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ON THE ORIGIN OF ANOMALIES IN THE QUANTUM NON-LOCAL CHARGE FOR THE GENERALIZED NON-LINEAR SIGMA MODELS

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A general criterion for the absence or presence of anomalies in the quantum non-local charge of the non-linear σ-model on a riemannian symmetric space is presented.

1. Introduction

Classically, the two-dimensional generalized non-linear σ-models are known to be integrable and to possess higher conservation laws, both non-local and local, whenever the field takes values in a riemannian symmetric space [1–3]. At the quantum level, however, the situation is more involved because even if one is able to quantize the (first) non-local charge and define it as a genuine operator, this charge may develop an anomaly and need no longer be conserved. For example, in the S^{N-1} model (usually called the O(N)-invariant non-linear σ-model) as well as in the $\mathbb{C}P^{N-1}$ model, the quantum non-local charge may be defined and analyzed within the $1/N$ expansion, and it turns out to be conserved in the former [4], while it develops an anomaly in the latter [5]. As a consequence, the S-matrix of the S^{N-1} model factorizes [4] and can be calculated exactly [6], while the S-matrix of the $\mathbb{C}P^{N-1}$ model does not factorize and is still unknown.

In this paper, we give a simple and general criterion for the absence or presence of anomalies in the quantum non-local charge of the non-linear σ-model on an irreducible riemannian globally symmetric space M of the compact type. This means, in particular, that we may represent M as a quotient space $\mathrm{M} = \mathrm{G}/\mathrm{H}$, where G is a compact connected semisimple Lie group with Lie algebra $\mathfrak{g}$ and $\mathrm{H} \subset \mathrm{G}$ is a closed (hence compact) subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. For simplicity, we also assume

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that G is simply connected – which forces H to be connected – and that G acts almost effectively on M. (Thus, for example, the complex grassmannians should be represented in the form $SU(p+q)/S(U(p)\times U(q))$, and not in the form $U(p+q)/(U(p)\times U(q))$, in order for our criterion below to be applicable. For more details on the mathematics, the reader is referred to the books by Helgason [7] and Kobayashi and Nomizu [8].) Under these circumstances, our criterion is a simple condition on (the Lie algebra $\mathfrak{h}$ of) the stability group H:

(i) Anomalies are forbidden if $\mathfrak{h}$ is simple. (This is understood to include the 1-dimensional abelian case $\mathfrak{h}\cong\mathbb{R}$, which occurs for the non-linear σ-model on $S^2\cong\mathbb{C}P^1$.

(ii) Anomalies are allowed, and are to be expected, if $\mathfrak{h}$ contains non-trivial ideals. In particular, this condition excludes anomalies in the S^{N-1} model, where $\mathfrak{h}=\mathrm{so}(N-1)$, but allows anomalies in the $\mathbb{C}P^{N-1}$ model, where $\mathfrak{h}=\mathrm{s}(\mathrm{u}(1)\times\mathrm{u}(N-1))\cong\mathrm{u}(N-1)$, as long as $N>2$. It also excludes anomalies in the "irreducible" principal chiral models, i.e. the non-linear σ-models on compact simple Lie groups, in agreement with arguments based on higher local charges [9].

2. The model

We begin by briefly reviewing the formulation of the classical two-dimensional non-linear σ-model on a riemannian globally symmetric space $M=G/H$, subject to the restrictions mentioned in the introduction.

First of all, the Lie algebra $\mathfrak{g}$ admits an orthogonal, Ad(H)-invariant direct decomposition

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m} \tag{2.1}$$

into the Lie algebra $\mathfrak{h}$ of the stability group H and a complementary subspace $\mathfrak{m}$, with commutation relations

$$[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h}\,,\qquad [\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}\,,\qquad [\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}\,, \tag{2.2}$$

and the corresponding decomposition of elements $X\in\mathfrak{g}$ will be written

$$X=X_{\mathfrak{h}}+X_{\mathfrak{m}}\,. \tag{2.3}$$

Moreover, the stability group H being compact, its Lie algebra $\mathfrak{h}$ admits a further orthogonal, Ad(H)-invariant direct decomposition

$$\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_r \tag{2.4}$$

into its center $\mathfrak{h}_0$ and r simple ideals $\mathfrak{h}_1,\ldots,\mathfrak{h}_r$, with commutation relations

$$[\mathfrak{h}_i,\mathfrak{h}_j]=\{0\}\,,\qquad \text{for } i\neq j\,, \tag{2.5}$$

and the corresponding decomposition of elements $X\in\mathfrak{h}$ will be written

$$X=X^{(0)}+X^{(1)}+\cdots+X^{(r)}\,. \tag{2.6}$$

We assume in addition that $r \leq 2$ and that the center $\mathscr{h}_0$ of $\mathscr{h}$ is at most one-dimensional, which can be justified, e.g., simply by going through the list of all irreducible riemannian globally symmetric spaces [7]. Note also that, M being irreducible, the subspace $\mathscr{m}$ does not admit any non-trivial Ad(H)-invariant subspaces. Thus

$$\mathscr{g} = \mathscr{h}_0 \oplus \mathscr{h}_1 \oplus \cdots \oplus \mathscr{h}_r \oplus \mathscr{m} \tag{2.7}$$

constitutes an orthogonal, Ad(H)-invariant direct decomposition of $\mathscr{g}$ into Ad(H)-irreducible subspaces (some of which may be $\{0\}$).

