ as before (cf. (4.18)) and set

$$Y(\delta) = -\frac{n}{\pi}(\gamma - 1 + \ln(\tfrac{1}{2}m\delta)) , \qquad Y'(\delta) = -1 . \tag{4.29}$$

Moreover, we may check that $Q$ is now conserved:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0 , \tag{4.30}$$

a result again valid to all orders in $1/n$.

## 5. On the determination of the exact $S$-matrix

As a first step towards computing the $S$-matrix, we have to get an idea about the particle content of the models under consideration - at least on the level of "fundamental particles" (as opposed to bound states).

In the pure model, the gauge field propagator has a pole at $p^2 = 0$, and hence the gauge field generates a long-range force between the partons, leading to confinement. The fundamental fields $z_i^a$, $\bar{z}_i^a$ therefore do not correspond to particle states, but gauge-invariant composite fields such as $z_i^a \bar{z}_j^a$ should. There is of course no reason to expect the corresponding meson bound state $S$-matrix to factorize, and calculating this $S$-matrix remains an important open problem, even in the $\mathbb{C}P^{n-1}$-case.

When fermions are coupled to the model in a minimal or supersymmetric way, the gauge field propagator loses its pole at $p^2 = 0$, and the long-range force disappears. The partons are therefore deconfined, and the fermions stay unconfined. This however does not exclude the phenomenon of screening, which eliminates certain fermionic quantum numbers, such as charge and chirality, from the physical spectrum, i.e. these quantum numbers do not appear in the physical states. In fact, this spurionization occurs in the massless Schwinger model [25] and in the chiral Gross-Neveu model minimally coupled to a U(1) gauge theory [26], which are obtained as low-energy effective theories from the minimal and the supersymmetric

$\mathbb{C}P^{n-1}$ model, respectively. As far as the fermionic degrees of freedom are concerned, we therefore expect that in the minimal model, they are completely spurious, i.e., physically, they do not show up at all as asymptotic states, while in the supersymmetric model, they do appear asymptotically, but lead to constraints of the type that antiparticles can be identified with bound states of particles. This also applies when the projective spaces $\mathbb{C}P^{n-1}$ are replaced by the Grassmann manifolds $G_{\mathbb{C}}(p, n-p)$ because the spurionization mechanism seems to survive the transition from abelian (U(1)) to non-abelian (U($p$)) gauge fields [27, 28]. However, the details of this picture certainly deserve further investigation.

In view of the preceding discussion, we introduce asymptotic states described by symbols $b_i^a(\theta)$ ($\bar{b}_i^a(\theta)$) for the bosonic (antibosonic) degrees of freedom. In the supersymmetric model, there are additional asymptotic states described by symbols $f_i^a(\theta)$ ($\bar{f}_i^a(\theta)$) for the fermionic (antifermionic) degrees of freedom. Here, $\theta$ is the rapidity, related to the on-shell momentum $p$ by

$$p = m(\cosh\theta, \sinh\theta)\,. \tag{5.1}$$

The whole $S$-matrix should exhibit a $(\mathrm{U}(n)\times\mathrm{U}(p))$ symmetry. In particular, the corresponding symmetry for the 2-body $S$-matrix leads us to introducing the notation

$$\begin{aligned}{}^{ab}_{ij}A^{cd}_{kl}(\theta) &= (a_1(\theta)\delta^{ac}\delta^{bd} + a_2(\theta)\delta^{ad}\delta^{bc})\delta_{ik}\delta_{jl}\\ &\quad + (a_3(\theta)\delta^{ac}\delta^{bd} + a_4(\theta)\delta^{ad}\delta^{bc})\delta_{il}\delta_{jk}\,,\end{aligned} \tag{5.2}$$

and to adopt the convention that (5.2) remains in force when $A$ is replaced by an arbitrary capital letter and $a$ (in $a_1, \ldots, a_4$) is replaced by the corresponding lower case letter. Then with the abbreviation $\theta = \theta_1 - \theta_2$, we can write the bosonic 2-body scattering amplitudes in the form

$$\begin{aligned}&{}^{\mathrm{out}}\langle b_k^c(\tilde\theta_1)b_l^d(\tilde\theta_2)|b_i^a(\theta_1)b_j^b(\theta_2)\rangle^{\mathrm{in}}\\ &\quad = \delta(\tilde\theta_1-\theta_1)\delta(\tilde\theta_2-\theta_2)\,{}^{ab}_{ij}U^{cd}_{kl}(\theta) + \delta(\tilde\theta_1-\theta_2)\delta(\tilde\theta_2-\theta_1)\,{}^{ab}_{ij}U^{dc}_{lk}(\theta)\,,\end{aligned} \tag{5.3}$$

