Local Symmetries in Gauge Theories in a Finite-Dimensional Setting

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Abstract

It is shown that the correct mathematical implementation of symmetry in the geometric formulation of classical field theory leads naturally beyond the concept of Lie groups and their actions on manifolds, out into the realm of Lie group bundles and, more generally, of Lie groupoids and their actions on fiber bundles. This applies not only to local symmetries, which lie at the heart of gauge theories, but is already true even for global symmetries when one allows for fields that are sections of bundles with (possibly) non-trivial topology or, even when these are topologically trivial, in the absence of a preferred trivialization.

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1 Introduction

Gauge theories constitute a class of models of central importance in field theory since they provide the conceptual basis for our present understanding of three of the four fundamental interactions – strong, weak and electromagnetic. At the very heart of gauge theories lies the principle of gauge invariance, according to which physics is invariant under symmetry transformations even if one is allowed to perform different symmetry transformations at different points of space-time: such transformations have come to be known as local symmetries, as opposed to rigid transformations which are the same at all points of space-time and are commonly referred to as global symmetries.[1]

One of the reasons why gauge theories are so natural is that there is a standard procedure, due to Hermann Weyl, for "gauging" a global symmetry so as to promote it to a local symmetry, or to put it differently, for constructing a field theory with local symmetries out of any given field theory with global symmetries. A salient feature of this method is that it requires the introduction of a new field, the gauge potential, which is needed to define covariant derivatives that replace ordinary (partial) derivatives: such a prescription, known as "minimal coupling", is already familiar from general relativity. (A subsequent step is to provide the gauge potential with a dynamics of its own.) In his original proposal [1], Weyl explored the possibility to apply this construction to scale transformations and, by converting scale invariance into a local symmetry, arrive at a unified theory of gravity and electromagnetism. Although this version was almost immediately dismissed[2] after Einstein had argued that it leads to physically unacceptable predictions, the method as such persisted. It became fruitful after the advent of quantum mechanics, when in his modified proposal [2], Weyl applied the same construction to phase transformations and showed that converting phase invariance, which is a characteristic feature of quantum mechanics, into a local symmetry, the electromagnetic field (or better, the electromagnetic potential) emerges naturally. In this way, Weyl created the concept of a gauge theory and established electromagnetism (coupled to matter) as its first example. In the 1950's, these ideas were extended from the abelian group $U(1)$ of quantum mechanical phases to the nonabelian isospin group $SU(2)$ [3] and, soon after, to general compact connected Lie groups G [4].

Another aspect that deserves to be mentioned in this context is that the field theory governing the only one of the four fundamental interactions not covered by gauge theories of the standard Yang-Mills type, namely Einstein's general relativity, also exhibits a kind of local symmetry (even though of a slightly different type), namely general coordinate invariance. The same type of local symmetry, going under the name of reparametrization invariance, prevails in string theory and membrane theory. Thus we may say that the concept of local symmetry pervades all of fundamental physics.

[1] In this paper, when speaking of symmetries (local or global), we are tacitly assuming that we are dealing with continuous symmetries, not with discrete ones.

[2] Somewhat ironically, an important remnant of this very first attempt at a unification between the fundamental interactions (the only two known ones at that time) is the persistent use of the word "gauge".

Unfortunately, there is one basic mathematical aspect of local symmetries which is the source of innumerous difficulties: the relevant symmetry groups are infinite-dimensional. For example, on an arbitrary space-time manifold M, gauging a field theory which is invariant under the action of some connected compact Lie group G will lead to a field theory which, in the simplest case where all fiber bundles involved are globally trivial, is invariant under the action of the infinite-dimensional group $C^\infty(M,G)$. Similarly, in general relativity, we find invariance under the action of the infinite-dimensional group $\mathrm{Diff}(M)$ – the diffeomorphism group of space-time [5]. The same type of local symmetry group also appears in string theory and membrane theory, although in this case M is to be interpreted as the parametrizing manifold and not as space-time. As is well known, the mathematical difficulties one has to face when dealing with such infinite-dimensional groups and their actions on infinite-dimensional spaces of field configurations or of solutions to the equations of motion (covariant phase space) are enormous and often insurmountable, in particular when M is not compact, as is the case for physically realistic models of space-time [5].

In view of this situation, it would be highly desirable to recast the property of invariance of a field theory under local symmetries into a form where one deals exclusively with finite-dimensional objects. That such a reformulation might be possible is suggested by observing that gauge transformations are, in a very specific sense, localized in space-time: according to the principle of relativistic causality, performing a gauge transformation in a certain region can have no effect in other, causally disjoint regions. Intuitively speaking, gauge transformations are "spread out" over space-time, and this should make it possible to eliminate all reference to infinite-dimensional objects if one looks at what happens at each point of space-time separately and only fits the results together at the very end.

This idea can be readily implemented in mechanics, where space-time M is reduced to a copy of the real line $\mathbb{R}$ representing the time axis. In the context of the lagrangian formulation, the procedure works as follows. Consider an autonomous mechanical system [3] with configuration space Q and lagrangian L, which is a given function on the tangent bundle TQ of Q: its dynamics is specified by postulating the solutions of the equations of motion of the system to be the stationary points of the action functional S associated with an arbitrary time interval $[t_0, t_1]$, defined by

$$S[\boldsymbol{q}] \;=\; \int_{t_0}^{t_1} dt\; L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) \tag{1}$$

for curves $\boldsymbol{q} \in C^\infty(\mathbb{R}, Q)$ in Q. To implement the notion of symmetry, we must fix a Lie group G together with an action

$$\begin{array}{ccc} G \times Q & \longrightarrow & Q \\ (g,q) & \longmapsto & g \cdot q \end{array} \tag{2}$$

[3] The generalization to non-autonomous systems is straightforward.

of G on Q and note that this induces an action

$$\begin{array}{ccc} G \times TQ & \longrightarrow & TQ \\ (g,(q,\dot{q})) & \longmapsto & g \cdot (q,\dot{q}) = (g \cdot q, g \cdot \dot{q}) \end{array} \tag{3}$$

of G on the tangent bundle TQ of Q as well as, more generally, an action

$$\begin{array}{ccc} TG \times TQ & \longrightarrow & TQ \\ ((g,Xg),(q,\dot{q})) & \longmapsto & (g,Xg) \cdot (q,\dot{q}) = (g \cdot q\ ,\ g \cdot \dot{q} + X_Q(g \cdot q)) \end{array} \tag{4}$$

of the tangent group TG of G on the tangent bundle TQ of Q. (Here, we use that the tangent bundle TG of a Lie group G is again a Lie group, whose group multiplication is simply the tangent map to the original one, and that we can use, e.g., right translations to establish a global trivialization

$$\begin{array}{ccc} TG & \longrightarrow & G \times \mathfrak{g} \\ (g,\dot{g}) & \longmapsto & (g,\dot{g}g^{-1}) \end{array} , \tag{5}$$

which shows that TG is isomorphic to the semidirect product of G with its own Lie algebra $\mathfrak{g}$ and allows to bring the induced action of TG on TQ, which is simply the tangent map to the original action of G on Q, into the form given in eqn (4).) Then the system will exhibit a global symmetry under G if S is invariant under the induced action of G on curves in Q, that is, if $S[g \cdot \mathsf{q}] = S[\mathsf{q}]$ where $(g \cdot \mathsf{q})(t) = g \cdot \mathsf{q}(t)$; obviously, this will be the case if and only if the lagrangian L is invariant under the action (3) of G on TQ. On the other hand, the system will exhibit a local symmetry under G if S is invariant under the induced action of curves in G on curves in Q, that is, if $S[\mathsf{g} \cdot \mathsf{q}] = S[\mathsf{q}]$ where $(\mathsf{g} \cdot \mathsf{q})(t) = \mathsf{g}(t) \cdot \mathsf{q}(t)$. Now it is easily verified that this will be so if and only if the lagrangian L is invariant under the action (4) of TG on TQ (see, e.g., Ref. [6]). In other words, the condition of invariance of the action functional under the infinite-dimensional group $C^\infty(\mathbb{R}, G)$ can be reformulated as a condition of invariance of the lagrangian under a finite-dimensional Lie group, which is simply the tangent group TG of the original global symmetry group G. Moreover, it is also shown in Ref. [6] how one can use this approach to "gauge" a given global symmetry to promote it to a local symmetry, provided one replaces the configuration space Q by its cartesian product with the Lie algebra $\mathfrak{g}$ and studies the dynamics of curves (q, A) where A is a Lagrange multiplier: the mechanical analogue of the gauge potential. Of course, in mechanics there is no natural dynamics for such a Lagrange multiplier, since the "curvature" of this "connection" vanishes identically.