Next, following [1–3], the field $q = q(x)$ taking values in $\mathrm{M} = \mathrm{G}/\mathrm{H}$ is (locally) lifted to a field $g = g(x)$ taking values in G, subject to the natural gauge equivalence

$$g_2(x) \sim g_1(x) \Leftrightarrow q_2(x) = q_1(x) \Leftrightarrow \begin{array}{l} \text{there exists a field } h = h(x) \\ \text{taking values in H such that} \\ g_2(x) = g_1(x) h(x)\,, \end{array} \tag{2.8}$$

under H. As usual, we consider the (left translated) derivative field $g^{-1}\partial_\mu g$ (taking values in $\mathscr{g}$) and split it into its vertical part, which is the gauge potential A_μ (taking values in $\mathscr{h}$), and its horizontal part, which is the (left-translated) covariant derivative field $k_\mu \equiv g^{-1} D_\mu g$ (taking values in $\mathscr{m}$):

$$A_\mu = (g^{-1}\partial_\mu g)_{\mathscr{h}}\,, \qquad k_\mu \equiv g^{-1} D_\mu g = (g^{-1}\partial_\mu g)_{\mathscr{m}}\,. \tag{2.9}$$

The gauge potential can be further split into its components along the various ideals $\mathscr{h}_i$:

$$A_\mu = A_\mu^{(0)} + A_\mu^{(1)} + \cdots + A_\mu^{(r)}\,. \tag{2.10}$$

[Cf. (2.3) and (2.6) for the notation.] Indeed, it follows from the Ad(H)-invariance of the direct decompositions (2.1) and (2.4) that under gauge transformations $g \to gh$, A_μ and $A_\mu^{(i)}$ transform as gauge potentials (i.e. $A_\mu \to h^{-1} A_\mu h + h^{-1}\partial_\mu h$ and $A_\mu^{(i)} \to h^{-1} A_\mu^{(i)} h + (h^{-1}\partial_\mu h)^{(i)}$), and k_μ is covariant (i.e. $k_\mu \to h^{-1} k_\mu h$). This motivates the introduction of gauge fields (curvature tensors) $F_{\mu\nu}$ for A_μ and $F_{\mu\nu}^{(i)}$ for $A_\mu^{(i)}$, and of a covariant derivative $D_\mu k_\nu$ for k_μ:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,, \tag{2.11}$$

$$F_{\mu\nu}^{(i)} = \partial_\mu A_\nu^{(i)} - \partial_\nu A_\mu^{(i)} + [A_\mu^{(i)}, A_\nu^{(i)}]\,, \tag{2.12}$$

$$D_\mu k_\nu = \partial_\mu k_\nu + [A_\mu, k_\nu]\,. \tag{2.13}$$

Observe that due to (2.5) and (2.10), $F_{\mu\nu}$ is simply the sum of the $F_{\mu\nu}^{(i)}$:

$$F_{\mu\nu} = F_{\mu\nu}^{(0)} + F_{\mu\nu}^{(1)} + \cdots + F_{\mu\nu}^{(r)}\,. \tag{2.14}$$

Moreover, as a consequence of the symmetric space structure of M, the identities

$$F_{\mu\nu} = -[k_\mu, k_\nu]\,, \tag{2.15}$$

$$D_\mu k_\nu = D_\nu k_\mu \tag{2.16}$$

hold for any field configuration; in fact, according to (2.2), eq. (2.15) [(2.16)] is simply the vertical part ($\mathscr{k}$-component) [horizontal part ($\mathscr{m}$-component)] of the identity

$$\partial_\mu(g^{-1}\partial_\nu g)+g^{-1}\partial_\mu g g^{-1}\partial_\nu g=\partial_\nu(g^{-1}\partial_\mu g)+g^{-1}\partial_\nu g g^{-1}\partial_\mu g\,.$$

