Nuclear Physics B164 (1980) 528–535

MORE ABOUT NON-LINEAR SIGMA MODELS ON SYMMETRIC SPACES

H. EICHENHERR
Fakultät für Physik der Universität Freiburg, Hermann-Herder-Str. 3, D-7800 Freiburg i. Br.

M. FORGER
Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 3, D-1000 Berlin 33

Received 15 October 1979

For two-dimensional non-linear σ-models on riemannian symmetric spaces G/H, there exists a natural formulation in terms of a single gauge-invariant G-valued field. The solution spaces of the G/H models are subspaces of the solution space of the principal G model. For hermitian symmetric spaces, the (anti-)instanton solutions are fixed points under the dual symmetry.

1. Introduction

The non-linear σ-models are simple and prominent examples for field theories of geometric nature. Their prototypes are the non-linear σ-models on the spheres S^N, whose classical as well as quantum theoretical properties have been investigated by a number of authors over the last couple of years.

In a recent paper [1], we have started a general analysis of the classical structure of the two-dimensional non-linear σ-models on homogeneous spaces G/H. Our aim was to disentangle certain structures of these models from their superficial dependence on properties of the orthogonal group and to find their appropriate differential geometric setting. We formulated the non-linear σ-model on G/H in terms of a G-valued field with gauge symmetry, where H is the gauge group and G is the global symmetry group. For the case of riemannian symmetric spaces [2,3], we were then able to give a general construction of the dual symmetry and the related set of infinitely many conserved non-local charges [4].

In the present communication, we continue our analysis in the framework of riemannian symmetric spaces. As our main result, we show that in this case there exists a natural gauge-invariant formulation of the σ-model dynamics in terms of a single G-valued field Q which is subject to a quadratic constraint. This is based on an embedding of the symmetric space G/H into the group G – going back to

Cartan [5] – which realizes the former as a closed totally geodesic submanifold of the latter, characterized by a simple quadratic condition. As an immediate consequence, the solution spaces of all σ-models on symmetric spaces with the same global symmetry group G appear as subspaces of the solution space of the principal G model. This means that a complete description of the space of solutions to the principal G-model – as intended in [6] – implies a complete description of the spaces of solutions to all G/H models where G/H is symmetric. In this respect, our work clarifies and generalizes an idea of Zakharov and Michailov [6]. It also sheds new light on the role of the gauge group H (for the classical dynamics) because in the gauge-invariant formulation the latter is coded into the constraint.

For the quantum dynamics, the situation is less clear: whereas the global symmetry group G as well as the employed representation are essential ingredients in the computation of bootstrap *S*-matrices [7–9], the influence of the stability group H is obscure. For example, lacking a reliable field-theoretic check, we do not know which of the complex grassmannian σ-models, if any, leads to the *S*-matrix with adjoint SU(*N*) symmetry [9].

In the second part of the paper, we explain how in the case of hermitian symmetric spaces, the differential equations of the dual symmetry can be integrated explicitly on the (anti-)instanton solutions. An immediate consequence of the resulting formula is the fact that (anti-)instantons are fixed points under the dual symmetry.

2. The role of the principal field

First of all, we briefly describe the geometric framework we are dealing with (the reader will find all necessary background information in the books by Helgason [2] and Kobayashi-Nomizu [3]).

Let us start with a riemannian locally symmetric space $\tilde{M}$. From $\tilde{M}$ we may go over to a riemannian globally symmetric space $\hat{M}$ which is locally isometric with $\tilde{M}$ and which, without loss of generality, may be assumed to be simply connected. $\hat{M}$ splits into a direct product

$$\hat{M} = M_0 \times M_- \times M_+ ,$$

where M_0 is a euclidean space and M_- and M_+ are riemannian globally symmetric spaces of the compact and non-compact type, respectively. Since M_0 is flat, the corresponding sector of the σ-model is a free field theory, and consequently we shall ignore its contribution completely in the following. We stress, however, that there is no reason to drop the non-compact part, as there are important examples of physical interest: for example, the Sp(2, $\mathbb{R}$) invariant Gross-Neveu model is closely related to the non-linear σ-model on O(2, 1)/O(2) which is a symmetric space of the non-compact type [10]. So we consider