The main goal of the present paper, which is based on the PhD thesis of the second author [7], is to show how one can implement the same program – recasting local symmetries in a purely finite-dimensional setting – in field theory, that is, for full-fledged gauge theories and in a completely geometric setup. As we shall see, this requires an important extension of the mathematical tools employed to describe symmetries: the passage from Lie groups and their actions on manifolds to Lie group bundles and their actions on fiber bundles (over the same base manifold).

The formulation of (classical) gauge theories in the language of modern differential geometry is an extensive subject that, since its beginnings in the 1970's, has been addressed by many authors; some references in this direction which have been useful in the course of our work are [8–13]. It should also be mentioned that Lie group bundles can be regarded as a special class of Lie groupoids, namely, locally trivial Lie groupoids for which the source projection and the target projection coincide [14], and the use of Lie groupoids in gauge theories has been advocated before by some authors [15, 16]. However, we believe that the theory of Lie group bundles should not be regarded as just a special case of the theory of general Lie groupoids but has a flavor of its own. In particular, there are various constructions which are familiar from the theory of Lie groups that can be extended in a fairly straightforward manner to Lie group bundles but are far more difficult to formulate for general Lie groupoids. As an example which is relevant here, consider the fact used above that the tangent bundle TG of a Lie group G is a Lie group and the tangent map to an action of a Lie group G is an action of the tangent Lie group TG: its natural extension to Lie group bundles is formulated in Theorems 1 and 2 below, but the question of how to extend it to general Lie groupoids – i.e., how to define jet groupoids of Lie groupoids and their actions – does not seem to have been addressed anywhere in the literature. Another example would be the question of how to define the notion of invariance of a tensor field under the action of a Lie groupoid. And finally, similar statements hold concerning the question of applications: there are lots of lagrangians in physics that are gauge invariant, but none that are gauge plus space-time diffeomorphism invariant. True Lie groupoids may become relevant if one wants to unify internal symmetries with space-time symmetries, but strong restrictions will have to be imposed on the space-time part: the question of how to formulate this as elegantly and concisely as possible is presently under investigation. At any rate, we think that, intuitively speaking, Lie group bundles occupy a place "half way in between" Lie groups and general Lie groupoids and that they deserve a separate treatment and a place in their own right.

2 Jet bundles and the connection bundle

The theory of jet bundles – as exposed, e.g., in Ref. [17] – is an important tool in differential geometry and, in particular, plays a central role for the understanding of symmetries in gauge theories as advocated in this paper. The general definition of jets (as equivalence classes of local maps whose Taylor expansions coincide up to a certain order) is somewhat complicated but will not be needed here since in first and second order (which is all we are going to consider) there are alternative definitions that are simpler [18]. In fact, given a fiber bundle E over a manifold M with projection $\pi : E \longrightarrow M$, we can define its first order jet bundle JE and its linearized first order jet bundle $\vec{J}E$ as follows: for any point e in E with base point $m = \pi(e)$ in M, consider the affine space

$$J_eE \;=\; \{\, u \in L(T_mM, T_eE) \mid T_e\pi \circ u \;=\; \mathrm{id}_{T_mM} \,\} \;, \tag{6}$$

and its difference vector space

$$\vec{J}_e E \;=\; L(T_m M, V_e E) \;=\; T_m^* M \otimes V_e E \;, \tag{7}$$

where $V_e E = \ker T_e \pi$ is the vertical space of E at e, and note that this defines JE as an affine bundle over E and $\vec{J}E$ as a vector bundle over E with respect to the target projection (which takes $J_e E$ and $\vec{J}_e E$ to e) but also defines both of them as fiber bundles over M with respect to the source projection (which takes $J_e E$ and $\vec{J}_e E$ to $m = \pi(e)$). Moreover, composition with the appropriate tangent maps provides a canonical procedure for associating with every strict homomorphism $f : E \longrightarrow F$ of fiber bundles E and F over M a homomorphism $Jf : JE \longrightarrow JF$ of affine bundles (sometimes called the jet prolongation of f) and a homomorphism $\vec{J}f : \vec{J}E \longrightarrow \vec{J}F$ of vector bundles covering f: in this way, J and $\vec{J}$ become functors. Iterating the construction, we can also define the second order jet bundle J^2E by (a) considering the iterated first order jet bundle $J(JE)$, (b) passing to the so-called semiholonomic second order jet bundle $\bar{J}^2E$, which is the affine subbundle of $J(JE)$ over JE defined by the condition

$$\bar{J}^2E \;=\; \{\, w \in J(JE) \,|\, \tau_{JE}(w) \;=\; J\tau_E(w) \,\} \;, \tag{8}$$

where τ_E and τ_{JE} are the target projections of JE and of $J(JE)$, respectively, while $J\tau_E : J(JE) \longrightarrow JE$ is the jet prolongation of $\tau_E : JE \longrightarrow E$, and (c) decomposing this, as a fiber product of affine bundles over JE, into a symmetric part and an antisymmetric part: the former is precisely J^2E and is an affine bundle over JE, with difference vector bundle equal to the pull-back to JE of the vector bundle $\pi^*\big(\bigvee^2 T^*M\big) \otimes VE$ over E by the target projection τ_E, whereas the latter is a vector bundle over JE, namely the pull-back to JE of the vector bundle $\pi^*\big(\bigwedge^2 T^*M\big) \otimes VE$ over E by the target projection τ_E:

$$\begin{aligned} \bar{J}^2E \;&\cong\; J^2E \;\times_{JE}\; \tau_E^*\Big(\pi^*\big(\textstyle\bigwedge^2 T^*M\big) \otimes VE\Big) \\ \vec{J}^2E \;&\cong\; \tau_E^*\Big(\pi^*\big(\textstyle\bigvee^2 T^*M\big) \otimes VE\Big) \end{aligned} \tag{9}$$

Turning to gauge theories, we begin by recalling that the starting point for the formulation of a gauge theory is the choice of (a) a Lie group G, with Lie algebra $\mathfrak{g}$, (b) a principal G-bundle P over the space-time manifold M with projection $\rho : P \longrightarrow M$ and carrying a naturally defined right action of G that will be written in the form

$$\begin{array}{ccc} P \times G & \longrightarrow & P \\ (p\,,g) & \longmapsto & p \cdot g \end{array} \tag{10}$$

and (c) a manifold Q carrying an action of G as in eqn (2) above, so that we can form the associated bundle $E = P \times_G Q$ over M as well as the connection bundle $CP = JP/G$ over M. Sections of E represent the (multiplet of all) matter fields present in the theory, whereas sections of CP represent the gauge potentials (connections). The group G is usually referred to as the structure group of the model: it contains a compact "internal part" (e.g., a $U(1)$ factor for electrodynamics or, more generally, an $SU(2)\times U(1)$ factor for the electroweak theory, an $SU(3)$ factor for chromodynamics, etc.) but possibly also a non-compact "space-time part" which is an appropriate spin group, in order to accomodate tensor and spinor fields.

The constructions of the associated bundle $E = P \times_G Q$ and of the connection bundle $CP = JP/G$ are standard (both can be obtained as quotients by a free action of G), and we just recall a few basic aspects in order to fix notation.

The first arises from the cartesian product $P \times Q$, on which G acts according to

$$g \cdot (p\,,q) \;=\; (p \cdot g^{-1}, g \cdot q)\;,$$

and we denote the equivalence class (orbit) of a point $(p\,,q) \in P \times Q$ by $[p\,,q] \in P \times_G Q$.