Passing to gauge-invariant quantities (taking values in $\mathscr{g}$), we have the Noether current

$$j_\mu=-gk_\mu g^{-1}=-D_\mu g g^{-1}\,, \tag{2.17}$$

as well as the symmetric tensor

$$J_{\mu\nu}=-gD_\mu k_\nu g^{-1} \tag{2.18}$$

(cf. (2.16)) and the antisymmetric tensors

$$G_{\mu\nu}=gF_{\mu\nu}g^{-1}\,, \tag{2.19}$$

$$G^{(i)}_{\mu\nu}=gF^{(i)}_{\mu\nu}g^{-1}\,. \tag{2.20}$$

Observe that due to (2.14), $G_{\mu\nu}$ is simply the sum of the $G^{(i)}_{\mu\nu}$:

$$G_{\mu\nu}=G^{(0)}_{\mu\nu}+G^{(1)}_{\mu\nu}+\cdots+G^{(r)}_{\mu\nu}\,. \tag{2.21}$$

Moreover, as a consequence of the symmetric space structure of M, the identities

$$G_{\mu\nu}=-[j_\mu,j_\nu]\,, \tag{2.22}$$

$$\partial_\mu j_\nu=J_{\mu\nu}+G_{\mu\nu} \tag{2.23}$$

hold for any field configuration.

The classical two-dimensional non-linear σ-model on M is defined in terms of its action functional

$$S=\tfrac{1}{2}\int d^2x\,(\partial_\mu q(x),\partial^\mu q(x))=\tfrac{1}{2}\int d^2x\,(D_\mu g(x),D^\mu g(x))\,, \tag{2.24}$$

which by the usual variational principle leads to the field equations

$$D_\mu D^\mu g-D_\mu g g^{-1}D^\mu g=0\,. \tag{2.25}$$

These imply that the current is conserved, i.e.

$$\partial_\mu j^\mu=0\,, \tag{2.26}$$

and conversely, (2.26) implies (2.25) [10]. Thus in the form of the conservation law (2.26), and together with the identity

$$\partial_\mu j_\nu-\partial_\nu j_\mu+2[j_\mu,j_\nu]=0\,, \tag{2.27}$$

which results from (2.22), (2.23), the equations of motion are equivalent to the integrability, for any value of the (real) parameter λ, of the following system of first-order linear differential equations:

$$\partial_\mu U^{(\lambda)} = U^{(\lambda)}\{j_\mu(1-\cosh\lambda) - \varepsilon_{\mu\nu} j^\nu \sinh\lambda\}. \tag{2.28}$$

Similarly, one can check that they imply conservation (i.e. time-independence) of the (first) non-local charge:

$$Q^{(1)}(t) = \int \mathrm{d}y_1\, \mathrm{d}y_2\, \theta(y_1-y_2)[j_0(t,y_1), j_0(t,y_2)] - \int \mathrm{d}y\, j_1(t,y). \tag{2.29}$$

3. Quantum non-local charge and anomalies

For the purposes of quantization, we shall work in some faithful N-dimensional representation of G by unitary matrices, which yields a faithful N-dimensional representation of $\mathfrak{g}$ by antihermitian matrices. The basic fields of the model are then the $(N\times N)$-matrix fields g and g^+ ($+$ denoting hermitian adjoint) which, classically, satisfy the unitarity condition

$$g^+ g = 1 = g g^+, \tag{3.1}$$

and are subject to a local H-invariance $g\to gh$, $g^+\to h^+g^+$ which enforces the use of covariant derivatives

$$\begin{aligned} D_\mu g &= \partial_\mu g - g A_\mu, & D_\mu D_\nu g &= \partial_\mu D_\nu g - D_\nu g A_\mu, \\ D_\mu g^+ &= \partial_\mu g^+ + A_\mu g^+, & D_\mu D_\nu g^+ &= \partial_\mu D_\nu g^+ + A_\mu D_\nu g^+, \end{aligned} \tag{3.2}$$

etc. Differentiating (3.1) gives

$$D_\mu g^+ g + g^+ D_\mu g = 0 = D_\mu g g^+ + g D_\mu g^+. \tag{3.3}$$

In the quantum theory, products of field operators at the same point will in general not be well-defined, and one has to use some definite normal product prescription for subtracting the singularities. We suppose here that such a normal product prescription $\mathcal{N}[\cdots]$ does exist, and that it is "reasonable" in the sense of maintaining the constraints (up to possible renormalization-dependent constants) and preserving the internal symmetry properties. Thus the definitions of the various composite fields in sect. 2 [eqs. (2.9)–(2.13) and (2.17)–(2.20)] and above [eq. (3.2)] can be transferred from the classical to the quantum theory by writing* g^+ for g^{-1} and applying a normal product symbol to any product or commutator. Moreover,

* For symmetric spaces of the non-compact type, where G is non-compact and does not admit any faithful finite-dimensional unitary representations, a quantum definition of g^{-1} is much more involved because g^{-1} will depend non-linearly on g. Thus, although in some cases (such as the duals of the real, complex or quaternionic grassmannians) this problem can be circumvented by using a suitable pseudo-unitary representation, we have for simplicity restricted ourselves to symmetric spaces of the compact type.