$$\begin{aligned}&{}^{\mathrm{out}}\langle b_k^c(\tilde\theta_1)\bar{b}_l^d(\tilde\theta_2)|b_i^a(\theta_1)\bar{b}_j^b(\theta_2)\rangle^{\mathrm{in}}\\ &\quad = \delta(\tilde\theta_1-\theta_1)\delta(\tilde\theta_2-\theta_2)\,{}^{ad}_{il}U^{cb}_{kj}(i\pi-\theta) + \delta(\tilde\theta_1-\theta_2)\delta(\tilde\theta_2-\theta_1)\,{}^{ad}_{il}R^{cb}_{kj}(\theta)\,,\end{aligned} \tag{5.4}$$

plus an amplitude obtained from (5.3) by substituting $\bar{b}$ for $b$. In the supersymmetric model, there are additional 2-body scattering amplitudes, both fermionic and mixed, which take the form

$$\begin{aligned}&{}^{\mathrm{out}}\langle f_k^c(\tilde\theta_1)f_l^d(\tilde\theta_2)|f_i^a(\theta_1)f_j^b(\theta_2)\rangle^{\mathrm{in}}\\ &\quad = \delta(\tilde\theta_1-\theta_1)\delta(\tilde\theta_2-\theta_2)\,{}^{ab}_{ij}V^{cd}_{kl}(\theta) - \delta(\tilde\theta_1-\theta_2)\delta(\tilde\theta_2-\theta_1)\,{}^{ab}_{ij}V^{dc}_{lk}(\theta)\,,\end{aligned} \tag{5.5}$$

$$\begin{aligned}&{}^{\mathrm{out}}\langle f_k^c(\tilde\theta_1)\bar{f}_l^d(\tilde\theta_2)|f_i^a(\theta_1)\bar{f}_j^b(\theta_2)\rangle^{\mathrm{in}}\\ &\quad = \delta(\tilde\theta_1-\theta_1)\delta(\tilde\theta_2-\theta_2)\,{}^{ad}_{il}V^{cb}_{kj}(i\pi-\theta) - \delta(\tilde\theta_1-\theta_2)\delta(\tilde\theta_2-\theta_1)\,{}^{ad}_{il}S^{cb}_{kj}(\theta)\,,\end{aligned} \tag{5.6}$$

plus an amplitude obtained from (5.5) by substituting $\bar{f}$ for $f$, and

$$
{}^{\text{out}}\langle b_k^c(\tilde\theta_1) f_l^d(\tilde\theta_2) | b_i^a(\theta_1) f_j^b(\theta_2)\rangle^{\text{in}}
$$
$$
= \delta(\tilde\theta_1-\theta_1)\delta(\tilde\theta_2-\theta_2){}_{ij}^{ab}C_{kl}^{cd}(\theta) - \delta(\tilde\theta_1-\theta_2)\delta(\tilde\theta_2-\theta_1){}_{ij}^{ab}D_{kl}^{cd}(\theta)\,, \tag{5.7}
$$
$$
{}^{\text{out}}\langle b_k^c(\tilde\theta_1) \bar f_l^d(\tilde\theta_2) | b_i^a(\theta_1) \bar f_j^b(\theta_2)\rangle^{\text{in}}
$$
$$
= \delta(\tilde\theta_1-\theta_1)\delta(\tilde\theta_2-\theta_2){}_{il}^{ad}C_{kj}^{cb}(i\pi-\theta) - \delta(\tilde\theta_1-\theta_2)\delta(\tilde\theta_2-\theta_1){}_{il}^{ad}E_{kj}^{cb}(\theta)\,, \tag{5.8}
$$
$$
{}^{\text{out}}\langle b_k^c(\tilde\theta_1) \bar b_l^d(\tilde\theta_2) | f_i^a(\theta_1) \bar f_j^b(\theta_2)\rangle^{\text{in}}
$$
$$
= \delta(\tilde\theta_1-\theta_1)\delta(\tilde\theta_2-\theta_2){}_{il}^{ad}D_{kj}^{cb}(i\pi-\theta) - \delta(\tilde\theta_1-\theta_2)\delta(\tilde\theta_2-\theta_1){}_{il}^{ad}F_{kj}^{cb}(\theta)\,, \tag{5.9}
$$

plus two amplitudes obtained from (5.7) and (5.8) by exchanging particles with antiparticles. (Note that in (5.4), (5.6), (5.8), (5.9) we have already used crossing symmetry to eliminate the transmission amplitudes in favor of the particle-particle amplitudes $U$, $V$, $C$, $D$.)