The second is obtained from the first order jet bundle JP of P, which carries a natural right G-action induced from the given right G-action on P by taking derivatives (jets) of local sections: for any point $m \in M$,

$$(j_m\sigma) \cdot g \;=\; j_m(\sigma \cdot g)\;,$$

and we denote the equivalence class (orbit) of a point $j_m\sigma \in JP$ by $[j_m\sigma] \in CP$. Such an equivalence class can be identified with a horizontal lifting map on the fiber P_m of P at m, that is, a G-equivariant homomorphism

$$\Gamma_m : P_m \times T_mM \longrightarrow TP \,|\, P_m$$

of vector bundles over P_m which, when composed with the restriction to $TP \,|\, P_m$ of the tangent map $T\rho : TP \longrightarrow TM$ to the principal bundle projection $\rho : P \longrightarrow M$, gives the identity. Still another representation, and the most useful one for practical purposes, is in terms of a connection form on the fiber P_m of P at m, that is, a G-equivariant 1-form A_m on P_m with values in the Lie algebra $\mathfrak{g}$ whose restriction to the vertical subspace V_pP at each point p of P_m coincides with the canonical isomorphism of V_pP with $\mathfrak{g}$ given by the standard formula for fundamental vector fields,

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & V_pP \\ X & \longmapsto & \hat{X}_P(p) \end{array} \qquad \text{where} \qquad \hat{X}_P(p) \;=\; \frac{d}{dt}\, p \cdot \exp(tX)\,\Big|_{t=0}\;,$$

the relation between the two being that the image of Γ_m coincides with the kernel of A_m. Since JP is an affine bundle over P, with difference vector bundle $\vec{J}P \cong \rho^*(T^*M) \otimes VP$ where VP is the vertical bundle of P which in turn is canonically isomorphic to the trivial vector bundle $P \times \mathfrak{g}$ over P, and since G acts by fiberwise affine transformations on JP and by fiberwise linear transformations on $\vec{J}P$, it follows that CP is an affine bundle over M, with difference vector bundle $\vec{C}P \cong T^*M \otimes (P \times_G \mathfrak{g})$.

Finally, we shall also need to consider the first order jet bundle of the connection bundle, which turns out to admit a canonical decomposition into a symmetric and an antisymmetric part. To derive this, note that when we lift the given right G-action from P first to JP as above and then to $J(JP)$ by applying the same prescription again, the semiholonomic second order jet bundle $\bar{J}^2 P$ will be a G-invariant subbundle of $J(JP)$ and its quotient by G will be precisely $J(CP)$:

$$J(CP) \;\cong\; \bar{J}^2 P / G \, . \tag{11}$$

But the canonical decomposition of $\bar{J}^2 P$ into symmetric and antisymmetric part as given in eqn (9), when specialized to principal bundles, is manifestly G-invariant, so we can divide by the G-action to arrive at the desired canonical decomposition of $J(CP)$, as a fiber product of affine bundles over CP, into a symmetric part (an affine bundle) and an antisymmetric part (vector bundle):

$$\begin{aligned} J(CP) \;&\cong\; (J^2 P)/G \;\times_{CP}\; \pi_{CP}^* \big(\textstyle\bigwedge^2 T^* M \otimes (P \times_G \mathfrak{g}) \big) \\ (\vec{J}^2 P)/G \;&\cong\; \pi_{CP}^* \big(\textstyle\bigvee^2 T^* M \otimes (P \times_G \mathfrak{g}) \big) \end{aligned} \tag{12}$$

The projection onto the second summand in this decomposition is called the **curvature map** because, at the level of sections, it maps the 1-jet of a connection form A to its curvature form F_A. See Theorem 1 of Ref. [12] and Theorems 5.3.4 and 5.3.5 of Ref. [17].

3 Gauge group bundles and their actions

In this section we introduce the basic object needed to describe symmetries in gauge theories within a purely finite-dimensional setting: the gauge group bundle and its descendants. All of these are Lie group bundles which admit certain natural actions on various of the bundles introduced before and/or their descendants: our aim here will be to define them rigorously. In order to do so, we must first introduce the notion of a Lie group bundle and of an action of a Lie group bundle on a fiber bundle (over the same base manifold): this involves the concept of a locally constant structure on a fiber bundle.

Let E be a fiber bundle over a base manifold M with projection $\pi : E \to M$ and typical fiber E_0, and suppose that both E_0 and every fiber E_m of E are equipped with some determined geometric structure of the same kind. We say that such a geometric structure is **locally constant along** M if there exists a family $(\Phi_\alpha)_{\alpha \in A}$ of local trivializations $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times E_0$ of E whose domains U_α cover M and such that for every point m in U_α the diffeomorphism $(\Phi_\alpha)_m : E_m \to E_0$ is structure preserving: any (local) trivialization of this kind will be called **compatible**.

Definition 1 *A **Lie group bundle** (often abbreviated "LGB") over a given base manifold M is a fiber bundle $\bar{G}$ over M whose typical fiber is a Lie group G and which comes equipped with a strict bundle homomorphism over M*

$$\begin{array}{ccc} \bar{G} \times_M \bar{G} & \longrightarrow & \bar{G} \\ (h, g) & \longmapsto & hg \end{array} \tag{13}$$

called ***multiplication*** *or the* ***product****, a global section* 1 *of* $\bar{G}$ *called the* ***unit*** *and a strict bundle homomorphism over* M

$$\begin{array}{ccc} \bar{G} & \longrightarrow & \bar{G} \\ g & \longmapsto & g^{-1} \end{array} \tag{14}$$

called ***inversion*** *which, taken together, define a Lie group structure in each fiber of* $\bar{G}$ *that is locally constant along* M*.*

This definition coincides with the one adopted in Ref. [14, p. 11].

Example 1 Let P be a principal bundle over a given manifold M with structure group G and bundle projection $\rho : P \to M$. Then the associated bundle $P \times_G G$, where G is supposed to act on itself by conjugation, is a Lie group bundle over M, if we define multiplication by

$$[p\,,h][p\,,g] \;=\; [p\,,hg]\;, \tag{15}$$

the unit by

$$1_{\rho(p)} \;=\; [p\,,1]\;, \tag{16}$$

and inversion by

$$[p\,,g]^{-1} \;=\; [p\,,g^{-1}]\;. \tag{17}$$

We call it the **gauge group bundle** associated with P because the group of its sections is naturally isomorphic to the group of strict automorphisms of P, which is usually referred to as the group of gauge transformations (or sometimes simply, though somewhat misleadingly, the gauge group) associated with P:

$$\Gamma(P \times_G G) \;\cong\; \mathrm{Aut}_s(P)\;. \tag{18}$$

Definition 2 *An* ***action*** *of a Lie group bundle* $\bar{G}$ *on a fiber bundle* E*, both over the same given base manifold* M*, is a strict bundle homomorphism over* M

$$\begin{array}{ccc} \bar{G} \times_M E & \longrightarrow & E \\ (g,e) & \longmapsto & g \cdot e \end{array} \tag{19}$$

which defines an action of each fiber of $\bar{G}$ *on the corresponding fiber of* E *that is locally constant along* M*.*

Example 2 Let P be a principal bundle over a given manifold M with structure group G and bundle projection $\rho : P \to M$ and Q be a manifold carrying an action of G as in eqn (2) above. Then the gauge group bundle $P \times_G G$ acts naturally on the associated bundle $P \times_G Q$, according to

$$[p\,,g] \cdot [p\,,q] \;=\; [p\,,g \cdot q]\;. \tag{20}$$

A particular case occurs if we take Q to be G itself, but this time letting G (the structure group) act on G (the typical fiber) by left translation: using the fact that the resulting associated bundle is canonically isomorphic to P itself,[4] we see that the gauge group bundle $P \times_G G$ acts naturally on P itself, according to

$$[p\,,g] \cdot (p \cdot g_0) \;=\; p \cdot (g g_0)\;. \tag{21}$$

Note that this (left) action of $P \times_G G$ on P commutes with the (right) action of G on P specified in eqn (10) above – a fact which can be viewed as a natural generalization, from Lie groups to principal bundles and bundles of Lie groups, of the well known statement that left translations commute with right translations (and to which it reduces when M is a single point).

Bundles of Lie groups and their actions also behave naturally under taking derivatives. For example, we have the following

Proposition 1 *An action of a Lie group bundle $\bar{G}$ on a fiber bundle E, both over the same base manifold M, induces in a natural way actions of $\bar{G}$ on the vertical bundle VE of E and on the linearized jet bundle $\vec{J}E$ of E.*

Explicitly, the first of these induced actions is defined by

$$g \cdot \frac{d}{dt}\, e(t)\Big|_{t=0} \;=\; \frac{d}{dt}\,\big(g \cdot e(t)\big)\Big|_{t=0} \qquad \text{for } m \in M,\ g \in \bar{G}_m,\ e \text{ any smooth curve in } E_m\ , \tag{22}$$

whereas the second is obtainded by taking its tensor product with the trivial action of $\bar{G}$ on the cotangent bundle T^*M of M, using the canonical isomorphism $\vec{J}E \cong \pi^*(T^*M) \otimes VE$. In a slightly different direction, we note the following

Theorem 1 *Let $\bar{G}$ be a Lie group bundle over M. Then for any $r \geqslant 1$, its r^{th} order jet bundle $J^r\bar{G}$ is also a Lie group bundle over M.[5] (When $r = 1$, we often omit the superfix r.)*

Theorem 2 *An action of a Lie group bundle $\bar{G}$ on a fiber bundle E, both over the same base manifold M, induces in a natural way an action of $J^r\bar{G}$ on J^rE, for any $r \geqslant 1$. (When $r = 1$, we often omit the superfix r.)*

Proof: Both theorems are direct consequences of the fact that taking r^{th} order jets is a functor, together with the fact that for any two fiber bundles E and F over the same base manifold M, there is a canonical isomorphism

$$J^r(E \times_M F) \;\cong\; J^rE \times_M J^rF$$

induced by the identification of (local) sections of $E \times_M F$ with pairs of (local) sections of E and of F. More specifically, we define the product, the unit and the inversion in $J^r\bar{G}$

[4] Explicitly, this isomorphism is given by mapping $[p, g]$ to p.

