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ON THE DUAL SYMMETRY OF THE NON-LINEAR SIGMA MODELS

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We analyze the dual symmetry which is responsible for the existence of infinitely many conserved non-local charges in the classical two-dimensional non-linear σ models. For compact global symmetry groups, we prove that the σ model has the dual symmetry if and only if the field takes values in a symmetric space.

1. Introduction

One of the most interesting features of the classical two-dimensional O(N) invariant non-linear σ models is the existence of a highly non-trivial hidden symmetry which was discovered in 1975 by Pohlmeyer [1]. Starting with one solution of the classical equations of motion, this "dual symmetry" generates a whole one-parameter family of new solutions and gives rise to the existence of infinitely many conserved non-local charges [2].

It is well-known that these charges can properly be defined as operators in the quantized σ model [3]. Their action on the asymptotic states strongly constrains the physics of the σ models: in each scattering process, the number of particles is conserved. On the basis of general axiomatic assumptions this fact leads to the factorization of the S-matrix into two-body amplitudes [4] which have been calculated exactly [5].

The existence of the dual symmetry is not specific to the O(N) invariant σ models. In fact, the question whether all σ models based on homogeneous spaces G/H should be integrable was one of the motivations for the introduction of the $\mathbb{C}P^N$ σ models [6]. The outcome of these investigations was that there are very special orbits of the adjoint representation of SU(N) which admit a dual symmetry transformation, namely the complex Grassmann manifolds

$$U(m+n)/(U(m) \times U(n)) .$$

On the other hand, the principal field introduced in [8] – it corresponds to the homogenous space G/{1} – has the symmetry also. So, up to now, we have a variety of integrable σ models but we lack a unifying point of view connecting all these models.

This rather confusing situation was the starting point of our investigation. In sect. 2, beginning with an intrinsic definition of the σ model on G/H, we derive necessary and sufficient conditions for the existence of the dual symmetry, and consequently of the higher conserved charges, in terms of the Lie algebras of G and H. These conditions involve two, at first sight different, situations. In sect. 3 we discuss the principal field and thereby show that both situations are essentially identical. Our general considerations are illustrated in sect. 4 by the example of the Grassmannian fields containing the O(N) invariant σ models as well as the $\mathbb{C}P^N$ σ models.

2. The general result

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and H $\subset$ G be a closed subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ such that the group $\mathrm{Ad}_{\mathfrak{g}}(\mathrm{H})$ of linear transformations on $\mathfrak{g}$ of the form Ad(h), $h \in$ H, is compact *. Then there exists an $\mathrm{Ad}_{\mathfrak{g}}(\mathrm{H})$ – invariant inner product (. , .) on $\mathfrak{g}$ †, and writing $\mathfrak{k}$ for the orthogonal complement $\mathfrak{h}^{\perp}$ of $\mathfrak{h}$ and π, $1-\pi$ for the orthogonal projection from $\mathfrak{g}$ onto $\mathfrak{h}$, $\mathfrak{k}$ with kernel $\mathfrak{k}$, $\mathfrak{h}$, respectively, we have $\mathrm{Ad}_{\mathfrak{g}}(\mathrm{H})\ \mathfrak{k} \subset \mathfrak{k}$, and in particular

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}\,, \qquad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}\,. \tag{2.1}$$

Let G/H be the homogenous space of left cosets gH of H, $g \in$ G, and consider the natural projection ρ: G $\to$ G/H as a fibration: this defines a principal H-bundle on which G operates transitively from the left. In other words, G acts on G and on G/H by left multiplication (transitively on both), while H acts on G and on G/H by right multiplication (trivially on G/H), and both actions are compatible with the projection ρ. This principal H-bundle carries a natural left-invariant connection, as well as other additional structures. On the one hand, the $\mathrm{Ad}_{\mathfrak{g}}(\mathrm{H})$-invariant inner product (. , .) on $\mathfrak{g}$ and its restriction to $\mathfrak{k}$ extend uniquely to a left G-invariant and right H-invariant Riemannian metric on G and to a left G-invariant Riemannian metric on G/H, respectively, which will both be denoted by (. , .) as well. On the other hand, there exist two distinguished 1-forms on G with values in $\mathfrak{g}$, namely the left-invariant Maurer-Cartan form $g^{-1}dg$ and the right-invariant Maurer-Cartan form dgg^{-1}. The natural left-invariant connection mentioned above can now be described in several equivalent ways:

* Ad denotes the adjoint representation.

† Observe that we do not have to assume G itself, or even H itself, to be compact, although H will have to be compact if it does not meet the center of G except at the unit element. However, if G is compact, the inner product (. , .) on $\mathfrak{g}$ will be Ad(G)-invariant, and the Riemannian metric (. , .) on G will be bi-invariant.