we require that

$$\mathcal{N}[\mathcal{O}_1 g^+ g \mathcal{O}_2] = c\mathcal{N}[\mathcal{O}_1\mathcal{O}_2]\,,$$
$$\mathcal{N}[\mathcal{O}_1 g g^+ \mathcal{O}_2] = c\mathcal{N}[\mathcal{O}_1\mathcal{O}_2]\,, \tag{3.4}$$

and, differentiating (3.4), that

$$\mathcal{N}[\mathcal{O}_1 D_\mu g^+ g \mathcal{O}_2] + \mathcal{N}[\mathcal{O}_1 g^+ D_\mu g \mathcal{O}_2] = 0\,,$$
$$\mathcal{N}[\mathcal{O}_1 D_\mu g g^+ \mathcal{O}_2] + \mathcal{N}[\mathcal{O}_1 g D_\mu g^+ \mathcal{O}_2] = 0\,, \tag{3.5}$$

for all formal products $\mathcal{O}_1$, $\mathcal{O}_2$ of g, g^+ and their covariant derivatives, where c is a renormalization-dependent constant.

Formulas (3.4) and (3.5) are the quantum analogues of constraints (3.1) and (3.3), respectively, and should be considered as part of the defining properties of the model. Other constraints, defining G as a closed subgroup of U(N), should be handled similarly. Finally, we require that under global G-transformations $g \to g_0 g$, $g^+ \to g^+ g_0^+$ and under local H-transformations $g \to gh$, $g^+ \to h^+ g^+$, any normal product behaves precisely like its classical counterpart (i.e. satisfies the correct Ward identities), and that, in particular, identities (2.15), (2.16), (2.22) and (2.23) are preserved in the quantum theory [with a normal product symbol in front of the commutators on the r.h.s. of (2.15) and (2.22)].

It should be mentioned at this point that in cases where standard techniques can be applied to construct normal products within the framework of renormalized perturbation theory [11], these requirements are indeed satisfied [5, 12].

The correct definition of the (first) quantum non-local charge, which is to be the quantum analogue of (2.29), requires the examination of the short-distance behavior of the commutator between two currents. This behavior is supposed to take the form of a Wilson expansion

$$[j_\mu(x+\varepsilon), j_\nu(x-\varepsilon)] = \sum_k C_{\mu\nu}^{(k)}(\varepsilon)\mathcal{N}[\mathcal{O}_k(x)]\,, \qquad (\varepsilon^2 < 0)\,, \tag{3.6}$$

where k labels a complete, linearly independent set of composite local operators $\mathcal{N}[\mathcal{O}_k(x)]$ of (canonical) dimension ≤ 2. This is justified in view of the asymptotic freedom of this class of models [13]. Moreover, due to $\varepsilon^2 < 0$ and locality, these operators should take values in $\mathfrak{g}$, and they should be globally G-covariant and locally H-invariant. But the only operators which satisfy all these requirements are the following:

Dimension 0: – ;

Dimension 1: $j_\mu(x)$;

Dimension 2: $J_{\mu\nu}(x)$ and $G_{\mu\nu}^{(0)}(x), \ldots, G_{\mu\nu}^{(r)}(x)$.

In the proof, we shall for simplicity omit the normal product symbols:

First, observe that g, g^+ being dimensionless, any composite local operator must be constructed from a chain of the type

$$L_1 g L_2 g^+ \cdots L_{2k-1} g L_{2k} g^+\,, \tag{3.7}$$

if it is to be globally G-covariant and locally H-invariant, and from a chain of the type

$$L_1 g^+ L_2 g \cdots L_{2k-1} g^+ L_{2k} g\,, \tag{3.8}$$

if it is to be globally G-invariant and locally H-covariant. Here and below, the L's are either the identity or products of covariant derivatives, and the total number of derivatives is equal to the dimension of the composite operator under consideration. Moreover, using the constraints (3.1) and (3.3), we can eliminate superfluous products g^+g, gg^+ and transfer the covariant derivatives from g^+ to g, so that the chains (3.7) and (3.8) can be rewritten in the form

$$L_1 gg^+ \cdots L_k gg^+\,, \tag{3.9}$$

$$g^+ L_1 g \cdots g^+ L_k g\,, \tag{3.10}$$

respectively. Note also that because of $\mathfrak{g} \subset \mathrm{u}(N)$, we have to eliminate the hermitian parts of (3.9) and (3.10), and at least for operators of dimension ≤ 2, it turns out that this is in fact sufficient to construct operators which take values in $\mathfrak{g}$ [and not just in $\mathrm{u}(N)$]. Finally, the resulting operators – insofar as they are globally G-invariant and locally H-covariant – may be decomposed into irreducible parts, without spoiling their internal symmetry properties, by using the Ad(H)-invariant decomposition (2.7).

In more concrete terms, this strategy proceeds as follows:

Dimension 0: There is no candidate.