[5] One should note that an analogous statement for principal bundles would be false: the fact that P is a principal bundle does not imply that J^rP is a principal bundle.

by extending the product, the unit and the inversion in $\bar{G}$, respectively, pointwise to local sections and then taking r^{th} order jets, and similarly, we define the action of $J^r\bar{G}$ on J^rE by extending the action of $\bar{G}$ on E pointwise to local sections and then taking r^{th} order jets. □

Example 3 Let P be a principal bundle over a given manifold M with structure group G and bundle projection $\rho : P \to M$. Then for any $r \geqslant 1$, the r^{th} order jet bundle $J^r(P \times_G G)$ of the gauge group bundle $P \times_G G$ is a Lie group bundle over M which we shall call the (r^{th} order) **derived gauge group bundle** associated with P. (When $r = 1$, we often omit the prefix "first order".)

With these tools at our disposal, we proceed to define various actions of the gauge group bundle and the (first and second order) derived gauge group bundles that play an important role in the analysis of symmetries in gauge theories. Starting with the action

$$(P \times_G G) \times (P \times_G Q) \;\longrightarrow\; P \times_G Q \tag{23}$$

of the gauge group bundle $P \times_G G$ on the matter field bundle $P \times_G Q$ already mentioned in Example 2 (cf. eqn (20)), consider first the induced action

$$(P \times_G G) \times V(P \times_G Q) \;\longrightarrow\; V(P \times_G Q) \tag{24}$$

of $P \times_G G$ on the vertical bundle $V(P \times_G Q)$ of $P \times_G Q$ obtained by taking derivatives along the fibers (this corresponds to the transition from the action (2) to the action (3), using that, for each point m in M, the fiber of $V(P \times_G Q)$ over m is precisely the tangent bundle of the fiber of $P \times_G Q$ over m), and extend it trivially to an action

$$(P \times_G G) \times \vec{J}(P \times_G Q) \;\longrightarrow\; \vec{J}(P \times_G Q) \tag{25}$$

of $P \times_G G$ on the linearized first order jet bundle $\vec{J}(P \times_G Q)$ of $P \times_G Q$, using the canonical isomorphism

$$\vec{J}(P \times_G Q) \;\cong\; \pi^*(T^*M) \otimes V(P \times_G Q)$$

and taking the tensor product with appropriate identities on $\pi^*(T^*M)$. Similarly, the action (23) induces an action

$$J(P \times_G G) \times J(P \times_G Q) \;\longrightarrow\; J(P \times_G Q) \tag{26}$$

of $J(P \times_G G)$ on the first order jet bundle $J(P \times_G Q)$ of $P \times_G Q$, obtained by applying Theorem 2 (with $r = 1$). On the other hand, starting with the action

$$(P \times_G G) \times P \;\longrightarrow\; P$$

of the gauge group bundle $P \times_G G$ on the principal bundle P itself already mentioned in Example 2 (cf. eqn (20)), which commutes with the (right) action of G on P, Theorem 2 with $r = 1$) provides an action

$$J(P \times_G G) \times JP \;\longrightarrow\; JP$$

of $J(P \times_G G)$ on the first order jet bundle JP of P which commutes with the induced (right) action of G on JP and therefore factors through the quotient to yield an action

$$J(P \times_G G) \times CP \longrightarrow CP \tag{27}$$

of $J(P \times_G G)$ on the connection bundle CP of P. Finally, applying Theorem 2 (with $r = 1$) once more and using the fact that $J^2(P \times_G G)$ can be realized as a Lie group subbundle of the iterated Lie group bundle $J(J(P \times_G G))$, we arrive at an action

$$J^2(P \times_G G) \times J(CP) \longrightarrow J(CP) \tag{28}$$

of $J^2(P \times_G G)$ on the first order jet bundle $J(CP)$ of CP.

4 Local expressions

Our next goal will be to derive local expressions for the actions (23-28) to show that they are global versions of well known and intuitively obvious constructions used by physicists.

In order to build this bridge, the first obstacle to be overcome is the fact that physicists usually write these actions in terms of fields, that is, of (local) sections of the bundles involved, rather than the bundles themselves. In mathematical terms, this corresponds to thinking in terms of sheaves, rather than bundles. Now a Lie group bundle corresponds to a "locally constant" sheaf of Lie groups and an action of a Lie group bundle on a fiber bundle corresponds to a "locally constant" action of a "locally constant" sheaf of Lie groups on a "locally constant" sheaf of manifolds. Explicitly, these sheaves are obtained by associating with each open subset U of the base manifold M the group $\Gamma(U, \bar{G})$ of sections $\boldsymbol{g}$ of $\bar{G}$ over U and the space $\Gamma(U, E)$ of sections φ of E over U, and with each pair of open subsets of M, one of which is contained in the other, the appropriate restriction maps. The requirement of local triviality then means that each point of M admits an open neighborhood such that for every open subset U of M contained in it, we have isomorphisms $\Gamma(U, \bar{G}) \cong C^\infty(U, G)$ and $\Gamma(U, E) \cong C^\infty(U, Q)$, and the requirement of local constancy means that these isomorphisms can be chosen such that the product, the unit and the inversion in $\Gamma(U, \bar{G})$ correspond to the pointwise defined product, unit and inversion in $C^\infty(U, G)$ and the action of $\Gamma(U, \bar{G})$ on $\Gamma(U, E)$ corresponds to the pointwise defined action of $C^\infty(U, G)$ on $C^\infty(U, Q)$:

$$(\boldsymbol{g}_1 \boldsymbol{g}_2)(x) = \boldsymbol{g}_1(x)\, \boldsymbol{g}_2(x) \quad , \quad \boldsymbol{g}^{-1}(x) = (\boldsymbol{g}(x))^{-1} \qquad \text{for } x \in U$$

$$(\boldsymbol{g} \cdot \varphi)(x) = \boldsymbol{g}(x) \cdot \varphi(x) \qquad \text{for } x \in U$$

This interpretation is particularly useful for considering induced actions of jet bundles of $\bar{G}$ on jet bundles of E, because it allows to state their definition in the simplest possible way: the action of $J\bar{G}$ on JE, say, induced by an action of $\bar{G}$ on E, when translated from an action of fibers on fibers to an action of local sections on local sections, is simply given by taking the derivative, according to the standard rules.

Let us apply this strategy to the actions of the gauge group bundle and its descendants introduced at the end of the previous section. To this end, we assume throughout the rest of this section that U is an arbitrary but fixed coordinate domain in M over which P is trivial and that we have chosen a section σ of P over U, together with a system of coordinates x^μ on U: together, these will induce, for each of the bundles appearing in eqns (23-28), a trivialization over U which in turn provides an isomorphism between its space of sections over U and an appropriate function space.[6]

