Hamiltonian Vector Fields on Multiphase Spaces of Classical Field Theory *

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Abstract

We present a classification of hamiltonian vector fields on multisymplectic and polysymplectic fiber bundles closely analogous to the one known for the corresponding dual jet bundles that appear in the multisymplectic and polysymplectic approach to first order classical field theories.

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1 Introduction

The quest for a fully covariant hamiltonian formulation of classical field theory has a long history. In particular, the search for a first order formalism analogous to that for classical mechanics in terms of concepts from symplectic geometry has stimulated the development of new geometric tools usually referred to as "multisymplectic" or "polysymplectic" structures whose real significance is emerging only gradually. In fact, for many years there has not even been a convincing general definition, although a standard class of examples in terms of duals of jet bundles has long been known and widely used.[1] This defect has recently been overcome [1], and it has been realized that both multisymplectic and polysymplectic structures (which are not the same thing) play an important role in the formalism; in particular, a multisymplectic structure always induces a special kind of polysymplectic structure by means of a construction called the "symbol". But the fact that both types come together to form a pair has been anticipated almost 20 years ago [2], when it became apparent that the covariant hamiltonian formulation of first order classical field theories requires the simultaneous use of two types of "multiphase space" that we shall refer to as "ordinary multiphase space" and "extended multiphase space", respectively.

To be more precise, let us briefly recall the cornerstones of the construction of the two types of multiphase space for first order lagrangian field theories; for more details, the reader is referred to [2–4]. The starting point is the choice of a fiber bundle E over the space-time manifold M called the configuration bundle because its sections represent the basic fields of the theory at hand. Next, one takes the first order jet bundle JE of E to accomodate first order derivatives of these fields: this is an affine bundle over E and is also the domain of definition of the lagrangian. Besides, one also considers the linearized first order jet bundle $\vec{J}E$ of E: this is a vector bundle over E defined as the difference vector bundle of JE. Finally, as in mechanics, one uses appropriate versions of the Legendre transformation induced by the given lagrangian to pass to the (twisted) affine dual $J^{\circledast}E$ of JE and to the (twisted) linear dual $\vec{J}^{\circledast}E$ of $\vec{J}E$: the former is the extended multiphase space and the latter is the ordinary multiphase space of the theory. Note that the former is an affine line bundle over the latter and that the hamiltonian obtained from the given lagrangian through Legendre transformation is not a function but rather a section of this affine line bundle, so both of these multiphase spaces are essential ingredients for defining the concept of a hamiltonian system in field theory! Moreover, it is well known that $J^{\circledast}E$ carries a naturally defined multisymplectic form ω. However, what does not seem to have been so widely noticed is the fact that $\vec{J}^{\circledast}E$ carries a naturally defined polysymplectic form $\hat{\omega}$ – even though this form already appears explicitly in Ref. [5]. As has been shown more recently [1], it can be derived from the multisymplectic form ω on $J^{\circledast}E$ by taking its symbol, which turns out to be degenerate precisely along the fibers of the aforementioned

[1] This situation has been strikingly similar to that in classical mechanics before it was realized that symplectic manifolds, rather than just cotangent bundles, provide an adequate framework if one wants to accomodate phenomena such as half-integral spin within classical mechanics.

affine line bundle, and then passing to the corresponding quotient of $J^{\circledast}E$ by the kernel of ω, which is precisely $\vec{J}^{\circledast}E$. Note that this polysymplectic form $\hat{\omega}$ on $\vec{J}^{\circledast}E$ is canonical, whereas the form $\omega_{\mathcal{H}}$ on $\vec{J}^{\circledast}E$ obtained as the pull-back of ω by means of a hamiltonian section $\mathcal{H} : \vec{J}^{\circledast}E \to J^{\circledast}E$ is not, since it depends on the choice of hamiltonian.[2]