Next, we make use of the fact that due to conservation of the quantum non-local charge $Q$, the $S$-matrix factorizes. This is proved just as in the case of spheres (pure model) and of complex projective spaces (minimal and supersymmetric model), namely by evaluating the action of $Q$ on the asymptotic states [9, 29]. The resulting factorization equations also lead to severe restrictions on the 2-body scattering amplitudes: in particular, they imply that all reflection amplitudes $R$, $S$, $E$, $F$ vanish, and they impose certain relations between the particle-particle amplitudes. For the bosonic amplitudes, these relations read

$$
u_3(\theta) = -\frac{2\pi i}{n\theta}u_1(\theta)\,, \qquad u_4(\theta) = -\frac{2\pi i}{n\theta}u_2(\theta)\,. \tag{5.10}
$$

In the supersymmetric model, there are additional relations between the particle-particle amplitudes, which read

$$
v_3(\theta) = +\frac{2\pi i}{n\theta}v_1(\theta)\,, \qquad v_4(\theta) = +\frac{2\pi i}{n\theta}v_2(\theta)\,, \tag{5.11}
$$

and

$$
c_3(\theta) = -\frac{2\pi i}{n\theta}c_1(\theta)\,, \qquad c_4(\theta) = -\frac{2\pi i}{n\theta}c_2(\theta)\,, \tag{5.12}
$$

$$
d_3(\theta) = -\frac{2\pi i}{n\theta}d_1(\theta)\,, \qquad d_4(\theta) = -\frac{2\pi i}{n\theta}d_2(\theta)\,. \tag{5.13}
$$

This reduces the determination of the 2-body $S$-matrix for the minimal model and for the supersymmetric model to the determination of the two functions $u_1(\theta)$, $u_2(\theta)$ and the eight functions $u_1(\theta)$, $u_2(\theta)$, $v_1(\theta)$, $v_2(\theta)$, $c_1(\theta)$, $c_2(\theta)$, $d_1(\theta)$, $d_2(\theta)$, respectively, which are further constrained by unitarity and analyticity.

In the $\mathbb{C}P^{n-1}$ case, the number of functions involved is automatically reduced by a factor of two, simply because in that case, only the sums $a_1+a_2$ and not the individual coefficients $a_1$, $a_2$ appear (cf. (5.2)). On these grounds, an explicit solution can then be found [29]. The Grassmann case, however, is more complicated. We have tried an ansatz of the form

$$a_2(\theta) = \pm \frac{2\pi i}{p\theta} a_1(\theta)\,, \tag{5.14}$$

which does lead to a completely explicit solution; however, we do not believe that it is correct because the ansatz (5.14) turns out to be incompatible with a lowest-order $1/n$ calculation (where $a_2$ is of order $1/n$ while $a_1$ is of order 1). The complete solution, therefore, remains to be found.

## Appendix

### DIRAC MATRICES, CHIRAL SYMMETRY AND FIERZ IDENTITIES IN TWO DIMENSIONS

Throughout this paper, we work either in flat Minkowski space or in flat euclidean space, with metric tensor $g_{\mu\nu}$ and determinant tensor $\varepsilon_{\mu\nu}$ given by $g_{00} = +1$, $g_{11} = -1$, $\varepsilon_{01} = -1$, $\varepsilon_{10} = +1$ or by $g_{\mu\nu} = \delta_{\mu\nu}$, $\varepsilon_{12} = +1$, $\varepsilon_{21} = -1$, respectively. In both cases, the description of Dirac spinors involves one and the same complex two-dimensional vector space $S$, equipped with the positive definite inner product

$$\begin{aligned} &S \times S \to \mathbb{C}\,, \\ &(u, v) \mapsto u^+ v\,, \end{aligned} \tag{A.1}$$

and with a conjugation*

$$\begin{aligned} &S \to S\,, \\ &u \mapsto u^*\,, \end{aligned} \tag{A.2}$$

which should be antiunitary:

$$(u^*)^+ v^* = v^+ u \quad \text{for} \quad u, v \in S\,. \tag{A.3}$$

Usually, $S$ is identified with $\mathbb{C}^2$ by choosing a fixed basis in $S$ which is simultaneously orthonormal and real (such a basis exists due to (A.3)). Then the standard spin representation of the Clifford algebra associated with the given metric is determined by some set of Dirac matrices which satisfy the standard anticommutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}\,, \tag{A.4}$$

and which may, and will, be assumed unitary:

$$\gamma_\mu^+ = \gamma_\mu^{-1}\,. \tag{A.5}$$

* A conjugation on a complex vector space V is an antilinear transformation on V which is involutive, i.e. its own inverse.