Under the identifications $\Gamma(U, P \times_G G) \cong C^\infty(U,G)$, $\Gamma(U, P \times_G Q) \cong C^\infty(U,Q)$, $\Gamma(U, V(P \times_G Q)) \cong C^\infty(U,TQ)$ and $\Gamma(U, \vec{J}(P \times_G Q)) \cong \mathrm{Hom}(TU,TQ)$, for example, the actions (23), (24) and (25) correspond, respectively, to an action

$$\begin{array}{ccc} C^\infty(U,G) \times C^\infty(U,Q) & \longrightarrow & C^\infty(U,Q) \\ (\boldsymbol{g}, \varphi) & \longmapsto & \boldsymbol{g} \cdot \varphi \end{array} \tag{29}$$

defined pointwise, i.e., by

$$(\boldsymbol{g} \cdot \varphi)(x) \;=\; \boldsymbol{g}(x) \cdot \varphi(x) \qquad \text{for } x \in U \ , \tag{30}$$

as above, where the symbol $\cdot$ on the rhs stands for the original action (2) of G on Q, to an action

$$\begin{array}{ccc} C^\infty(U,G) \times C^\infty(U,TQ) & \longrightarrow & C^\infty(U,TQ) \\ (\boldsymbol{g}, \delta\varphi) & \longmapsto & \boldsymbol{g} \cdot \delta\varphi \end{array} \tag{31}$$

defined pointwise, i.e., by

$$(\boldsymbol{g} \cdot \delta\varphi)(x) \;=\; \boldsymbol{g}(x) \cdot \delta\varphi(x) \qquad \text{for } x \in U \ , \tag{32}$$

and to an action

$$\begin{array}{ccc} C^\infty(U,G) \times \mathrm{Hom}(TU,TQ) & \longrightarrow & \mathrm{Hom}(TU,TQ) \\ (\boldsymbol{g}, D\varphi) & \longmapsto & \boldsymbol{g} \cdot D\varphi \end{array} \tag{33}$$

defined pointwise, i.e., by

$$(\boldsymbol{g} \cdot D\varphi)(x,u) \;=\; \boldsymbol{g}(x) \cdot D\varphi(x,u) \qquad \text{for } x \in U,\ u \in T_x M \ , \tag{34}$$

where the symbol $\cdot$ on the rhs now stands for the induced action (3) of G on TQ and we have used the symbols $\delta\varphi$ and $D\varphi$ to indicate that the transformation laws of these objects correspond to those of variations of sections[7] and of covariant derivatives

[6] Among these function spaces we will find spaces of the form $\mathrm{Hom}(E,F)$, where E and F are vector bundles over (possibly different) manifolds M and N, respectively, consisting of all vector bundle homomorphisms from E to F, that is, of all smooth maps from the manifold E to the manifold F which are fiber preserving and fiberwise linear. As an example, note that for f in $C^\infty(M,N)$, its tangent map Tf, which we shall also denote by ∂f, is in $\mathrm{Hom}(TM,TN)$.

[7] Variations $\delta\varphi$ of a section φ of a fiber bundle E over M are formal first order derivatives of one-parameter families $(\varphi_s)_{-\epsilon<s<\epsilon}$ of sections of E around φ with respect to the parameter ($\varphi_0 = \varphi$, $d\varphi_s/ds\,|_{s=0} = \delta\varphi$) and are therefore sections of the pull-back $\varphi^*(VE)$ of the vertical bundle VE of E to M via φ. If, formally, one considers the space $\Gamma(E)$ of sections of E as a manifold, they form its tangent space at φ: $T_\varphi(\Gamma(E)) = \Gamma(\varphi^*(VE))$.

of sections [8]. Similarly, under the identifications $\Gamma(U, J(P \times_G G)) \cong \mathrm{Hom}(TU, TG)$ and $\Gamma(U, J(P \times_G Q)) \cong \mathrm{Hom}(TU, TQ)$, the action (26) corresponds to an action

$$\begin{array}{ccc} \mathrm{Hom}(TU, TG) \times \mathrm{Hom}(TU, TQ) & \longrightarrow & \mathrm{Hom}(TU, TQ) \\ (\partial \boldsymbol{g}, \partial \varphi) & \longmapsto & \partial \boldsymbol{g} \cdot \partial \varphi \end{array} \tag{35}$$

defined pointwise, i.e., by

$$(\partial \boldsymbol{g} \cdot \partial \varphi)(x, u) \;=\; \partial \boldsymbol{g}(x, u) \cdot \partial \varphi(x, u) \qquad \text{for } x \in U,\ u \in T_x M \;, \tag{36}$$

where the symbol $\cdot$ on the rhs now stands for the induced action (4) of TG on TQ and we have used the symbol $\partial \varphi$ to indicate that the transformation law of this object corresponds to that of ordinary derivatives of sections; we can also imagine the symbol $\partial \varphi$ to represent the collection of all partial derivatives $\partial_\mu \varphi$ of φ and, similarly, the symbol $\partial \boldsymbol{g}$ to represent the collection of all partial derivatives $\partial_\mu \boldsymbol{g}$ of $\boldsymbol{g}$.

In order to deal with the remaining two actions, as well as to rewrite the last of the previous ones in a different form, we use additional identifications provided by the trivialization (5) of the tangent bundle TG of G and by the interpretation of sections of CP as connection forms to represent the pertinent objects in a more manageable form. Writing $\mathcal{T}_1^0(U, \mathfrak{g}) = \Omega^1(U, \mathfrak{g})$ for the space of rank 1 tensor fields, or 1-forms, $\mathcal{T}_2^0(U, \mathfrak{g})$ for the space of rank 2 tensor fields and $\mathcal{T}_{2,s}^0(U, \mathfrak{g})$ for the space of symmetric rank 2 tensor fields on U with values in $\mathfrak{g}$, the pertinent isomorphisms for the derived gauge group bundles are

$$\Gamma(U, J(P \times_G G)) \;\cong\; \mathrm{Hom}(TU, TG) \;\cong\; \mathrm{Hom}(TU, G \times \mathfrak{g}) \;\cong\; C^\infty(U, G) \times \mathcal{T}_1^0(U, \mathfrak{g}) \;,$$

and

$$\Gamma(U, J^2(P \times_G G)) \;\cong\; C^\infty(U, G) \times \mathcal{T}_1^0(U, \mathfrak{g}) \times \mathcal{T}_{2,s}^0(U, \mathfrak{g}) \;,$$

while those for the connection bundle and its first order jet bundle are

$$\Gamma(U, CP) \cong \mathcal{T}_1^0(U, \mathfrak{g}) \;,$$

and

$$\Gamma(U, J(CP)) \cong \mathcal{T}_1^0(U, \mathfrak{g}) \times \mathcal{T}_2^0(U, \mathfrak{g}) \;,$$

respectively. Explicitly, in terms of a G-valued function $\boldsymbol{g}$ on U representing a section of $P \times_G G$ over U, the first two isomorphisms are given by the prescription of taking its 1-jet, symbolically represented by the pair $(\boldsymbol{g}, \partial \boldsymbol{g})$, to the pair $(\boldsymbol{g}, \partial \boldsymbol{g}\, \boldsymbol{g}^{-1})$ and its 2-jet, symbolically represented by the triple $(\boldsymbol{g}, \partial \boldsymbol{g}, \partial^2 \boldsymbol{g})$, to the triple $(\boldsymbol{g}, \partial \boldsymbol{g}\, \boldsymbol{g}^{-1}, \partial(\partial \boldsymbol{g}\, \boldsymbol{g}^{-1}))$, where

[8]Given an arbitrary connection in a fiber bundle E over M, which can be viewed as the choice of a horizontal bundle, or equivalently, a vertical projection, or equivalently, a horizontal projection, one has a notion of covariant derivative: the covariant derivative of a section is obtained from its ordinary derivative (tangent map) by composition with the vertical projection. Thus, for any section φ of E and any point $m \in M$, its ordinary derivative at m can be viewed as belonging to the affine space $J_{\varphi(m)}E$ (jet space) but its covariant derivative at m belongs to the vector space $\vec{J}_{\varphi(m)}E$ (linearized jet space).

we can imagine the expression $\partial^2 \boldsymbol{g}$ to represent the collection of all second order partial derivatives $\partial_\mu \partial_\nu \varphi$ of φ while the expression $\partial(\partial \boldsymbol{g}\, \boldsymbol{g}^{-1})$ is supposed to represent the collection of symmetrized partial derivatives $\partial_{(\mu}(\partial_{\nu)} \boldsymbol{g}\, \boldsymbol{g}^{-1})$, with expansions into components as follows:

$$\begin{aligned} \partial \boldsymbol{g} \boldsymbol{g}^{-1} &= (\partial_\mu \boldsymbol{g} \boldsymbol{g}^{-1})\; dx^\mu \\ \partial(\partial \boldsymbol{g}\, \boldsymbol{g}^{-1}) &= \tfrac{1}{2}\big(\partial_\mu(\partial_\nu \boldsymbol{g} \boldsymbol{g}^{-1}) + \partial_\nu(\partial_\mu \boldsymbol{g} \boldsymbol{g}^{-1})\big)\; dx^\mu \otimes dx^\nu \end{aligned} .$$

Similarly, the last two isomorphisms amount to representing a section of CP over U by a $\mathfrak{g}$-valued connection 1-form A on U and its 1-jet by the pair $(A, \partial A)$, where we can imagine the expression ∂A to represent the collection of all partial derivatives $\partial_\mu A_\nu$, with expansions into components as follows:

$$\begin{aligned} A &= A_\mu\; dx^\mu \\ \partial A &= \partial_\mu A_\nu\; dx^\mu \otimes dx^\nu \end{aligned} .$$