(a) Left translation of the various tangent spaces of G to the tangent space $\mathfrak{g}$ of G at the unit element takes the vertical spaces to $\mathfrak{h}$ and the horizontal spaces to $\mathfrak{k}$.

(b) The horizontal bundle is the orthogonal complement of the vertical bundle.

(c) The connection form is the vertical part of the left-invariant Maurer-Cartan form on G:

$$A = \pi(g^{-1} dg) . \tag{2.2}$$

For simplicity, we write π for the vertical projection and $1 - \pi$ for the horizontal projection at each point.

With this machinery at hand, we study non-linear σ models in two dimensions, where the field $q(x)$ takes values in G/H and is (locally) lifted to a field $g(x)$ taking values in G, which leads to a natural equivalence of two such liftings:

$g_2(x) \sim g_1(x) \Leftrightarrow$ there exists a field $h(x)$ taking values in H such that

$$g_2(x) = g_1(x)\, h(x) . \tag{2.3}$$

This is the by now familiar method [7,9] of introducing a gauge symmetry into a non-linear σ model, H being the gauge group. It implies that on gauge-invariant quantities such as the field $q(x)$, ordinary derivatives are relevant, while on gauge covariant quantities such as the field $g(x)$, they have to be replaced by covariant derivatives. In particular, we define *

$$D_\mu g = \partial_\mu g - g A_\mu , \tag{2.4}$$

which can also be written in the form

$$g A_\mu = \pi(\partial_\mu g) \qquad = \text{vertical part of } \partial_\mu g , \tag{2.5}$$

$$D_\mu g = (1 - \pi)(\partial_\mu g) = \text{horizontal part of } \partial_\mu g . \tag{2.6}$$

Using these covariant derivatives, we introduce a gauge-invariant current

$$j_\mu = -\alpha D_\mu g g^{-1} , \tag{2.7}$$

and a gauge-covariant current

$$k_\mu = \alpha g^{-1} D_\mu g = \alpha(1 - \pi)(g^{-1} \partial_\mu g) , \tag{2.8}$$

where $\alpha \in \mathbb{R}$ is some suitable constant introduced for later convenience. Again, the gauge transformation law ($g \to gh$ implies $j_\mu \to j_\mu$, $k_\mu \to h^{-1} k_\mu h$) dictates the kind of derivatives to be used; in particular

$$D_\mu k_\nu = \partial_\mu k_\nu + [A_\mu, k_\nu] . \tag{2.9}$$

* In expressions like gA_μ, $g^{-1}\partial_\mu g$, $D_\mu g g^{-1}$ etc., the simplified product notation stands for the action of G on its tangent bundle TG by left and right translations.

The action for the non-linear σ model is

$$S = \tfrac{1}{2}\int d^2x(\partial_\mu q, \partial^\mu q) = \tfrac{1}{2}\int d^2x(D_\mu g, D^\mu g)\,, \tag{2.10}$$

and the field equations read

$$D_\mu D^\mu g - D_\mu g g^{-1} D^\mu g = 0\,. \tag{2.11}$$

They imply that j_μ is conserved and k_μ is covariantly conserved:

$$\partial_\mu j^\mu = 0\,, \tag{2.12}$$

$$D_\mu k^\mu = 0\,. \tag{2.13}$$

Due to (2.1) and

$$\partial_\mu A_\nu - \partial_\nu A_\mu = -\pi(g^{-1}\partial_\mu g g^{-1}\partial_\nu g) + \pi(g^{-1}\partial_\nu g g^{-1}\partial_\mu g)$$

$$= -\pi[g^{-1}D_\mu g + A_\mu, g^{-1}D_\nu g + A_\nu]\,,$$

the curvature form is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = -\pi[g^{-1}D_\mu g, g^{-1}D_\nu g]\,. \tag{2.14}$$

Observe that in this formulation of non-linear σ models, the two groups H and G play entirely different roles: H is a gauge group, while G is a global symmetry group. In particular, j is just the Noether current corresponding to the left G-invariance of the theory. To tie up with the usual notation, we introduce (local) coordinates ξ, η which are complex coordinates in the Euclidean case and light-cone coordinates in the Minkowski case:

$$\xi = \tfrac{1}{2}\bar{z} = \tfrac{1}{2}(x - iy)\,, \qquad \eta = \tfrac{1}{2}z = \tfrac{1}{2}(x + iy)\,, \qquad \text{with } x = x^0 = x_0, y = x^1 = x_1,$$

$$\partial_\xi = 2\partial/\partial\bar{z} = \partial_0 + i\partial_1\,, \qquad \partial_\eta = 2\partial/\partial z = \partial_0 - i\partial_1\,, \qquad \text{(Euclidean case)}\,,$$

$$\xi = \tfrac{1}{2}(t + x)\,, \qquad \eta = \tfrac{1}{2}(t - x)\,, \qquad \text{with } t = x^0 = x_0, x = x^1 = -x_1,$$

$$\partial_\xi = \partial_0 + \partial_1\,, \qquad \partial_\eta = \partial_0 - \partial_1\,, \qquad \text{(Minkowski case)}\,. \tag{2.15}$$