In terms of adapted local coordinates (x^μ, q^i, p_i^μ, p) for $J^{\circledast}E$ and (x^μ, q^i, p_i^μ) for $\vec{J}^{\circledast}E$, induced by local coordinates x^μ for M, local coordinates q^i for the typical fiber Q of E and a local trivialization of E [4], we have

$$\begin{array}{c}\text{extended multiphase space } J^{\circledast}E \\ \text{adapted local coordinates } (x^\mu, q^i, p_i^\mu, p) \\ \text{multisymplectic form} \quad \omega \;=\; dq^i \wedge dp_i^\mu \wedge d^n x_\mu \;-\; dp \wedge d^n x\end{array} \tag{1}$$

and

$$\begin{array}{c}\text{ordinary multiphase space } \vec{J}^{\circledast}E \\ \text{adapted local coordinates } (x^\mu, q^i, p_i^\mu) \\ \text{polysymplectic form} \quad \hat{\omega} \;=\; dq^i \wedge dp_i^\mu \otimes d^n x_\mu\end{array} \tag{2}$$

where p is (except for a sign) a scalar energy variable and $d^n x$ is the (local) volume form induced by the x^μ while $d^n x_\mu$ is the (local) $(n-1)$-form obtained by contracting $d^n x$ with $\partial_\mu \equiv \partial / \partial x^\mu$:

$$d^n x_\mu \;=\; i_{\partial_\mu} d^n x \;.$$

The same picture prevails in the general case if we replace adapted local coordinates by Darboux coordinates; see [1]. *Extended multiphase space is multisymplectic, ordinary multiphase space is polysymplectic.*

A crucial role in the development of the hamiltonian formalism is played by the notion of a hamiltonian vector field. According to the picture outlined above, this comes in two variants: a multisymplectic one and a polysymplectic one. We shall deal with the two versions separately, beginning with the pertinent definitions.

2 The multisymplectic case

According to Ref. [1], a multisymplectic fiber bundle of rank N can be defined as a fiber bundle P over an n-dimensional base manifold M equipped with a closed, non-degenerate $(n+1)$-form ω on its total space P which (a) is $(n-1)$-horizontal, i.e., such that its contraction with any three vertical vector fields vanishes, and (b) admits a multilagrangian distribution, i.e., an isotropic vector subbundle L of the vertical bundle VP of P of codimension N and dimension $Nn+1$. (It then turns out that P has dimension $(N+1)(n+1)$.) Assuming this distribution to be involutive, which is automatic as soon as $n \geqslant 3$ but has to be imposed as a separate condition when $n = 2$, Darboux's theorem assures that

[2] It should be noted that the form $\omega_{\mathcal{H}}$ is closed and non-degenerate but not multisymplectic in the sense of the definition given in Ref. [1].

there exist local coordinates, called canonical local coordinates or Darboux coordinates, in which ω assumes the form

$$\omega \;=\; dq^i \wedge dp_i^\mu \wedge d^n x_\mu \;-\; dp \wedge d^n x \;. \tag{3}$$

Locally, ω is exact, i.e.,

$$\omega \;=\; -\,d\theta \;, \tag{4}$$

where d denotes the exterior derivative, with

$$\theta \;=\; p_i^\mu \; dq^i \wedge d^n x_\mu \;+\; p \; d^n x \;. \tag{5}$$

The standard example is that of the extended multiphase space $J^{\circledast}E$ mentioned above, for which ω is also globally exact, i.e., the so-called multicanonical form θ in equations (4) and (5) is globally defined, and L is the vector subbundle of VP generated by the vector fields $\partial/\partial p_i^\mu$ and $\partial/\partial p$, that is, the vertical bundle for the projection of $J^{\circledast}E$ onto E (with respect to which $J^{\circledast}E$ is a vector bundle).

Given this situation, we say that a vector field X on P is *locally hamiltonian* if $i_X\omega$ is closed, or equivalently, if

$$L_X\omega \;=\; 0 \;. \tag{6}$$

It is called *globally hamiltonian* if $i_X\omega$ is exact, that is, if there exists an $(n-1)$-form f on P such that

$$i_X\omega \;=\; df \;. \tag{7}$$

In this case, f is said to be a *hamiltonian form associated with* X. Finally, when ω is exact and given by equation (4), X is called *exact hamiltonian* if

$$L_X\theta \;=\; 0 \;. \tag{8}$$

The main theorem states that these vector fields can be classified in terms of their components with respect to canonical local coordinates, which are given by the expansion

$$X \;=\; X^\mu \, \frac{\partial}{\partial x^\mu} \;+\; X^i \, \frac{\partial}{\partial q^i} \;+\; X_i^\mu \, \frac{\partial}{\partial p_i^\mu} \;+\; X_0 \, \frac{\partial}{\partial p} \;, \tag{9}$$

whereas, locally, the hamiltonian form corresponding to such a vector field, which is determined up to an arbitrary closed form, can be assumed to have an expansion of the form