With these identifications, the action (35) can be rewritten as an action

$$\begin{array}{ccc} (C^\infty(U,G) \times \mathcal{T}_1^0(U,\mathfrak{g})) \times \mathrm{Hom}(TU,TQ) & \longrightarrow & \mathrm{Hom}(TU,TQ) \\ ((\boldsymbol{g}, \partial \boldsymbol{g} \boldsymbol{g}^{-1}), \partial\varphi) & \longmapsto & (\boldsymbol{g}, \partial \boldsymbol{g} \boldsymbol{g}^{-1}) \cdot \partial\varphi \end{array} \tag{37}$$

defined pointwise, i.e., by

$$\begin{array}{rcl} ((\boldsymbol{g}, \partial \boldsymbol{g} \boldsymbol{g}^{-1}) \cdot \partial\varphi)(x,u) &=& \boldsymbol{g}(x) \cdot (\partial\varphi(x,u)) \;+\; \big((\partial \boldsymbol{g} \boldsymbol{g}^{-1})(x,u)\big)_Q \big(\boldsymbol{g}(x) \cdot \varphi(x)\big) \\ && \text{for } x \in U,\ u \in T_x M \end{array} , \tag{38}$$

where the second symbol $\cdot$ on the rhs stands for the original action (2) of G on Q, the first symbol $\cdot$ on the rhs for the induced action (3) of G on TQ and, for any $X \in \mathfrak{g}$, X_Q denotes the fundamental vector field on Q associated with this generator: this is the precise meaning of the intuitive but formal "Leibniz rule for group actions":

$$\partial(\boldsymbol{g} \cdot \varphi) \;=\; \boldsymbol{g} \cdot \partial\varphi + \partial \boldsymbol{g} \cdot \varphi \;.$$

Similarly, the action (27) corresponds to an action

$$\big(C^\infty(U,G) \times \mathcal{T}_1^0(U,\mathfrak{g})\big) \times \mathcal{T}_1^0(U,\mathfrak{g}) \;\longrightarrow\; \mathcal{T}_1^0(U,\mathfrak{g}) \tag{39}$$

taking $\big((\boldsymbol{g}, \partial \boldsymbol{g} \boldsymbol{g}^{-1}), A\big)$ to $\big(\boldsymbol{g}, \partial \boldsymbol{g} \boldsymbol{g}^{-1}\big) \cdot A$, which is given by the well known transformation law

$$\big((\boldsymbol{g}, \partial \boldsymbol{g} \boldsymbol{g}^{-1}) \cdot A\big)_\mu \;=\; \boldsymbol{g} A_\mu \boldsymbol{g}^{-1} \;-\; \partial_\mu \boldsymbol{g} \boldsymbol{g}^{-1} \;, \tag{40}$$

and finally the action (28) corresponds to an action

$$\big(C^\infty(U,G) \times \mathcal{T}_1^0(U,\mathfrak{g}) \times \mathcal{T}_{2,s}^0(U,\mathfrak{g})\big) \times \big(\mathcal{T}_1^0(U,\mathfrak{g}) \times \mathcal{T}_2^0(U,\mathfrak{g})\big) \;\longrightarrow\; \mathcal{T}_1^0(U,\mathfrak{g}) \times \mathcal{T}_2^0(U,\mathfrak{g}) \tag{41}$$

taking $\big((\boldsymbol{g}, \partial \boldsymbol{g} \boldsymbol{g}^{-1}, \partial(\partial \boldsymbol{g}\, \boldsymbol{g}^{-1})), (A, \partial A)\big)$ to $\big(\boldsymbol{g}, \partial \boldsymbol{g} \boldsymbol{g}^{-1}, \partial(\partial \boldsymbol{g}\, \boldsymbol{g}^{-1})\big) \cdot \big(A, \partial A\big)$, which is given by differentiating eqn (40) and rearranging terms:

$$\begin{aligned} &\big((\boldsymbol{g}, \partial \boldsymbol{g} \boldsymbol{g}^{-1}, \partial(\partial \boldsymbol{g}\, \boldsymbol{g}^{-1})) \cdot (A, \partial A)\big)_{\mu\nu} \\ &\quad = \boldsymbol{g}\, \partial_\mu A_\nu\, \boldsymbol{g}^{-1} + [\partial_\mu \boldsymbol{g} \boldsymbol{g}^{-1}, \boldsymbol{g} A_\nu \boldsymbol{g}^{-1}] - \tfrac{1}{2}\, [\partial_\mu \boldsymbol{g} \boldsymbol{g}^{-1}, \partial_\nu \boldsymbol{g} \boldsymbol{g}^{-1}] \\ &\qquad - \tfrac{1}{2}\, \partial_\mu(\partial_\nu \boldsymbol{g} \boldsymbol{g}^{-1}) - \tfrac{1}{2}\, \partial_\nu(\partial_\mu \boldsymbol{g} \boldsymbol{g}^{-1}) \;. \end{aligned} \tag{42}$$

For later use, we decompose this transformation law into its symmetric part

$$\begin{aligned} &\big((g, \partial g g^{-1}, \partial(\partial g\, g^{-1})) \cdot (A, \partial A)\big)_{\mu\nu} + \big((g, \partial g g^{-1}, \partial(\partial g\, g^{-1})) \cdot (A, \partial A)\big)_{\nu\mu} \\ &\quad = g\big(\partial_\mu A_\nu + \partial_\nu A_\mu\big)\, g^{-1} + [\partial_\mu g g^{-1}, g A_\nu g^{-1}] + [\partial_\nu g g^{-1}, g A_\mu g^{-1}] \\ &\qquad - \tfrac{1}{2}\, \partial_\mu(\partial_\nu g g^{-1}) - \tfrac{1}{2}\, \partial_\nu(\partial_\mu g g^{-1})\ , \end{aligned} \tag{43}$$

and its antisymmetric part

$$\begin{aligned} &\big((g, \partial g g^{-1}, \partial(\partial g\, g^{-1})) \cdot (A, \partial A)\big)_{\mu\nu} - \big((g, \partial g g^{-1}, \partial(\partial g\, g^{-1})) \cdot (A, \partial A)\big)_{\nu\mu} \\ &\quad = g\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big)\, g^{-1} + [\partial_\mu g g^{-1}, g A_\nu g^{-1}] - [\partial_\nu g g^{-1}, g A_\mu g^{-1}] \\ &\qquad - [\partial_\mu g g^{-1}, \partial_\nu g g^{-1}]\ , \end{aligned} \tag{44}$$

the latter being equivalent to the simple transformation law

$$\big(g \cdot F\big)_{\mu\nu} = g F_{\mu\nu} g^{-1} \tag{45}$$

for the curvature form F of the connection form A, with components

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\ .$$

As is well known, the affine transformation law (40) implies that one can, at any given point of M, gauge the potential A to zero by an appropriate choice of gauge transformation, namely by assuming that at the given point x, the value of g is 1 and that of its first order partial derivatives is given by $\partial_\mu g(x) = A_\mu(x)$. In the language adopted here, this translates into

Proposition 2 *The action of $J(P \times_G G)$ on CP is fiber transitive.*

Similarly, the affine transformation laws (42-44) imply that one can, at any given point of M, gauge the potential A to zero, the symmetric part of its derivative to zero and the antisymmetric part of its derivative to be equal to its curvature by an appropriate choice of gauge transformation, namely by assuming that at the given point x, the value of g is 1, that of its first order partial derivatives is given by $\partial_\mu g(x) = A_\mu(x)$ and that of its second order partial derivatives is given by $\big(\partial_\mu(\partial_\nu g\, g^{-1}) + \partial_\nu(\partial_\mu g\, g^{-1})\big)(x) = \big(\partial_\mu A_\nu + \partial_\nu A_\mu\big)(x)$. In the language adopted here, this translates into

Proposition 3 *The action of $J^2(P \times_G G)$ on $J(CP)$ preserves the decomposition (12) into symmetric and antisymmetric part, is fiber transitive on the symmetric part and under the curvature map (which is the projection to the antisymmetric part) is taken to the natural action of $P \times_G G$ on $\bigwedge^2 T^*M \otimes (P \times_G \mathfrak{g})$ induced by the adjoint representation of G on $\mathfrak{g}$.*

5 Global and local invariance

In the usual geometric formulation of a gauge theory with structure group G and underlying principal G-bundle P over space-time M, gauge potentials are represented by connection forms A on P, which can be reinterpreted as sections of the bundle CP of principal connections on P, and the matter fields (all assembled into one big multiplet) by sections of a fiber bundle $P \times_G Q$ associated to P. The configuration bundle of the whole theory is thus the fiber product $E = CP \times_M (P \times_G Q)$. The group $\mathrm{Aut}(P)$ of automorphisms of P and the subgroup $\mathrm{Gau}(P)$ of gauge transformations, or in mathematical terms, of strict automorphisms of P, act naturally on $\Gamma(CP)$ and on $\Gamma(P \times_G Q)$ and, therefore, also on $\Gamma(E)$, and it is to this group and its actions that one usually refers to when speaking about gauge invariance. Unfortunately, the group $\mathrm{Gau}(P)$ and all the spaces on which it acts are infinite-dimensional, which makes this kind of symmetry very hard to handle. As observed in the introduction, this happens already in the case of mechanics. And precisely as in that case, there is a way out: we can convert the principle of gauge invariance into a completely finite-dimensional setting by making use of the natural actions of Lie group bundles introduced in the preceding two sections. In fact, in a completely geometric formulation of field theory, even the correct implementation of global symmetries, in the matter field sector, already requires the use of actions of Lie group bundles over space-time, rather than just ordinary Lie groups!