The action for the non-linear σ model becomes

$$S = \tfrac{1}{2}\int d^2x(\partial_\xi q, \partial_\eta q) = \tfrac{1}{2}\int d^2x(D_\xi g, D_\eta g)\,, \tag{2.10}$$

and the field equations read

$$D_\xi D_\eta g + D_\eta D_\xi g - D_\xi g g^{-1} D_\eta g - D_\eta g g^{-1} D_\xi g = 0\,, \tag{2.11}$$

while the conservation laws take the form

$$\partial_\xi j_\eta + \partial_\eta j_\xi = 0\,, \tag{2.12}$$

$$D_\xi k_\eta + D_\eta k_\xi = 0\,, \tag{2.13}$$

and the curvature form is

$$F_{\xi\eta} = \partial_\xi A_\eta - \partial_\eta A_\xi + [A_\xi, A_\eta] = -\pi[g^{-1}D_\xi g, g^{-1}D_\eta g] \ . \tag{2.15}$$

According to the standard strategy [1,2,6], our definition of the dual symmetry is based on the following one-parameter family of differential equations

$$\begin{aligned} \partial_\xi U^{(\gamma)} &= (1-\gamma^{-1})\, U^{(\gamma)} j_\xi = -\alpha(1-\gamma^{-1})\, U^{(\gamma)} D_\xi g g^{-1} \ , \\ \partial_\eta U^{(\gamma)} &= (1-\gamma)\, U^{(\gamma)} j_\eta \quad = -\alpha(1-\gamma)\, U^{(\gamma)} D_\eta g g^{-1} \ . \end{aligned} \tag{2.16}$$

j being gauge invariant, so is the G-valued function $U^{(\gamma)}$. Existence and uniqueness of $U^{(\gamma)}$ (up to a constant factor which is specified by setting $U^{(\gamma)} = \mathbb{1}$ at some prescribed point of space-time) are guaranteed by the Frobenius theorem [10] if the expression

$$(1-\gamma)\,\partial_\xi j_\eta - (1-\gamma^{-1})\,\partial_\eta j_\xi + (1-\gamma^{-1})(1-\gamma)[j_\xi, j_\eta]$$

vanishes. Due to (2.12), this integrability condition reduces to the γ-independent equation

$$\partial_\xi j_\eta - \partial_\eta j_\xi + 2[j_\xi, j_\eta] = 0 \ . \tag{2.17}$$

But

$$\partial_\xi j_\eta - \partial_\eta j_\xi + 2[j_\xi, j_\eta] = \alpha g(2\alpha - 1 - \pi)[g^{-1}D_\xi g, g^{-1}D_\eta g]\, g^{-1} \ .$$

This leads us to distinguish two cases:

Case A: $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$. This implies $\pi[g^{-1}D_\xi g, g^{-1}D_\eta g] = [g^{-1}D_\xi g, g^{-1}D_\eta g]$, and to satisfy (2.17), we choose $\alpha = 1$.

Case B: $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$. This implies $\pi[g^{-1}D_\xi g, g^{-1}D_\eta g] = 0$, and to satisfy (2.17), we choose $\alpha = \frac{1}{2}$. Notice that in this case, $F = 0$ and $[\mathfrak{g}, \mathfrak{k}] \subset \mathfrak{k}$, so that the natural connection in ρ: G $\to$ G/H is flat and right-invariant as well as left-invariant.

Before commenting on the significance of these two cases, let us complete our discussion of the dual symmetry.

Case A. The $U^{(\gamma)}$ generate a one-parameter family of transformations on the space of solutions of the equations of motion simply by left translation

$$g \to g^{(\gamma)} = U^{(\gamma)} g \ . \tag{2.18}$$

We have $g^{(1)} = g$ and

$$\partial_\xi g^{(\gamma)} = \gamma^{-1} U^{(\gamma)} D_\xi g + g^{(\gamma)} A_\xi \ ,$$

where $U^{(\gamma)} D_\xi g$ is horizontal and $g^{(\gamma)} A_\xi$ is vertical, which implies

$$D_\xi^{(\gamma)} g^{(\gamma)} = \gamma^{-1} U^{(\gamma)} D_\xi g \ , \qquad A_\xi^{(\gamma)} = A_\xi \ . \tag{2.19}$$

Similarly

$$D_\eta^{(\gamma)} g^{(\gamma)} = \gamma U^{(\gamma)} D_\eta g\,, \qquad A_\eta^{(\gamma)} = A_\eta\,. \tag{2.20}$$

Thus the action (2.10) is invariant under the transformations (2.18), and solutions of the field equations are taken to solutions of the field equations.