Dimension 1: There is a unique candidate, namely

$$D_\mu gg^+ = -j_\mu = gk_\mu g^+\,.$$

This is already antihermitian and does indeed take values in $\mathfrak{g}$ [rather than just in $\mathrm{u}(N)$]. The decomposition of k_μ into irreducible parts is trivial and shows that j_μ is the basic composite operator of dimension 1.

Dimension 2: There are two linearly independent candidates, namely

$$D_\mu gg^+ D_\nu gg^+\,,$$

$$D_\mu D_\nu gg^+ - D_\mu gg^+ D_\nu gg^+ = \partial_\mu (D_\nu gg^+) = -\partial_\mu j_\nu\,.$$

Due to

$$(D_\mu gg^+ D_\nu gg^+)^+ = gD_\nu g^+ gD_\mu g^+ = D_\nu gg^+ D_\mu gg^+\,,$$

$$(\partial_\mu j_\nu)^+ = -\partial_\mu j_\nu\,,$$

the antihermitian parts are

$$\tfrac{1}{2}(D_\mu gg^+ D_\nu gg^+ - D_\nu gg^+ D_\mu gg^+) = \tfrac{1}{2}[j_\mu, j_\nu] = \tfrac{1}{2}G_{\mu\nu} = \tfrac{1}{2}gF_{\mu\nu}g^+\,,$$

$$\partial_\mu j_\nu = J_{\mu\nu} + G_{\mu\nu} = g(D_\mu k_\nu + F_{\mu\nu})g^+\,,$$

and do indeed take values in $\mathfrak{g}$ [rather than just in $\mathrm{u}(N)$]. The decomposition of

$F_{\mu\nu}$ and of $D_\mu k_\nu + F_{\mu\nu}$ into irreducible parts then shows that $J_{\mu\nu}$ and $G^{(0)}_{\mu\nu}, \ldots, G^{(r)}_{\mu\nu}$ are the basic composite operators of dimension 2.

Using this result, together with (2.21) and the identity (2.23), we can write the Wilson expansion (3.6) in the form

$$[j_\mu(x+\varepsilon), j_\nu(x-\varepsilon)] = C^\rho_{\mu\nu}(\varepsilon) j_\rho(x) + D^{\sigma\rho}_{\mu\nu}(\varepsilon)(\partial_\sigma j_\rho)(x) + \sum_{i=0}^{r} D^{(i)\sigma\rho}_{\mu\nu}(\varepsilon) G^{(i)}_{\sigma\rho}(x), \qquad (\varepsilon^2 < 0), \tag{3.11}$$

with the subsidiary condition

$$\sum_{i=0}^{r} D^{(i)\sigma\rho}_{\mu\nu}(\varepsilon) = 0. \tag{3.12}$$

(Equivalently, we could have required $D^{\sigma\rho}_{\mu\nu}(\varepsilon)$ to be symmetric in σ and ρ.) The tensorial nature of the linearly divergent coefficient function $C^\rho_{\mu\nu}(\varepsilon)$ and the logarithmically divergent coefficient functions $D^{\sigma\rho}_{\mu\nu}(\varepsilon)$ and $D^{(i)\sigma\rho}_{\mu\nu}(\varepsilon)$ can be determined from general principles such as covariance (under the full Poincaré group, i.e. including parity and time reversal), current conservation, etc. This derivation proceeds along the same lines as for the S^{N-1} model [4] and $\mathbb{C}P^{N-1}$ model [14], and we shall not repeat it here.

Following [4, 5], we now define the (first) quantum non-local charge as the limit

$$Q^{(1)}(t) = \lim_{\delta \to 0} Q^{(1)}_\delta(t) \tag{3.13}$$

of a cutoff charge

$$Q^{(1)}_\delta(t) = \int_{|y_1 - y_2| \geq \delta} dy_1\, dy_2\, \theta(y_1 - y_2)[j_0(t, y_1), j_0(t, y_2)] - Z(\delta) \int dy\, j_1(t, y), \tag{3.14}$$

where

$$Z(\delta) = \text{const} \cdot \ln(\mu\delta). \tag{3.15}$$

Here, μ is a mass parameter, and the constant is chosen in such a way as to cancel the linear divergence (for $\delta \to 0$) in the first integrand on the r.h.s. of (3.14).

Concerning conservation of this charge, we distinguish two cases:

(i) $\mathscr{k}$ is simple. (As mentioned in the introduction, this is understood to include the case where $\mathscr{k}$ is one-dimensional and abelian.) There is only one non-zero summand in the decomposition (2.4), and (3.11) simplifies to

$$[j_\mu(x+\varepsilon), j_\nu(x-\varepsilon)] = C^\rho_{\mu\nu}(\varepsilon) j_\rho(x) + D^{\sigma\rho}_{\mu\nu}(\varepsilon)(\partial_\sigma j_\rho)(x), \qquad (\varepsilon^2 < 0). \tag{3.16}$$

Following [4], one may then verify that the charge $Q^{(1)}$ is indeed conserved.