$$M = M_- \times M_+ ,$$

and defining $G_+(G_-)$ to be the identity component of the group of isometries of $M_+(M_-)$ and $H_+(H_-)$ to be the stability group of some arbitrarily chosen point of $M_+(M_-)$, we may write

$$M = M_- \times M_+ = G_-/H_- \times G_+/H_+ = G/H ,$$

where

$$G = G_- \times G_+ , \qquad H = H_- \times H_+ .$$

There exists an involutive automorphism σ of G such that

$$(G_\sigma)_0 \subset H \subset G_\sigma ,$$

where G_σ is the set of fixed points of σ and $(G_\sigma)_0$ is its identity component. The tangent map $\dot{\sigma} = d\sigma|_1$ splits the Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$ of G into eigenspaces $\mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{h}_+$ and $\mathfrak{k} = \mathfrak{k}_- \oplus \mathfrak{k}_+$ corresponding to the eigenvalues +1 and −1 of $\dot{\sigma}$, respectively. On $\mathfrak{g}$, we have an Ad(G)-invariant non-degenerate symmetric bilinear form (,) which is positive definite on $\mathfrak{k}$. Take on $\mathfrak{g}_+$ the Killing form of G_+ and on $\mathfrak{g}_-$ the negative of the Killing form of G_-. This form on $\mathfrak{g}$ and its restriction to $\mathfrak{k}$ extend to a bi-invariant pseudo-riemannian metric (,) on G and to a left G-invariant riemannian metric (,) on G/H, respectively.

Now consider the non-linear σ-model on the riemannian symmetric space G/H. In terms of the gauge covariant G-valued field g [1] the action is given by

$$S = \tfrac{1}{2} \int d^2x \, (D_\mu g, D^\mu g) , \tag{1}$$

and the field equations read

$$D_\mu D^\mu g - D_\mu g g^{-1} D^\mu g = 0 , \tag{2}$$

where

$$D_\mu g = g \, \tfrac{1}{2}(1 - \dot{\sigma})(g^{-1} \partial_\mu g) \tag{3}$$

is the horizontal part of $\partial_\mu g$. Under the above assumptions, the invariance of S under global left G translations implies the conservation law

$$\partial_\mu j^\mu = 0 \tag{4}$$

for the Noether current

$$j_\mu = -D_\mu g g^{-1} . \tag{5}$$

In fact, (2) and (4) are equivalent.

Next, let us introduce the gauge-invariant formulation of these models. The elements of H being fixed points of the involution σ, the field

$$Q = \sigma(g) g^{-1} \tag{6}$$

is gauge invariant. With the help of the formula

$$\partial_\mu \sigma(g) = \sigma(g)\, \dot{\sigma}(g^{-1}\partial_\mu g)\,, \tag{7}$$

we compute

$$\tfrac{1}{2}\partial_\mu Q = -QD_\mu g g^{-1}\,, \tag{8}$$

so that

$$j_\mu = \tfrac{1}{2} Q^{-1}\partial_\mu Q\,. \tag{9}$$

Thus from (4), (9) and (6) we find that the gauge-invariant G-valued field Q obeys the following two equations:

$$\partial_\mu \partial^\mu Q - \partial_\mu Q Q^{-1} \partial^\mu Q = 0\,, \tag{10a}$$

$$\sigma(Q)Q = 1\,. \tag{10b}$$

Eq. (10a) being the field equation for the principal field, this tells us that the solution space of the G/H model appears as a subspace of the solution space of the principal G model due to the following theorem.