Case B. Due to the factor $\alpha = \frac{1}{2}$ in the definition of j, the transformation (2.18) does not leave the action invariant; thus it has to be replaced by a somewhat more complicated ansatz. To this end, let us consider the following one-parameter family of differential equations

$$D_\xi V^{(\gamma)} = (1-\gamma^{-1})\, V^{(\gamma)} k_\xi = \tfrac{1}{2}(1-\gamma^{-1})\, V^{(\gamma)} g^{-1} D_\xi g\,,$$

$$D_\eta V^{(\gamma)} = (1-\gamma)\, V^{(\gamma)} k_\eta = \tfrac{1}{2}(1-\gamma)\, V^{(\gamma)} g^{-1} D_\eta g\,. \tag{2.21}$$

k being gauge covariant ($g \to gh$ implies $k \to h^{-1}kh$), so is the G-valued function $V^{(\gamma)}$ ($g \to gh$ implies $V^{(\gamma)} \to V^{(\gamma)}h$). Thus (2.21) can be rewritten in the form

$$\partial_\xi V^{(\gamma)} = (1-\gamma^{-1})\, V^{(\gamma)} k_\xi + V^{(\gamma)} A_\xi = \tfrac{1}{2}(1-\gamma^{-1})\, V^{(\gamma)} g^{-1} D_\xi g + V^{(\gamma)} A_\xi\,,$$

$$\partial_\eta V^{(\gamma)} = (1-\gamma)\, V^{(\gamma)} k_\eta + V^{(\gamma)} A_\eta = \tfrac{1}{2}(1-\gamma)\, V^{(\gamma)} g^{-1} D_\eta g + V^{(\gamma)} A_\eta\,. \tag{2.22}$$

Again, existence and uniqueness of $V^{(\gamma)}$ (up to a constant factor which is specified by setting $V^{(\gamma)} = \mathbb{1}$ at some prescribed point of space-time) are guaranteed by the Frobenius theorem [10] if the expression

$$\partial_\xi((1-\gamma)\, k_\eta + A_\eta) - \partial_\eta((1-\gamma^{-1})\, k_\xi + A_\xi)$$

$$+ [(1-\gamma^{-1})\, k_\xi + A_\xi, (1-\gamma)\, k_\eta + A_\eta]$$

$$= (1-\gamma)\, D_\xi k_\eta - (1-\gamma^{-1})\, D_\eta k_\xi + (1-\gamma^{-1})(1-\gamma)\, [k_\xi, k_\eta] + F_{\xi\eta}$$

vanishes. Due to (2.13) and because of $F_{\xi\eta} = 0$, this integrability condition reduces to the γ-independent equation

$$D_\xi k_\eta - D_\eta k_\xi + 2\,[k_\xi, k_\eta] = 0\,. \tag{2.23}$$

But

$$D_\xi k_\eta - D_\eta k_\xi + 2\,[k_\xi, k_\eta] = -\tfrac{1}{2} F_{\xi\eta} = 0\,.$$

Now the $U^{(\gamma)}$ and $V^{(\gamma)}$ generate a one-parameter family of transformations on the space of solutions of the field equations by

$$g \to g^{(\gamma)} = U^{(\gamma)} g\, V^{(\gamma)-1}\, V^{(1)}\,. \tag{2.24}$$

We have $g^{(1)} = g$ and

$$\partial_\xi g^{(\gamma)} = \gamma^{-1}\, U^{(\gamma)} D_\xi g\, V^{(\gamma)-1}\, V^{(1)} + g^{(\gamma)} A_\xi\,,$$

where $U^{(\gamma)} D_\xi g V^{(\gamma)-1} V^{(1)}$ is horizontal and $g^{(\gamma)} A_\xi$ is vertical, which implies

$$D_\xi^{(\gamma)} g^{(\gamma)} = \gamma^{-1} U^{(\gamma)} D_\xi g V^{(\gamma)-1} V^{(1)} , \qquad A_\xi^{(\gamma)} = A_\xi . \tag{2.25}$$

Similarly

$$D_\eta^{(\gamma)} g^{(\gamma)} = \gamma U^{(\gamma)} D_\eta g V^{(\gamma)-1} V^{(1)} , \qquad A_\eta^{(\gamma)} = A_\eta . \tag{2.26}$$

Thus once again, the action (2.10) is invariant under the transformations (2.24), and solutions of the field equations are taken to solutions of the field equations.