Our definition of $\gamma_5$, motivated by the requirement that $\gamma_5^2 = 1$ and $\gamma_5^+ = \gamma_5$ always, is**

$$\gamma_5 = \gamma_0\gamma_1 \,, \tag{A.6$_\mathrm{M}$}$$

$$\gamma_5 = -i\gamma_1\gamma_2 \,. \tag{A.6$_\mathrm{E}$}$$

Thus (A.4) gives

$$\gamma_\mu\gamma_5 = \varepsilon_{\mu\nu}\gamma^\nu \,, \tag{A.7$_\mathrm{M}$}$$

$$\gamma_\mu\gamma_5 = -i\varepsilon_{\mu\nu}\gamma^\nu \,. \tag{A.7$_\mathrm{E}$}$$

This typically two-dimensional relation is responsible for many of the peculiar features of fermions in two-dimensional space-time; for example, it plays a central rôle in the derivation of non-local conservation laws for non-linear $\sigma$-models with fermions (see sect. 2 and [7]).

The positive definite inner product (A.1) and the conjugation (A.2) play only an auxiliary rôle - in contrast to the invariant inner product

$$\begin{aligned} S \times S &\to \mathbb{C} \,, \\ (u, v) &\mapsto \bar{u}v \,, \end{aligned} \tag{A.8}$$

and to charge conjugation

$$\begin{aligned} S &\to S \,, \\ u &\mapsto u^c \,. \end{aligned} \tag{A.9}$$

Writing

$$\bar{u}v = u^+\rho v \quad \text{for} \quad u, v \in S \,, \tag{A.10}$$

$$u^c = C^{-1}u^* \quad \text{for} \quad u \in S \,, \tag{A.11}$$

the latter are distinguished by demanding that

$$\rho^+ = \rho \,, \qquad \gamma_\mu^+ = \rho\gamma_\mu\rho^{-1} \,, \tag{A.12}$$

$$C^{-1}C^* = 1 \,, \qquad \gamma_\mu^* = \varepsilon C\gamma_\mu C^{-1} \,, \tag{A.13}$$

where $\varepsilon$ is a sign which, in two dimensions, turns out to be uniquely determined by the signature of the given metric. (In higher dimensions, the situation is more complicated [30].)

Explicit representations for the Dirac matrices in two dimensions can be given in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.14}$$

** In this appendix, the indices M and E in the formulae stand for "minkowskian case" and "euclidean case", respectively, while equations not carrying any such index are valid in both cases simultaneously.

In this context, we adhere to the standard terminology that the Dirac matrices are given in a

(i) chiral representation if $\gamma_5$ is diagonal,

(ii) Majorana representation if $C = 1$,

(iii) chiral Majorana representation if both conditions are met simultaneously (which is the case if and only if $\gamma_5$ and $C$ commute).

In the Minkowski case, there is a chiral Majorana representation:

$$\gamma_0 = \sigma_2 , \qquad \gamma_1 = i\sigma_1 , \qquad \gamma_5 = \sigma_3 ,$$
$$\rho = \gamma_0 , \qquad C = 1 , \qquad \varepsilon = -1 . \tag{A.15$_\mathrm{M}$}$$

In the euclidean case, we have to decide between a chiral representation, e.g.

$$\gamma_1 = \sigma_1 , \qquad \gamma_2 = \sigma_2 , \qquad \gamma_5 = \sigma_3 ,$$
$$\rho = 1 , \qquad C = \sigma_1 , \qquad \varepsilon = +1 , \tag{A.15a$_\mathrm{E}$}$$

and a Majorana representation, e.g.

$$\gamma_1 = \sigma_3 , \qquad \gamma_2 = \sigma_1 , \qquad \gamma_5 = \sigma_2 ,$$
$$\rho = 1 , \qquad C = 1 , \qquad \varepsilon = +1 . \tag{A.15b$_\mathrm{E}$}$$