Theorem [5]: The smooth mapping

$$\begin{aligned} &\Phi\colon \mathrm{G/H} \to \mathrm{G}\,, \\ &\quad g\mathrm{H} \mapsto \Phi(g\mathrm{H}) = \sigma(g) g^{-1} \end{aligned} \tag{11}$$

is a diffeomorphism of G/H onto the closed totally geodesic submanifold

$$\mathrm{M}_\sigma = \{Q \in \mathrm{G} \mid \sigma(Q)Q = 1\} \tag{12}$$

of G.

For the proof, let us first define a smooth mapping

$$\begin{aligned} &\Psi\colon \mathrm{G} \to \mathrm{G}\,, \\ &\quad Q \mapsto \Psi(Q) = \sigma(Q)Q\,. \end{aligned} \tag{13}$$

The kernel of its tangent map $T_Q\Psi\colon T_Q\mathrm{G} \to T_{\sigma(Q)Q}\mathrm{G}$ at Q is

$$\mathrm{Ker}\, T_Q\Psi = Q\mathfrak{k}_Q\,, \tag{14}$$

where

$$\mathfrak{k}_Q = \{X \in \mathfrak{g} \mid \dot{\sigma}(X) + \mathrm{Ad}(Q)X = 0\}\,. \tag{15}$$

Hence the rank of Ψ at Q is equal to the dimension of $\mathfrak{h}$ and independent of Q, so that by the constant rank theorem [2], M_σ is a closed submanifold of G. Next, take any geodesic τ in G which is tangent to M_σ at Q; it has the form $\tau_t = Q \exp tX$ with $X \in \mathfrak{g}$, where

$$QX \in T_Q \mathrm{M}_\sigma = \mathrm{Ker}\, T_Q\Psi = Q\mathfrak{k}_Q\,,$$

i.e.,

$$\dot{\sigma}(X) + \mathrm{Ad}(Q)X = 0\,,$$

implying

$$\sigma(\tau_t)\tau_t = \sigma(Q)\exp t\dot{\sigma}(X)Q\exp tX = 1\;.$$

Hence M_σ is totally geodesic. Obviously, M_σ contains the image of Φ, so that we are left with proving the converse inclusion $\mathrm{M}_\sigma \subset \mathrm{im}\,\Phi$. To this end, let Q_0 be any point in M_σ. There always exists a finite number of points Q_i in M_σ and of geodesics τ_i in M_σ joining Q_{i-1} with Q_i, $1 \leqslant i \leqslant N$, such that $Q_N = \sigma(g_N)g_N^{-1} \in \mathrm{im}\,\Phi$. As shown before, we can write

$$Q_{N-1} = Q_N \exp tX, \qquad \text{with} \qquad X \in \mathscr{k}_{Q_N} \subset \mathscr{g}\,,$$

for some suitable value of the parameter t. But this implies

$$\begin{aligned}
&\sigma(\exp(-\tfrac{1}{2}tX)g_N)(\exp(-\tfrac{1}{2}X)g_N)^{-1}\\
&\quad = \exp(-\tfrac{1}{2}t\dot{\sigma}(X))\,\sigma(g_N)g_N^{-1}\,\exp(\tfrac{1}{2}tX)\\
&\quad = \exp(\tfrac{1}{2}t\,\mathrm{Ad}(Q_N)X)\,Q_N\,\exp(\tfrac{1}{2}tX)\\
&\quad = Q_N \exp tX\\
&\quad = Q_{N-1}\,,
\end{aligned}$$

i.e., $Q_{N-1} \in \mathrm{im}\,\Phi$. Repeating the argument for the Q_i with $0 \leqslant i \leqslant N$, we see that $Q_0 \in \mathrm{im}\,\Phi$, i.e., $M_\sigma \subset \mathrm{im}\,\Phi$, and the proof is complete.