In both cases, the current transforms according to

$$j_\xi \to j_\xi^{(\gamma)} = \gamma^{-1} U^{(\gamma)} j_\xi U^{(\gamma)-1} ,$$

$$j_\eta \to j_\eta^{(\gamma)} = \gamma U^{(\gamma)} j_\eta U^{(\gamma)-1} , \tag{2.27}$$

and performing a Taylor expansion in $w = (\gamma - 1)/(\gamma + 1)$ around $w = 0$, one obtains the desired infinite series of conserved (non-local) currents and charges [2] starting with

$$Q_1 = \int_{-\infty}^{+\infty} \mathrm{d}x \, j_0(t, x) ,$$

$$Q_2 = 2 \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{x} \mathrm{d}x \, j_0(t, y) \, j_0(t, x) - \int_{-\infty}^{+\infty} \mathrm{d}x \, j_1(t, x) .$$

For the convenience of the reader, we rewrite the most important formulae in terms of the standard (local) coordinates x^μ, using the Hodge star operator which on 1-forms in two dimensions is given by ${}^*\omega^\mu = \epsilon^{\mu\nu} \omega_\nu$.

Euclidean case: with the conventions $\gamma = \mathrm{e}^{i\lambda}$ and $\epsilon^{01} = \epsilon_{01} = 1$, and observing that $\epsilon^{\mu\kappa} \epsilon_{\kappa\nu} = -\delta^\mu_\nu$, i.e., $*^2 = -1$, we find

$$\partial_\mu U^{(\gamma)} = U^{(\gamma)} \{ j_\mu (1 - \cos \lambda) - {}^*j_\mu \sin \lambda \} , \tag{2.16}$$

$$D_\mu V^{(\gamma)} = V^{(\gamma)} \{ k_\mu (1 - \cos \lambda) - {}^*k_\mu \sin \lambda \} , \tag{2.21}$$

$$j_\mu \to j_\mu^{(\gamma)} = U^{(\gamma)} \{ j_\mu \cos \lambda + {}^*j_\mu \sin \lambda \} U^{(\gamma)-1} . \tag{2.27}$$

Minkowski case: with the conventions $\gamma = \mathrm{e}^\lambda$ and $-\epsilon^{01} = \epsilon_{01} = 1$, and observing that $\epsilon^{\mu\kappa} \epsilon_{\kappa\nu} = \delta^\mu_\nu$, i.e., $*^2 = 1$, we find

$$\partial_\mu U^{(\gamma)} = U^{(\gamma)} \{ j_\mu (1 - \cosh \lambda) - {}^*j_\mu \sinh \lambda \} , \tag{2.16}$$

$$D_\mu V^{(\gamma)} = V^{(\gamma)} \{ k_\mu (1 - \cosh \lambda) - {}^*k_\mu \sinh \lambda \} , \tag{2.21}$$

$$j_\mu \to j_\mu^{(\gamma)} = U^{(\gamma)} \{ j_\mu \cosh \lambda + {}^*j_\mu \sinh \lambda \} U^{(\gamma)-1} . \tag{2.27}$$

In both cases

$$\partial_\mu {}^*j^\mu + [j_\mu, {}^*j^\mu] = 0 \,, \tag{2.17}$$

$$D_\mu {}^*k^\mu + [k_\mu, {}^*k^\mu] = 0 \,. \tag{2.23}$$

These formulae provide a certain amount of motivation for the term "dual symmetry" because its definition is based on mixing the current with its Hodge dual *via* a rotation in the plane spanned by the two.

We still have to clarify the role of the two cases A and B for which we have now shown the existence of the dual symmetry, and we claim that in both cases we are actually dealing with a symmetric space situation [11,12].

In case A, this is obvious because if G/H is a Riemannian symmetric space, we have (2.1) and

$$[\mathscr{k}, \mathscr{k}] \subset \mathscr{h} \,,$$

and conversely, whenever we have (2.1) and

$$[\mathscr{k}, \mathscr{k}] \subset \mathscr{h} \,,$$

the linear map $\dot{\sigma}$: $\mathscr{g} \to \mathscr{g}$ which is +1 on $\mathscr{h}$ and −1 on $\mathscr{k}$ is an isometric Lie algebra automorphism, and if it can be lifted to an isometric Lie group automorphism σ: G → G (e.g., always if G is simply connected), G/H is a Riemannian symmetric space.