$$f \;=\; f^\mu \; d^n x_\mu \;+\; \tfrac{1}{2} \, f_i^{\mu\nu} \; dq^i \wedge d^n x_{\mu\nu} \;, \tag{10}$$

where

$$d^n x_{\mu\nu} \;=\; i_{\partial_\nu} i_{\partial_\mu} \, d^n x \;.$$

An easy calculation gives

$$\begin{aligned} i_X\omega \;=\; & X^\nu \; dq^i \wedge dp_i^\mu \wedge d^n x_{\mu\nu} \;-\; X_i^\mu \; dq^i \wedge d^n x_\mu \;+\; X^i \; dp_i^\mu \wedge d^n x_\mu \\ & +\; X^\mu \; dp \wedge d^n x_\mu \;-\; X_0 \; d^n x \;, \end{aligned} \tag{11}$$

and in the exact case

$$i_X\theta \;=\; (p_i^\mu X^i \,+\, p\,X^\mu)\; d^n x_\mu \,-\, p_i^\mu X^\nu \; dq^i \wedge d^n x_{\mu\nu} \; . \tag{12}$$

These formulas constitute the starting point for the proof of the following

Theorem 1 *A vector field X on P is locally hamiltonian if and only if its components X^μ, X^i, X_i^μ and X_0 with respect to canonical local coordinates, as defined by equation (9), satisfy the following conditions:*

1. *the coefficients X^μ and X^i are independent of the multimomentum variables p_k^κ and of the energy variable p, with the coefficients X^μ depending only on the local coordinates x^κ of the base manifold M as soon as $N > 1$,*
2. *the remaining coefficients X_i^μ and X_0 can be expressed in terms of the previous ones and of new coefficients f_0^μ which are also independent of the multimomentum variables p_k^κ and of the energy variable p, according to*

$$X_i^\mu \;=\; -\,p\,\frac{\partial X^\mu}{\partial q^i} \,-\, p_j^\mu\,\frac{\partial X^j}{\partial q^i} \,+\, p_i^\nu\,\frac{\partial X^\mu}{\partial x^\nu} \,-\, p_i^\mu\,\frac{\partial X^\nu}{\partial x^\nu} \,+\, \frac{\partial f_0^\mu}{\partial q^i} \; , \tag{13}$$

(the first term being absent as soon as $N > 1$) and

$$X_0 \;=\; -\,p\,\frac{\partial X^\mu}{\partial x^\mu} \,-\, p_i^\mu\,\frac{\partial X^i}{\partial x^\mu} \,+\, \frac{\partial f_0^\mu}{\partial x^\mu} \; . \tag{14}$$

The components of the corresponding hamiltonian form f are given by

$$f^\mu \;=\; p_i^\mu X^i \,+\, p\,X^\mu \,+\, f_0^\mu \; , \tag{15}$$

and

$$f_i^{\mu\nu} \;=\; p_i^\nu X^\mu \,-\, p_i^\mu X^\nu \; . \tag{16}$$

In addition, when ω is exact and given by equation (4), X is exact hamiltonian if and only if the coefficients f_0^μ vanish.

This theorem has been explicitly stated in Ref. [6] and an explicit proof of a more general theorem (where vector fields are replaced by multivector fields) can be found in Ref. [7].

3 The polysymplectic case

According to Ref. [1], a polysymplectic fiber bundle of rank N can be defined as a fiber bundle P over an n-dimensional base manifold M equipped with a vertically closed, non-degenerate vertical 2-form $\hat{\omega}$ on its total space P which (a) takes values in (the pull-back $\pi^*\hat{T}$ to P of) some given $\hat{n}$-dimensional coefficient vector bundle $\hat{T}$ over M

and (b) admits a polylagrangian distribution, i.e., an isotropic vector subbundle L of the vertical bundle VP of P of codimension N and dimension $N\hat{n}$. (It then turns out that P has dimension $N\hat{n} + N + \hat{n}$.) Assuming this distribution to be involutive, which is automatic as soon as $\hat{n} \geqslant 3$ but has to be imposed as a separate condition when $\hat{n} = 2$, Darboux's theorem assures that given any basis $\{\, \hat{e}_a \,/\, 1 \leqslant a \leqslant \hat{n} \,\}$ of local sections of $\hat{T}$, there exist local coordinates, called canonical local coordinates or Darboux coordinates, in which $\hat{\omega}$ assumes the form