(ii) $\mathscr{k}$ has non-trivial ideals. (As mentioned in sect. 2, we are assuming the center $\mathscr{k}_0$ of $\mathscr{k}$ to be at most one-dimensional.) There are at least two non-zero summands in the decomposition (2.4), and the last term on the r.h.s. of (3.11) provides an anomaly, so that the charge $Q^{(1)}$ will in general no longer be conserved. This happens, for example, in the $\mathbb{C}P^{N-1}$ model, where the coefficients have been calculated within the $1/N$ expansion and have been shown to be non-zero to leading order in $1/N$ [5] and, more recently, to all orders in $1/N$ [14].

4. Examples

To conclude, we want to facilitate the comparison of our result with earlier work [4, 5] by exhibiting the explicit form of the fields $j_\mu, J_{\mu\nu}$ and $G^{(0)}_{\mu\nu}, \dots, G^{(r)}_{\mu\nu}$ – the basic building blocks for the Wilson expansion (3.11) – in the case of real and complex grassmannians.

For the complex grassmannians $\mathrm{SU}(N)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$, where $N=p+q$, $\mathscr{g}$ is the Lie algebra $\mathrm{su}(N)$ of all traceless antihermitian complex $(N\times N)$ matrices, for which we use the block matrix notation:

$$\underset{\overleftrightarrow{N}}{(\cdots)} \updownarrow N = \begin{pmatrix} \underset{}{(\cdots)} & (\cdots) \\ \underset{\overleftrightarrow{p}}{(\cdots)} & \underset{\overleftrightarrow{q}}{(\cdots)} \end{pmatrix} \begin{matrix} \updownarrow p \\ \updownarrow q \end{matrix} . \tag{4.1}$$

Then (2.1) holds with

$$\mathscr{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Big/ \begin{matrix} A^+ = -A,\, B^+ = -B \\ \operatorname{tr} A + \operatorname{tr} B = 0 \end{matrix} \right\}, \tag{4.2}$$

$$\mathscr{m} = \left\{ \begin{pmatrix} 0 & -R^+ \\ R & 0 \end{pmatrix} \right\}, \tag{4.3}$$

and (2.4) holds with $r=2$ and

$$\mathscr{k}_0 = \left\{ i\lambda \begin{pmatrix} 1/p & 0 \\ 0 & -1/q \end{pmatrix} \Big/ \lambda \in \mathbb{R} \right\} \cong \mathbb{R}, \tag{4.4}$$

$$\mathscr{k}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \Big/ \begin{matrix} A^+ = -A \\ \operatorname{tr} A = 0 \end{matrix} \right\} \cong \mathrm{su}(p), \tag{4.5}$$

$$\mathscr{k}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \Big/ \begin{matrix} B^+ = -B \\ \operatorname{tr} B = 0 \end{matrix} \right\} \cong \mathrm{su}(q). \tag{4.6}$$

Furthermore, the field g is written in the form $g=(X, Y)$, where all matrices have N rows and g, X, Y have N, p, q columns, respectively. In these terms, the constraints

$$g^+g = 1_N, \qquad gg^+ = 1_N,$$

become

$$X^+X = 1_p, \qquad X^+Y = 0, \qquad Y^+X = 0, \qquad Y^+Y = 1_q, \tag{4.7}$$

$$XX^{+}+YY^{+}=1_{N}, \tag{4.8}$$

respectively. Next, using covariant derivatives

$$\begin{aligned} D_{\mu}X&=\partial_{\mu}X-XX^{+}\partial_{\mu}X, \qquad & D_{\mu}D_{\nu}X&=\partial_{\mu}D_{\nu}X-D_{\nu}XX^{+}\partial_{\mu}X,\\ D_{\mu}Y&=\partial_{\mu}Y-YY^{+}\partial_{\mu}Y, \qquad & D_{\mu}D_{\nu}Y&=\partial_{\mu}D_{\nu}Y-D_{\nu}YY^{+}\partial_{\mu}Y, \end{aligned} \tag{4.9}$$

etc., we get

$$A_{\mu}=\begin{pmatrix} A_{\mu}^{X} & 0\\ 0 & A_{\mu}^{Y}\end{pmatrix}, \qquad \text{with} \quad \begin{matrix} A_{\mu}^{X}=X^{+}\partial_{\mu}X\\ A_{\mu}^{Y}=Y^{+}\partial_{\mu}Y \end{matrix}, \tag{4.10}$$

$$k_{\mu}=\begin{pmatrix} 0 & X^{+}D_{\mu}Y\\ Y^{+}D_{\mu}X & 0\end{pmatrix}, \tag{4.11}$$