In case B, on the other hand, the relation is less direct: The first step is to observe that whenever we have (2.1) and

$$[\mathscr{k}, \mathscr{k}] \subset \mathscr{k} \,,$$

G/H is canonically isomorphic to the connected normal Lie subgroup K in G generated by the ideal $\mathscr{k}$ in $\mathscr{g}$, and if the action of H on $\mathscr{k}$ (by Lie algebra automorphisms) can be lifted to an action of H on K (by Lie group automorphisms) (e.g., always if K is simply connected), G is the semidirect product [14] H ⊗ K of H and K with respect to the latter *. Thus in case B, we are actually dealing with the principal σ model [8] where the field takes values in a connected Lie group K, decorated with a gauge symmetry which is more or less superfluous; in other words, there is no loss of generality in assuming $\mathscr{h} = 0$ right from the start. The second step is to realize that K is canonically isomorphic to the symmetric space

$$\mathrm{K} \times \mathrm{K}/\Delta\mathrm{K} \,,$$

and that under this isomorphism, the principal σ model appears as a special example of case A, at least under the physically reasonable assumption that K is compact, which makes K × K/ΔK a Riemannian symmetric space; we give the details in sect. 3.

* However, if G is compact, we have $[\mathscr{h}, \mathscr{k}] = 0$, implying that modulo a discrete central subgroup, G is the direct product H × K of H and K.

Keeping in mind our additional assumptions which have been made for technical reasons—G is simply connected in case A, K is compact in case B—we may summarize our result in the following theorem:

A non-linear σ model in two dimensions, where the field takes values in a homogenous space G/H as explained at the beginning of this section, possesses the dual symmetry if and only if G/H is a symmetric space.

Of course, this theorem does not rule out the existence of higher conserved charges in other types of non-linear σ models. However, this would require a generalization of the concept of dual symmetry going beyond (2.16), (2.18), and so far, nothing in this direction seems to be known.

3. The principal field

In this section, we discuss in detail the principal field [8]. We show that this theory can be reformulated as a σ model on a symmetric space.

The principal field model is by definition the σ model on G/H where H = $\{\mathbb{1}\}$. So we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} ,$$

where

$$\mathfrak{h} = \{0\} , \qquad \mathfrak{k} \simeq \mathfrak{g} ,$$

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} .$$

Writing Γ for the field which takes values in G, the gauge fields vanish. The dual symmetry is given by

$$\Gamma \to \Gamma^{(\gamma)} = U^{(\gamma)} \Gamma V^{(\gamma)-1} , \tag{3.1}$$

where

$$\begin{aligned}
\partial_\xi U^{(\gamma)} &= (1 - \gamma^{-1})\, U^{(\gamma)} [-\tfrac{1}{2}\partial_\xi \Gamma \Gamma^{-1}] , \\
\partial_\eta U^{(\gamma)} &= (1 - \gamma)\, U^{(\gamma)} [-\tfrac{1}{2}\partial_\eta \Gamma \Gamma^{-1}] , \\
\partial_\xi V^{(\gamma)} &= (1 - \gamma^{-1})\, V^{(\gamma)} [\tfrac{1}{2}\Gamma^{-1} \partial_\xi \Gamma] , \\
\partial_\eta V^{(\gamma)} &= (1 - \gamma)\, V^{(\gamma)} [\tfrac{1}{2}\Gamma^{-1} \partial_\eta \Gamma] .
\end{aligned} \tag{3.2}$$

Now it is well-known [12] that every connected Lie group G can be considered as a symmetric space: let

$$\Delta \mathrm{G} = \{(g, g) \,|\, g \in \mathrm{G}\}$$

be the diagonal of G $\times$ G, and define σ: G $\times$ G $\to$ G $\times$ G by

$$\sigma(g, g') = (g', g) .$$

Then $G \times G/\Delta G$ is a symmetric space with involution σ. The set of fixed points of σ coincides with ΔG. G itself is diffeomorphic with $G \times G/\Delta G$, the diffeomorphism being induced by the map

$$G \times G \to G ,$$

$$(g_1, g_2) \to \Gamma = g_1 g_2^{-1} .$$

This motivates us to study the σ model on $G \times G/\Delta G$. In this case,

$$\mathcal{g} \oplus \mathcal{g} = \mathcal{h} \oplus \mathcal{k} ,$$

where

$$\mathcal{h} = \{(a, a) | a \in \mathcal{g}\} , \qquad \mathcal{k} = \{(a, -a) | a \in \mathcal{g}\} ,$$

$$[\mathcal{k}, \mathcal{k}] \subset \mathcal{h} .$$

Obviously,

$$\pi(a, b) = (\tfrac{1}{2}(a + b), \tfrac{1}{2}(a + b)) .$$

Writing $g = (g_1, g_2)$ for the field which takes values in $G \times G$, we have the gauge field

$$A_\mu = (\tfrac{1}{2}(g_1^{-1}\partial_\mu g_1 + g_2^{-1}\partial_\mu g_2), \tfrac{1}{2}(g_1^{-1}\partial_\mu g_1 + g_2^{-1}\partial_\mu g_2)) .$$