As an example, let us consider the complex Grassmann manifold $\mathrm{U}(m+n)/\mathrm{U}(m)\times\mathrm{U}(n)$. Its involution is given by

$$\sigma(g) = \Theta g\Theta^{-1}, \qquad \Theta = \begin{pmatrix} \mathbb{1}_m & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}\,, \tag{16}$$

and writing $g = (X, Y) \in U(m+n)$, where $X(Y)$ is a matrix with $m+n$ rows and $m(n)$ columns, we find from (6)

$$Q = \Theta(2P - \mathbf{1})\,,$$

and $P = XX^+$ projects onto the m-dimensional subspace of $\mathbb{C}^{m+n}$ spanned by the column vectors of X. Thus we arrive at the well-known formulation of the grassmannian σ-models in terms of projector fields [6,1]. Eq. (10b) is the generalization of the "reduction condition" $g^2 = 1$ of [6].

The equations for Q have the obvious discrete symmetry

$$Q \mapsto Q^{-1}\,. \tag{17a}$$

In terms of the g fields, it is given simply by

$$g \mapsto \sigma(g)\,. \tag{17b}$$

In the light of (10a,b), the dual symmetry and the non-local charges of the G/H model are merely restrictions of the respective structures in the principal G model. For example, the dual symmetry [1] is given by

$$Q \mapsto Q^{(\gamma)} = \sigma(U^{(\gamma)}) Q U^{(\gamma)^{-1}} , \tag{18}$$

where $U^{(\gamma)}$ is defined by the compatible differential equations

$$\partial_\mu U^{(\gamma)} = \tfrac{1}{2} U^{(\gamma)} Q^{-1} \{(1 - c(\lambda))\partial_\mu Q - s(\lambda)\epsilon_{\mu\nu}\partial^\nu Q\} . \tag{19}$$

Here, $\gamma = e^{i\lambda}$, $c(\lambda) = \cos\lambda$, $s(\lambda) = \sin\lambda$ in the euclidean case and $\gamma = e^{\pm\lambda}$, $c(\lambda) = \pm\cosh\lambda$, $s(\lambda) = \pm\sinh\lambda$ in the Minkowski case. Observe that the transformation law (18) is compatible with (10b).

In addition, we note the following property of (19):

$$U^{(-\gamma)} = \sigma(U^{(\gamma)})\, U^{(-1)} \tag{20}$$

(*cf.* [4]), where an appropriate normalization of $U^{(\gamma)}$ has been assumed. Eq. (20) is easily derived from the formula

$$\dot\sigma(Q^{-1}\partial_\mu Q) + \partial_\mu Q Q^{-1} = 0 ,$$

which follows from (10b). In particular,

$$U^{(-1)} = Q, \qquad Q^{(-1)} = Q^{-1} . \tag{21}$$

3. Instantons and dual symmetry on hermitian symmetric spaces

Finally, let us turn to the question of instantons in the framework of symmetric spaces. As is well known [11,12], all σ-models on Kähler manifolds possess instantons solving self-duality equations which are nothing but the Cauchy-Riemann equations. Concerning the relationship of instantons and dual symmetry, we therefore confine our attention to the case where G/H is a hermitian symmetric space rather than just a riemannian one. Thus we have an $\mathrm{Ad}_{\mathfrak{g}}(\mathrm{H})$-invariant complex structure (i.e., a concept of multiplication by i) on $\mathfrak{k}$ which extends uniquely to a left G-invariant and right H-invariant complex structure on the horizontal bundle of $G \to G/H$ and to a left G-invariant complex structure on G/H.

(Anti-)instantons are the solutions of the (anti-)self duality equations

$$\epsilon_{\mu\nu} D_\nu g = \underset{(-)}{+} i D_\mu g. \tag{22}$$

Under the additional assumptions that H is compact, G is semisimple and acts effectively on G/H, there exists [3] an element J in the center of $\mathfrak{h}$ which induces the complex structure according to

$$[J, X] = 0, \quad \text{for } X \in \mathfrak{h} , \qquad [J, X] = iX, \quad \text{for } X \in \mathfrak{k} . \tag{23}$$

With the help of J, we can explicitly integrate the differential equations for the dual symmetry [1] on (anti-)instantons g:

$$U^{(\gamma)} = g\, \exp(\underset{(-)}{+}\lambda J) g^{-1} \,. \tag{24}$$

In fact, using (22) and (23), it is a straightforward exercise to check that (24) satisfies the correct differential equations.