$$\hat{\omega} \;=\; dq^i \wedge dp_i^a \otimes \hat{e}_a \; . \tag{17}$$

Locally, $\hat{\omega}$ is vertically exact, i.e.,

$$\hat{\omega} \;=\; -\, d_V \hat{\theta} \; , \tag{18}$$

where d_V denotes the vertical exterior derivative, with

$$\hat{\theta} \;=\; p_i^a \, dq^i \otimes \hat{e}_a \; . \tag{19}$$

The standard example is that of the ordinary multiphase space $\vec{J}^{\circledast} E$ mentioned above, with $\hat{T} = \bigwedge^{n-1} T^* M$ and $e_a = d^n x_\mu$, for which $\hat{\omega}$ is also globally vertically exact, i.e., the so-called polycanonical form $\hat{\theta}$ in equations (18) and (19) is globally defined, and L is the vector subbundle of VP generated by the vector fields $\partial/\partial p_i^\mu$, that is, the vertical bundle for the projection of $\vec{J}^{\circledast} E$ onto E (with respect to which $\vec{J}^{\circledast} E$ is a vector bundle).

Given this situation, we say that a vertical vector field X on P is *locally hamiltonian* if $i_X \hat{\omega}$ is vertically closed, or equivalently, if

$$L_X \hat{\omega} \;=\; 0 \; . \tag{20}$$

It is called *globally hamiltonian* if $i_X \hat{\omega}$ is vertically exact, that is, if there exists a section f of the vector bundle $\pi^* \hat{T}$ over P such that

$$i_X \hat{\omega} \;=\; d_V f \; . \tag{21}$$

In this case, f is said to be a *hamiltonian section associated with* X. Finally, when $\hat{\omega}$ is vertically exact and given by equation (18), X is called *exact hamiltonian* if

$$L_X \hat{\theta} \;=\; 0 \; . \tag{22}$$

The main theorem states that these vector fields can be classified in terms of their components with respect to canonical local coordinates, which are given by the expansion

$$X \;=\; X^i \, \frac{\partial}{\partial q^i} \,+\, X_i^a \, \frac{\partial}{\partial p_i^a} \tag{23}$$

whereas, locally, the hamiltonian section corresponding to such a vector field, which is determined up to (the pull-back to P of) an arbitrary section of $\hat{T}$, can be assumed to have an expansion of the form

$$f \;=\; f^a \, \hat{e}_a \; . \tag{24}$$

An easy calculation gives

$$i_X \hat{\omega} \;=\; - X_i^a \, dq^i \otimes \hat{e}_a \, + \, X^i \, dp_i^a \otimes \hat{e}_a \;, \tag{25}$$

and in the exact case

$$i_X \hat{\theta} \;=\; X^i p_i^a \, \hat{e}_a \;. \tag{26}$$

These formulas constitute the starting point for the proof of the following

Theorem 2 *A vector field X on P is locally hamiltonian if and only if its components X^i and X_i^μ with respect to canonical local coordinates, as defined by equation (23), satisfy the following conditions:*

1. *the coefficients X^i are independent of the multimomentum variables p_k^κ,*
2. *the remaining coefficients X_i^a can be expressed in term of the previous ones and of new coefficients f_0^a which are also independent of the multimomentum variables p_k^κ, according to*

$$X_i^a \;=\; - \, p_j^a \, \frac{\partial X^j}{\partial q^i} \, + \, \frac{\partial f_0^a}{\partial q^i} \;. \tag{27}$$

The components of the corresponding hamiltonian section f are given by

$$f^a \;=\; p_i^a X^i \, + \, f_0^a \;, \tag{28}$$

In addition, when $\hat{\omega}$ is vertically exact and given by equation (18), X is exact hamiltonian if and only if the coefficients f_0^a vanish.

The proof of this theorem is entirely analogous to that of the previous one, except that it is much simpler. The essence of the argument can already be found in Ref. [8, 9], but the proper global context of the result is not adequately treated there.

4 Outlook

The analogous problem of determining hamiltonian vector fields with respect to the form $\omega_{\mathcal{H}}$ on ordinary multiphase space mentioned in the introduction has been addressed and solved in Ref. [10], but the results are somewhat complicated and not very enlightening. We now believe this to be related to the fact that, according to the structurally natural definition given in Ref. [1], $\omega_{\mathcal{H}}$ is *not* multisymplectic.

One problem that, for the time being, remains open is to give a global, coordinate independent formulation of the results of Theorems 1 and 2. This question is presently under investigation.

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