$$F_{\mu\nu}=\begin{pmatrix} F_{\mu\nu}^{X} & 0\\ 0 & F_{\mu\nu}^{Y}\end{pmatrix}, \qquad \text{with} \quad \begin{matrix} F_{\mu\nu}^{X}=D_{\mu}X^{+}D_{\nu}X-D_{\nu}X^{+}D_{\mu}X\\ F_{\mu\nu}^{Y}=D_{\mu}Y^{+}D_{\nu}Y-D_{\nu}Y^{+}D_{\mu}Y \end{matrix}, \tag{4.12}$$

$$D_{\mu}k_{\nu}=\begin{pmatrix} 0 & X^{+}D_{\mu}D_{\nu}Y\\ Y^{+}D_{\mu}D_{\nu}X & 0\end{pmatrix}, \tag{4.13}$$

while (2.10) and (2.14) hold with

$$A_{\mu}^{(0)}=(\operatorname{tr}A_{\mu}^{X})\begin{pmatrix} 1/p & 0\\ 0 & -1/q\end{pmatrix}=(\operatorname{tr}A_{\mu}^{Y})\begin{pmatrix} -1/p & 0\\ 0 & 1/q\end{pmatrix}, \tag{4.14}$$

$$A_{\mu}^{(1)}=\begin{pmatrix} A_{\mu}^{X}-1/p\operatorname{tr}A_{\mu}^{X} & 0\\ 0 & 0\end{pmatrix}, \tag{4.15}$$

$$A_{\mu}^{(2)}=\begin{pmatrix} 0 & 0\\ 0 & A_{\mu}^{Y}-1/q\operatorname{tr}A_{\mu}^{Y}\end{pmatrix}, \tag{4.16}$$

and

$$F_{\mu\nu}^{(0)}=(\operatorname{tr}F_{\mu\nu}^{X})\begin{pmatrix} 1/p & 0\\ 0 & -1/q\end{pmatrix}=(\operatorname{tr}F_{\mu\nu}^{Y})\begin{pmatrix} -1/p & 0\\ 0 & 1/q\end{pmatrix}, \tag{4.17}$$

$$F_{\mu\nu}^{(1)}=\begin{pmatrix} F_{\mu\nu}^{X}-1/p\operatorname{tr}F_{\mu\nu}^{X} & 0\\ 0 & 0\end{pmatrix}, \tag{4.18}$$

$$F_{\mu\nu}^{(2)}=\begin{pmatrix} 0 & 0\\ 0 & F_{\mu\nu}^{Y}-1/q\operatorname{tr}F_{\mu\nu}^{Y}\end{pmatrix}. \tag{4.19}$$

Now introducing the antisymmetric tensors

$$\begin{aligned} G_{\mu\nu}^{X}&=XF_{\mu\nu}^{X}X^{+}=XD_{\mu}X^{+}D_{\nu}XX^{+}-XD_{\nu}X^{+}D_{\mu}XX^{+}\\ &=D_{\mu}YD_{\nu}Y^{+}-D_{\nu}YD_{\mu}Y^{+}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} G_{\mu\nu}^{Y}&=YF_{\mu\nu}^{Y}Y^{+}=YD_{\mu}Y^{+}D_{\nu}YY^{+}-YD_{\nu}Y^{+}D_{\mu}YY^{+}\\ &=D_{\mu}XD_{\nu}X^{+}-D_{\nu}XD_{\mu}X^{+}, \end{aligned} \tag{4.21}$$

which obviously satisfy

$$G^X_{\mu\nu}+G^Y_{\mu\nu}=G_{\mu\nu}=G^{(0)}_{\mu\nu}+G^{(1)}_{\mu\nu}+G^{(2)}_{\mu\nu}\,, \tag{4.22}$$

we can rewrite the gauge-invariant fields j_μ, $J_{\mu\nu}$ and $G^{(0)}_{\mu\nu}$, $G^{(1)}_{\mu\nu}$, $G^{(2)}_{\mu\nu}$ purely in terms of the field X or purely in terms of the field Y:

$$\begin{aligned} j_\mu &= XD_\mu X^+ - D_\mu XX^+ \\ &= YD_\mu Y^+ - D_\mu YY^+\,, \end{aligned} \tag{4.23}$$

$$\begin{aligned} J_{\mu\nu} &= D_\mu D_\nu XX^+ - XD_\mu D_\nu X^+ + G^X_{\mu\nu} \\ &= D_\mu D_\nu YY^+ - YD_\mu D_\nu Y^+ + G^Y_{\mu\nu}\,, \end{aligned} \tag{4.24}$$

$$\begin{aligned} G^{(0)}_{\mu\nu} &= \frac{1}{pq}(\operatorname{tr} F^X_{\mu\nu})(NXX^+ - p\,1_N) \\ &= \frac{1}{pq}(\operatorname{tr} F^Y_{\mu\nu})(NYY^+ - q\,1_N)\,, \end{aligned} \tag{4.25}$$