At first sight, the reader will probably find the last statement rather surprising (we certainly did when we first stumbled over it), partly because it seems to go against widespread belief (but as the reader will be able to convince himself as we go along, this is not really the case), partly also because it is not immediately visible in the mechanical analogue discussed in the introduction. However, its origins are really quite easy to understand. Consider, for example, a function V on the total space $P \times_G Q$, which may represent a potential term for some matter field lagrangian: then since it is not the Lie group G itself but rather the Lie group bundle $P \times_G G$ that acts naturally on this total space,[9] the only way to formulate the condition that V be globally invariant is to require V to be invariant under the action of $P \times_G G$. For a truly dynamical theory, with configuration bundle $P \times_G Q$, we must of course include derivatives of fields that are sections of $P \times_G Q$. For simplicity, we shall assume, as always in this paper, that the dynamics of the theory is governed by a first order "matter field lagrangian", for which we contemplate three slightly different options regarding the choice of its domain,[10] depending on whether we use the linearized first order jet bundle $\vec{J}(P \times_G Q)$ or the full first order jet bundle $J(P \times_G Q)$ of $P \times_G Q$ and whether we include an explicit dependence on connections or not:

[9]Recall that in the definition of $P \times_G Q$, we divide $P \times Q$ by the joint action of G, so $P \times_G Q$ no longer carries any remnants of the action of G on Q.

[10]For the range, we use volume forms (pseudoscalars) rather than scalars: this allows us to integrate lagrangians over regions of space-time without having to fix a separately defined volume form on M.

- $\vec{\mathcal{L}}_{\mathrm{mat}} : \vec{J}(P \times_G Q) \longrightarrow \bigwedge^n T^* M$: such a lagrangian will be called **globally invariant** if it is invariant under the action (25) of the Lie group bundle $P \times_G G$ on $\vec{J}(P \times_G Q)$.
- $\mathcal{L}_{\mathrm{mat}} : J(P \times_G Q) \longrightarrow \bigwedge^n T^* M$: such a lagrangian will be called **locally invariant** or **gauge invariant** if, for every compact subset K of space-time M, the action functional $(\mathcal{S}_{\mathrm{mat}})_K : \Gamma(P \times_G Q) \longrightarrow \mathbb{R}$ defined by integration of $\mathcal{L}_{\mathrm{mat}}$ over K, i.e.,

$$(\mathcal{S}_{\mathrm{mat}})_K[\varphi] \;=\; \int_K \mathcal{L}_{\mathrm{mat}}(\varphi, \partial\varphi) \;, \tag{46}$$

  is invariant under the action of the group $\mathrm{Gau}(P)$.
- $\mathcal{L}_{\mathrm{mat}} : CP \times_M J(P \times_G Q) \longrightarrow \bigwedge^n T^* M$: again, such a lagrangian will be called **locally invariant** or **gauge invariant** if, for every compact subset K of space-time M, the action functional $(\mathcal{S}_{\mathrm{mat}})_K : \Gamma(CP \times_M (P \times_G Q)) \longrightarrow \mathbb{R}$ defined by integration of $\mathcal{L}_{\mathrm{mat}}$ over K, i.e.,

$$(\mathcal{S}_{\mathrm{mat}})_K[A, \varphi] \;=\; \int_K \mathcal{L}_{\mathrm{mat}}(A, \varphi, \partial\varphi) \;, \tag{47}$$

  is invariant under the action of the group $\mathrm{Gau}(P)$.

Regarding gauge invariance, we then have the following

Theorem 3 *The action functional $\mathcal{S}_{\mathrm{mat}}$ defined by integration of the lagrangian $\mathcal{L}_{\mathrm{mat}}$ over compact subsets of space-time is gauge invariant if and only if the lagrangian $\mathcal{L}_{\mathrm{mat}}$ is invariant under the action (26) of the Lie group bundle $J(P \times_G G)$ on $J(P \times_G Q)$, in the first case, and under the action of the Lie group bundle $J(P \times_G G)$ on $CP \times_M J(P \times_G Q)$ that results from combining its actions (27) on CP and (26) on $J(P \times_G Q)$, in the second case.*

Proof: As observed in the preceding sections, the action (23) of $P \times_G G$ on $P \times_G Q$, when lifted to sections, induces the standard action of strict automorphisms of P on sections φ of $P \times_G Q$, the action (26) of $J(P \times_G G)$ on $J(P \times_G Q)$, when lifted to sections, induces the standard action of strict automorphisms of P on sections of $P \times_G Q$ together with their first order derivatives, and finally the action (27) of $J(P \times_G G)$ on CP, when lifted to sections, induces the standard action of strict automorphisms of P on connection forms on P by pull-back. Now since gauge transformations do not move the points of space-time, invariance of the integral $(\mathcal{S}_{\mathrm{mat}})_K$ over any compact subset K of M is equivalent to invariance of the integrand $\mathcal{L}_{\mathrm{mat}}$. □

It should be noted at this point that without specifying further data, only the third of the above versions is physically meaningful. Indeed, the function $\vec{\mathcal{L}}_{\mathrm{mat}}$ of the first version, by itself, is not an acceptable lagrangian because lagrangians in field theory must be defined on the first order jet bundle and not on the linearized first order jet bundle. On the other hand, lagrangians $\mathcal{L}_{\mathrm{mat}}$ as in the second version, depending only on the matter fields and their first order partial derivatives, are impossible to construct directly: all known examples require additional data. (In particular, this holds when the fields are represented by sections of bundles which may be non-trivial or at least are not manifestly trivialized.) The only way to overcome these problems, all in one single stroke, is to introduce some connection in P: this may either be a fixed (preferrably, flat) background connection which allows us to identify $\vec{J}(P \times_G Q)$ and $J(P \times_G Q)$ (and is usually introduced tacitly, without ever being mentioned explicitly), or it may itself be a dynamical variable, as indicated in the third version above. In the next section, we shall show that when this is done, there is a natural prescription, called "minimal coupling", that allows us to pass from a globally invariant lagrangian $\vec{\mathcal{L}}_{\mathrm{mat}}$ on $\vec{J}(P \times_G Q)$ to a locally invariant lagrangian $\mathcal{L}_{\mathrm{mat}}$ on $CP \times_M J(P \times_G Q)$, and back.

In the gauge field sector, the situation is completely analogous. Now the configuration bundle is just CP, and the dynamics of the theory is assumed to be governed by a first order "gauge field lagrangian", which is a map $\mathcal{L}_{\mathrm{gauge}} : J(CP) \longrightarrow \bigwedge^n T^*M$. (The standard example is of course the Yang-Mills lagrangian.) Again, such a lagrangian will be called **locally invariant** or **gauge invariant** if, for every compact subset K of space-time M, the action functional $(\mathcal{S}_{\mathrm{gauge}})_K : \Gamma(CP) \longrightarrow \mathbb{R}$ defined by integration of $\mathcal{L}_{\mathrm{gauge}}$ over K, i.e.,

$$(\mathcal{S}_{\mathrm{gauge}})_K[A] \;=\; \int_K \mathcal{L}_{\mathrm{gauge}}(A, \partial A) \;, \tag{48}$$

is invariant under the action of the group $\mathrm{Gau}(P)$.