The dual symmetry is given by

$$(g_1, g_2) \to (g_1^{(\gamma)}, g_2^{(\gamma)}) = (U_1^{(\gamma)}, U_2^{(\gamma)})(g_1, g_2) = (U_1^{(\gamma)} g_1, U_2^{(\gamma)} g_2) , \tag{3.3}$$

where

$$(\partial_\xi U_1^{(\gamma)}, \partial_\xi U_2^{(\gamma)}) = (1 - \gamma^{-1})(U_1^{(\gamma)}, U_2^{(\gamma)})[-D_\xi(g_1, g_2) \cdot (g_1, g_2)^{-1}] ,$$

$$(\partial_\eta U_1^{(\gamma)}, \partial_\eta U_2^{(\gamma)}) = (1 - \gamma)(U_1^{(\gamma)}, U_2^{(\gamma)})[-D_\eta(g_1, g_2) \cdot (g_1, g_2)^{-1}] . \tag{3.4}$$

Evaluating the covariant derivative and using the embedding prescription $\Gamma = g_1 g_2^{-1}$, we arrive at

$$(\partial_\xi U_1^{(\gamma)}, \partial_\xi U_2^{(\gamma)}) = (1 - \gamma^{-1})(U_1^{(\gamma)}, U_2^{(\gamma)})(-\tfrac{1}{2}\partial_\xi \Gamma \Gamma^{-1}, \tfrac{1}{2}\Gamma^{-1}\partial_\xi \Gamma) ,$$

$$(\partial_\eta U_1^{(\gamma)}, \partial_\eta U_2^{(\gamma)}) = (1 - \gamma)(U_1^{(\gamma)}, U_2^{(\gamma)})(-\tfrac{1}{2}\partial_\eta \Gamma \Gamma^{-1}, \tfrac{1}{2}\Gamma^{-1}\partial_\eta \Gamma) . \tag{3.5}$$

With the identification $U^{(\gamma)} = U_1^{(\gamma)}$, $V^{(\gamma)} = U_2^{(\gamma)}$, and comparing (3.3), (3.5) with (3.1), (3.2), we see that the principal field is in fact an integrable σ model on a symmetric space.

Moreover, from these considerations and the discussion at the end of sect. 2, it follows that in the whole case B of sect. 2 we are dealing with integrable σ models on symmetric spaces.

Thus our general analysis has shown that all integrable σ models have a common property which is responsible for the existence of the dual symmetry: contrary to the claim expressed in [8], the field has to take its values not on an arbitrary homogenous space, but on a symmetric one. From this point of view, the trouble with the principal field does not arise at all: If the principal field is defined properly on the symmetric space $G \times G/\Delta G$, the field when lifted correctly to $G \times G$ admits the general dual transformation (2.16), (2.18). The fact that some special symmetric spaces are diffeomorphic with Lie groups enables us to simplify the corresponding principal field theory. However, the price one has to pay is the introduction of an auxiliary transformation $V^{(\gamma)}$.

4. The Grassmannian fields

The σ model on the complex Grassmann manifold $U(m+n)/(U(m) \times U(n))$ provides a useful example for the general structure discussed in sect. 2. For $n = 1$, it reduces to the $\mathbb{C}P^m$ σ model [6,7]. Following sect. 2, we start with a $U(m+n)$ valued field g

$$g = (X, Y)\,, \qquad g^+ g = \mathbb{1}\,,$$

where $X(Y)$ is a matrix with $m+n$ rows and $m(n)$ columns. We introduce the orthogonal projectors

$$P = XX^+\,, \qquad \bar{P} = YY^+\,, \qquad P + \bar{P} = \mathbb{1}\,,$$

which map $\mathbb{C}^{m+n}$ onto the $m(n)$ dimensional subspaces spanned by the column vectors of $X(Y)$. Gauge transformations act on g as

$$(X, Y) \to (Xh_1, Yh_2)\,,$$

where

$$\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \in (U(m) \times U(n))\,.$$

The gauge field is

$$A_\mu = \begin{pmatrix} X^+\partial_\mu X & 0 \\ 0 & Y^+\partial_\mu Y \end{pmatrix}\,,$$

and the covariant derivative is

$$D_\mu(X, Y) = (\bar{P}\partial_\mu X, P\partial_\mu Y) = (D_\mu X, D_\mu Y)\,.$$