Independently of any explicit form of the representation, chiral symmetry transformations constitute a one-parameter group of matrices given by

$$\exp (i\alpha\gamma_5) = \cos \alpha\, 1 + i \sin \alpha\gamma_5 , \tag{A.16$_\mathrm{M}$}$$

$$\exp (\alpha\gamma_5) = \cosh \alpha\, 1 + \sinh \alpha\gamma_5 . \tag{A.16$_\mathrm{E}$}$$

Spinors and their conjugates transform according to

$$u \to \exp (i\alpha\gamma_5)u , \qquad \bar{u} \to \bar{u} \exp (i\alpha\gamma_5) , \tag{A.17$_\mathrm{M}$}$$

$$u \to \exp (\alpha\gamma_5)u , \qquad \bar{u} \to \bar{u} \exp (\alpha\gamma_5) , \tag{A.17$_\mathrm{E}$}$$

so that $\bar{u}\gamma_\mu v$ is invariant, while

$$\bar{u}v \to \cos (2\alpha)\bar{u}v + i \sin (2\alpha)\bar{u}\gamma_5 v ,$$
$$\bar{u}\gamma_5 v \to i \sin (2\alpha)\bar{u}v + \cos (2\alpha)\bar{u}\gamma_5 v , \tag{A.18$_\mathrm{M}$}$$

$$\bar{u}v \to \cosh (2\alpha)\bar{u}v + \sinh (2\alpha)\bar{u}\gamma_5 v ,$$
$$\bar{u}\gamma_5 v \to \sinh (2\alpha)\bar{u}v + \cosh (2\alpha)\bar{u}\gamma_5 v . \tag{A.18$_\mathrm{E}$}$$

Fierz identities constitute a technique for rearranging terms in quartic polynomials in fermion fields and, in two dimensions, are based on the formula

$$\delta_{\alpha\delta}\delta_{\gamma\beta} = \tfrac{1}{2}\{\delta_{\alpha\beta}\delta_{\gamma\delta} + (\gamma_5)_{\alpha\beta}(\gamma_5)_{\gamma\delta} + g^{\mu\nu}(\gamma_\mu)_{\alpha\beta}(\gamma_\nu)_{\gamma\delta}\} , \tag{A.19}$$

which can be checked e.g. by an explicit calculation in each of the representations (A.15). More generally, if $\Gamma$ and $\Gamma'$ are arbitrary,

$$\Gamma_{\alpha\delta}\Gamma'_{\gamma\beta} = \tfrac{1}{2}\{\Gamma_{\alpha\beta}\Gamma'_{\gamma\delta} + (\Gamma\gamma_5)_{\alpha\beta}(\Gamma'\gamma_5)_{\gamma\delta} + g^{\mu\nu}(\Gamma\gamma_\mu)_{\alpha\beta}(\Gamma'\gamma_\nu)_{\gamma\delta}\} . \qquad (A.20)$$

A particular consequence are the Fierz identities*

$$g^{\mu\nu}(\bar\psi^a\gamma_\mu\psi^b)(\bar\psi^c\gamma_\nu\psi^d) = \pm\{(\bar\psi^a\psi^d)(\bar\psi^c\psi^b) - (\bar\psi^a\gamma_5\psi^d)(\bar\psi^c\gamma_5\psi^b)\} , \qquad (A.21)$$

$$\varepsilon^{\mu\nu}(\bar\psi^a\gamma_\mu\psi^b)(\bar\psi^c\gamma_\nu\psi^d) = \pm\{(\bar\psi^a\gamma_5\psi^d)(\bar\psi^c\psi^b) - (\bar\psi^a\psi^d)(\bar\psi^c\gamma_5\psi^b)\} , \qquad (A.22_M)$$

$$\varepsilon^{\mu\nu}(\bar\psi^a\gamma_\mu\psi^b)(\bar\psi^c\gamma_\nu\psi^d) = \pm i\{(\bar\psi^a\gamma_5\psi^d)(\bar\psi^c\psi^b) - (\bar\psi^a\psi^d)(\bar\psi^c\gamma_5\psi^b)\} , \qquad (A.22_E)$$

where the upper and lower sign should be used whenever the components of are commuting and anticommuting $c$-numbers, respectively.

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* Indices $\alpha, \beta, \gamma, \delta, \ldots = 1, 2$ label different spinor components, while indices $a, b, c, d, \ldots = 1, \ldots, N$ label different spinors.

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