$$\begin{aligned} G^{(1)}_{\mu\nu} &= G^X_{\mu\nu} - \frac{1}{p}(\operatorname{tr} F^X_{\mu\nu})XX^+ \\ &= D_\mu YD_\nu Y^+ - D_\nu YD_\mu Y^+ - \frac{1}{p}(\operatorname{tr} F^Y_{\mu\nu})(YY^+ - 1_N)\,, \end{aligned} \tag{4.26}$$

$$\begin{aligned} G^{(2)}_{\mu\nu} &= G^Y_{\mu\nu} - \frac{1}{q}(\operatorname{tr} F^Y_{\mu\nu})YY^+ \\ &= D_\mu XD_\nu X^+ - D_\nu XD_\mu X^+ - \frac{1}{q}(\operatorname{tr} F^X_{\mu\nu})(XX^+ - 1_N)\,. \end{aligned} \tag{4.27}$$

Thus we see that the true complex grassmannian model, with $p \geqslant 2$ and $q \geqslant 2$, contains two linearly independent operators which can give rise to anomalies. If $p=1$ or $q=1$, we are dealing with the $\mathbb{C}P^{N-1}$ model and have $\mathscr{k}_1=\{0\}$ or $\mathscr{k}_2=\{0\}$, respectively, so that writing z instead of X or Y, respectively,

$$G_{\mu\nu} = zz^+F^z_{\mu\nu} + D_\mu zD_\nu z^+ - D_\nu zD_\mu z^+ \tag{4.28}$$

is the curl of the current, but $zz^+F^z_{\mu\nu}$ (or $D_\mu zD_\nu z^+ - D_\nu zD_\mu z^+$) for itself is a linearly independent operator which can give rise to an anomaly. This does not, however, apply to the case of the $\mathbb{C}P^1$ model, where $p=q=1$ and $\mathscr{k}_1=\mathscr{k}_2=\{0\}$, so that

$$G_{\mu\nu} = F^z_{\mu\nu}(2zz^+ - 1)\,, \tag{4.29}$$

and so there can be no anomaly.

For the real grassmannians $SO(N)/SO(p)\times SO(q)$, where $N=p+q$, the previous analysis applies if we replace + (hermitian adjoint) by T (transpose), discard all imaginary parts and observe that now $\mathscr{k}_0=\{0\}$. Thus we see that the true real grassmannian model, with $p \geqslant 2$ and $q \geqslant 2$, contains one linearly independent operator which can give rise to an anomaly. If $p=1$ or $q=1$, we are dealing with the S^{N-1} model and have $\mathscr{k}_1=\{0\}$ or $\mathscr{k}_2=\{0\}$, respectively, so that writing q instead of X or Y, respectively,

$$G_{\mu\nu}=\partial_\mu q\,\partial_\nu q^{\mathrm{T}}-\partial_\nu q\,\partial_\mu q^{\mathrm{T}} \tag{4.30}$$

is the curl of the current, and so there can be no anomaly.

References

[1] H. Eichenherr and M. Forger, Nucl. Phys. B155 (1979) 381
[2] H. Eichenherr and M. Forger, Nucl. Phys. B164 (1980) 528
[3] H. Eichenherr and M. Forger, Comm. Math. Phys. 82 (1981) 227
[4] M. Lüscher, Nucl. Phys. B135 (1978) 1
[5] E. Abdalla, M.C.B. Abdalla and M. Gomes, Phys. Rev. D23 (1981) 1800
[6] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. B133 (1978) 525
[7] S. Helgason, Differential geometry, Lie groups, and symmetric spaces (Academic Press, New York, 1978)
[8] S. Kobayashi and K. Nomizu, Foundations of differential geometry, vol. 2 (Interscience, New York, 1969)
[9] Y.Y. Goldschmidt and E. Witten, Phys. Lett. 91B (1980) 392
[10] M. Forger, Differential geometric methods in nonlinear sigma models and gauge theories, PhD thesis, Freie Universität Berlin (1980) unpublished
[11] W. Zimmermann, Ann. of Phys. 77 (1973) 536, 569; J. Lowenstein, BPHZ renormalization, *in* Renormalization theory, Proc. Erice 1975, ed. G. Velo and A.S. Wightman (D. Reidel, Dordrecht, 1976)
[12] I. Ya. Aref'eva, Teor. Mat. Fiz. 36 (1978) 24; J. Lowenstein and E. Speer, Nucl. Phys. B158 (1979) 397
[13] E. Brézin, S. Hikami and J. Zinn-Justin, Nucl. Phys. B165 (1980) 528
[14] E. Abdalla, M.C.B. Abdalla and M. Gomes, On the non-local charge for the $\mathbb{C}P^{N-1}$ model and its supersymmetric extension to all orders, to appear