Consequently,

$$g^{(\gamma)} = g\, \exp(\underset{(-)}{+}\lambda J)\,,$$

which means that on (anti-)instantons the dual symmetry reduces to a gauge transformation (*cf.* [13,1]).

As an example, let us again consider $\mathrm{U}(m+n)/\mathrm{U}(m) \times \mathrm{U}(n)$. We have

$$\mathfrak{h} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle|\, A \in \mathrm{U}(m), B \in \mathrm{U}(n) \right\}$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & -C^{+} \\ C & 0 \end{pmatrix} \middle|\, C \text{ complex } m \times n \text{ matrix} \right\}.$$

Then in $\mathfrak{k}$, multiplication by i is given by $C \to iC$ and

$$\begin{pmatrix} 0 & -C^{+} \\ C & 0 \end{pmatrix} \to \begin{pmatrix} 0 & iC^{+} \\ iC & 0 \end{pmatrix}.$$

It is induced by

$$J = \begin{bmatrix} i\,\dfrac{n}{m+n} & 0 \\ 0 & -i\,\dfrac{m}{m+n} \end{bmatrix},$$

and (24) takes the form

$$U^{(\gamma)} = \mathrm{e}^{(\mp)i\lambda m/(m+n)}(\mathbb{1} - (1 - \overset{(-)}{\gamma})P)$$

(*cf.* [1]).

Conversely, however, not every fixed point of the dual symmetry is an (anti-)instanton with respect to the complex structure of the space under consideration. For example, take any instanton of the O(3)/O(2) model [14] with non-zero topological charge. It may equally well be regarded as a solution of the $\mathbb{C}\mathrm{P}^N$ model ($N \geqslant 2$) with vanishing gauge field and, consequently, vanishing topological charge. So it is not an instanton of the $\mathbb{C}\mathrm{P}^N$ model, but of course a fixed point of the dual symmetry both of the O(3)/O(2) model and the $\mathbb{C}\mathrm{P}^N$ model.

References

[1] H. Eichenherr and M. Forger, Nucl. Phys. B155 (1979) 381.
[2] S. Helgason, Differential geometry, Lie groups, and symmetric spaces (Academic Press, New York, 1978).

[3] S. Kobayashi and K. Nomizu, Foundations of differential geometry vol. 2 (Interscience, New York, 1969).
[4] M. Lüscher and K. Pohlmeyer, Nucl. Phys. B137 (1978) 46.
[5] E. Cartan, La théorie de groupes finis et continus et l'Analysis situs, Mém. Sci. Math. Fasc. XLII (1930).
[6] W.E. Zakharov and A.W. Michailov, JETP (Sov. Phys.) 47 (1978) 1017.
[7] A.B. Zamolodchikov and Al. B. Zamolodchikov, Nucl. Phys. B133 (1978) 525.
[8] B. Berg, M. Karowski, P. Weisz and V. Kurak, Nucl. Phys. B134 (1978) 125.
[9] B. Berg and P. Weisz, Comm. Math. Phys. 67 (1979) 241.
[10] A. Neveu and N. Papanicolaou, Comm. Math. Phys. 58 (1978) 31.
[11] M. Forger, Instantons in σ-models, gauge theories and general relativity, Proc. Diff. Geom. Math. Phys. (Clausthal-Zellerfeld, 1978).
[12] A.M. Perelomov, Comm. Math. Phys. 63 (1978) 237.
[13] A. D'Adda, M. Lüscher and P. Di Vecchia, Les Houches Lectures (1978).
[14] A.A. Belavin and A.M. Polyakov, JETP Lett. 22 (1975) 245.