Of course, the action and the field equations split into two separate parts:

$$S = \tfrac{1}{2}\int d^2x\{\mathrm{tr}[D_\mu X(D^\mu X)^+] + \mathrm{tr}[D_\mu Y(D^\mu Y)^+]\}\,, \tag{4.1}$$

$$D_\mu D^\mu X + X(D_\mu X)^+ D^\mu X = 0\,, \qquad D_\mu D^\mu Y + Y(D_\mu Y)^+ D^\mu Y = 0\,. \tag{4.2}$$

We may rewrite (4.1), (4.2) in terms of the projector fields to make contact with the formulation used in [6,8]:

$$S = \frac{1}{2}\int d^2x \; \mathrm{tr}(\partial_\mu P \partial^\mu P) = \frac{1}{2}\int d^2x \; \mathrm{tr}(\partial_\mu \bar{P} \partial^\mu \bar{P}) \tag{4.1}$$

$$[\Box P, P] = 0 \,, \qquad [\Box \bar{P}, \bar{P}] = 0 \,. \tag{4.2}$$

According to (2.16), (2.18), the dual symmetry is defined by

$$(X, Y) \to U^{(\gamma)}(X, Y) \,,$$

where

$$\partial_\xi U^{(\gamma)} = (1 - \gamma^{-1})\, U^{(\gamma)} [-D_\xi(X, Y) \cdot (X, Y)^+] \,,$$

$$\partial_\eta U^{(\gamma)} = (1 - \gamma)\, U^{(\gamma)} [-D_\eta(X, Y) \cdot (X, Y)^+] \,.$$

Observing that

$$-D_\xi(X, Y) \cdot (X, Y)^+ = [P, \partial_\xi P] = [\bar{P}, \partial_\xi \bar{P}] \,,$$

we recover the well-known transformation

$$P \to U^{(\gamma)} P U^{(\gamma)+} \,, \qquad \bar{P} \to U^{(\gamma)} \bar{P} U^{(\gamma)+} \,,$$

for the projector fields.

The same discussion applies to the real Grassmann manifold $SO(m + n)/SO(m) \times SO(n)$ if we replace + (hermitian adjoint) by T (transpose) throughout. For $n = 1$, we may thus rederive the dual symmetry in the $SO(m + 1)$ invariant σ model found by Pohlmeyer [1]. In this case, the projector $\bar{P}$ takes the form

$$P_{ij} = q_i q_j \,,$$

and $\boldsymbol{q} = (q_1, ..., q_{m+1})$, $\|\boldsymbol{q}\| = 1$, is the field vector of the previously studied $SO(m + 1)$ invariant σ model.

We should like to conclude this section with a remark concerning the influence of the dual transformation on instantons. As is well-known [15], the second homotopy group of $U(m + n)/(U(m) \times U(n))$ is isomorphic to $\mathbb{Z}$, and consequently the Euclidean P fields are labelled by their integer topological charge

$$Q(P) = \frac{i}{4\pi m}\int d^2x \; \epsilon_{\mu\nu} \; \mathrm{tr}(P[\partial_\mu P, \partial_\nu P]) \,.$$

The action is bounded from below by

$$S(P) \geqslant 2\pi m \, |Q(P)| \,,$$

and equality holds if and only if P is (anti) self-dual:

$$\partial P = \underset{(-)}{+} [\partial P, P] \,, \qquad \text{with } \partial = \partial_0 - i\partial_1 \,, \qquad \bar{\partial} = \partial_0 + i\partial_1 \,.$$

How does the dual symmetry act on a self-dual field P? From

$$\partial U^{(\gamma)} = -(1-\gamma)\, U^{(\gamma)} \partial P\,,$$

$$\bar{\partial} U^{(\gamma)} = (1-\bar{\gamma})\, U^{(\gamma)} \bar{\partial} P\,, \tag{4.3}$$

we guess the explicit solution

$$U^{(\gamma)} = \mathbb{1} - (1-\gamma)\, P\,,$$

where $U^{(\gamma)}$ has been suitably normalized. (For anti-self-dual P, the solution of

$$\partial U^{(\gamma)} = (1-\gamma)\, U^{(\gamma)} \partial P\,,$$

$$\bar{\partial} U^{(\gamma)} = -(1-\bar{\gamma})\, U^{(\gamma)} \bar{\partial} P\,, \qquad \gamma \in \mathbb{C}\,, \qquad |\gamma| = 1\,, \tag{4.3}$$

is

$$U^{(\gamma)} = \mathbb{1} - (1-\bar{\gamma})\, P\,.)$$

So we have $P^{(\gamma)} = P$ if P is (anti) self-dual.

Thus the Grassmannian instantons are invariant under the dual symmetry. For the O(3) invariant σ model, this fact was already noted in [13].

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