Theorem 4 *The action functional $\mathcal{S}_{\mathrm{gauge}}$ defined by integration of the lagrangian $\mathcal{L}_{\mathrm{gauge}}$ over compact subsets of space-time is gauge invariant if and only if the lagrangian $\mathcal{L}_{\mathrm{gauge}}$ is invariant under the action (28) of the Lie group bundle $J^2(P \times_G G)$ on $J(CP)$.*

Proof: As observed in the preceding sections, the action (27) of $J(P \times_G G)$ on CP, when lifted to sections, induces the standard action of strict automorphisms of P on connection forms A on P and the action (28) of $J^2(P \times_G G)$ on $J(CP)$, when lifted to sections, induces the standard action of strict automorphisms of P on connection forms on P together with their first order derivatives. Now since gauge transformations do not move the points of space-time, invariance of the integral $(\mathcal{S}_{\mathrm{mat}})_K$ over any compact subset K of M is equivalent to invariance of the integrand $\mathcal{L}_{\mathrm{mat}}$. $\square$

The general situation is handled by simply combining the previous constructions. The configuration bundle is now $E = CP \times_M (P \times_G Q)$, as stated at the beginning of the section, and the dynamics of the complete theory is governed by a total lagrangian $\mathcal{L} : JE \longrightarrow \bigwedge^n T^*M$ which is the sum of a "gauge field lagrangian" $\mathcal{L}_{\mathrm{gauge}}$, as before, and a "matter field lagrangian" $\mathcal{L}_{\mathrm{mat}} : CP \times_M J(P \times_G Q) \longrightarrow \bigwedge^n T^*M$:

$$\mathcal{L}(A, \partial A, \varphi, \partial\varphi) \;=\; \mathcal{L}_{\mathrm{gauge}}(A, \partial A) \,+\, \mathcal{L}_{\mathrm{mat}}(A, \varphi, \partial\varphi) \;. \tag{49}$$

(Note that the gauge fields do appear in the matter field lagrangian – otherwise, there would be no coupling between matter fields and gauge fields and hence no interaction – but appear only as auxiliary fields, or Lagrange multipliers, i.e., through the gauge potentials without any derivatives.) The hypothesis of local (or gauge) invariance is then understood to mean that both $\mathcal{L}_{\mathrm{gauge}}$ and $\mathcal{L}_{\mathrm{mat}}$ should be locally (or gauge) invariant, and so Theorems 3 and 4 apply as before.

An even stronger hypothesis than gauge invariance, which might be called gauge plus space-time diffeomorphism invariance, would be that these functionals are invariant under the action of the whole group $\mathrm{Aut}(P)$. However, this condition is really much too strong: it is realized to some extent in "topological field theories", the standard examples of which are Chern-Simons theories in 3-dimensional space-time, but it certainly does not hold for gauge theories that are of direct physical relevance. What might be considered is invariance under subgroups of $\mathrm{Aut}(P)$ that cover space-time isometries (Poincaré transformations, in the case of flat Minkowski space-time), rather than arbitrary diffeomorphisms.

6 Minimal Coupling

In this section, we want to show that the prescription of "minimal coupling", which plays a central role in the construction of gauge invariant lagrangians for the matter field sector, can be understood mathematically as a one-to-one correspondence between globally invariant matter field lagrangians

$$\vec{\mathcal{L}}_{\mathrm{mat}} : \vec{J}(P \times_G Q) \;\longrightarrow\; \textstyle\bigwedge^n T^*M \;, \tag{50}$$

and locally invariant matter field lagrangians

$$\mathcal{L}_{\mathrm{mat}} : CP \times_M J(P \times_G Q) \;\longrightarrow\; \textstyle\bigwedge^n T^*M \;. \tag{51}$$

The basic idea underlying this correspondence can be summarized in a simple commutative diagram,

$$\begin{tikzcd} J(P \times_G G) \arrow[r, dashed, "\text{acts}", "\text{on}"'] \arrow[d] & CP \times_M J(P \times_G Q) \arrow[d, "D"'] \arrow[dr, "\mathcal{L}_{\mathrm{mat}}"] & \\ P \times_G G \arrow[r, dashed, "\text{acts}", "\text{on}"'] & \vec{J}(P \times_G Q) \arrow[r, "\vec{\mathcal{L}}_{\mathrm{mat}}"] & \bigwedge^n T^*M \;, \end{tikzcd} \tag{52}$$

where the first vertical arrow is the target projection from $J(P \times_G G)$ to $P \times_G G$ while the second vertical arrow is the **covariant derivative map** defined by

$$D(A, \varphi, \partial\varphi) \;=\; (\varphi, D_A\varphi) \;, \tag{53}$$

where, for any local section φ of $P \times_G Q$, its covariant derivative $D_A\varphi$ with respect to A can be defined simply as the composition of its standard derivative $\partial\varphi$ with the corresponding vertical projection. As indicated in the diagram, this map is equivariant under the respective actions of the Lie group bundles $J(P \times_G G)$ and $P \times_G G$, and even more than that is true: it takes the $J(P \times_G G)$-orbits in $CP \times_M J(P \times_G Q)$ precisely onto the $(P \times_G G)$-orbits in $\vec{J}(P \times_G Q)$. This proves our claim that the formula

$$\mathcal{L}_{\mathrm{mat}}(A, \varphi, \partial\varphi) \;=\; \vec{\mathcal{L}}_{\mathrm{mat}}(\varphi, D_A\varphi) \;. \tag{54}$$

establishes a one-to-one correspondence between $(\bigwedge^n T^*M)$-valued functions $\mathcal{L}_{\mathrm{mat}}$ on $CP \times_M J(P \times_G Q)$ and $(\bigwedge^n T^*M)$-valued functions $\vec{\mathcal{L}}_{\mathrm{mat}}$ on $\vec{J}(P \times_G Q)$: this construction of the former from the latter is what is known as the prescription of **minimal coupling**.

7 Utiyama's Theorem

Another important fact concerning the construction of gauge invariant lagrangians, this time in the gauge field sector, is known as Utiyama's theorem: it states, roughly speaking, that any gauge invariant lagrangian in the gauge field sector must be a function only of the curvature tensor and its covariant derivatives. In the present context where we consider only first order lagrangians, it can be understood mathematically as a one-to-one correspondence between globally invariant lagrangians

$$\mathcal{L}_{\mathrm{curv}} : {\textstyle\bigwedge^2} T^*M \otimes (P \times_G \mathfrak{g}) \;\longrightarrow\; {\textstyle\bigwedge^n} T^*M \;, \tag{55}$$

and locally invariant gauge field lagrangians

$$\mathcal{L}_{\mathrm{gauge}} : J(CP) \;\longrightarrow\; {\textstyle\bigwedge^n} T^*M \;. \tag{56}$$

Again, the basic idea underlying this correspondence can be summarized in a simple commutative diagram

$$\begin{tikzcd}
J^2(P \times_G G) \arrow[r, dashed, "\text{acts}" above, "\text{on}" below] \arrow[d] & J(CP) \arrow[d, "F"'] \arrow[dr, "\mathcal{L}_{\mathrm{gauge}}"] & \\
P \times_G G \arrow[r, dashed, "\text{acts}" above, "\text{on}" below] & {\textstyle\bigwedge^2} T^*M \otimes (P \times_G \mathfrak{g}) \arrow[r, "\mathcal{L}_{\mathrm{curv}}"'] & {\textstyle\bigwedge^n} T^*M \;,
\end{tikzcd} \tag{57}$$

where the first vertical arrow is the target projection from $J^2(P \times_G G)$ to $P \times_G G$ while the second vertical arrow is the **curvature map** that to each connection form A associates its curvature form F_A. As indicated in the diagram, this map is equivariant under the respective actions of the Lie group bundles $J^2(P \times_G G)$ and $P \times_G G$, and as stated in

Proposition 3, it takes the $J^2(P \times_G G)$-orbits in $J(CP)$ precisely onto the $(P \times_G G)$-orbits in $\bigwedge^2 T^*M \otimes (P \times_G \mathfrak{g})$. This proves our claim that the formula

$$\mathcal{L}_{\mathrm{gauge}}(A, \partial A) \;=\; \mathcal{L}_{\mathrm{curv}}(F_A) \;. \tag{58}$$

establishes a one-to-one correspondence between $(\bigwedge^n T^*M)$-valued functions $\mathcal{L}_{\mathrm{gauge}}$ on $J(CP)$ and $(\bigwedge^n T^*M)$-valued functions $\mathcal{L}_{\mathrm{curv}}$ on $\bigwedge^2 T^*M \otimes (P \times_G \mathfrak{g})$.

8 Conclusions and Outlook

As we have tried to demonstrate in this paper, the appropriate mathematical concept for dealing with symmetries in classical field theory, when adopting a geometrical and at the same time purely finite-dimensional framework (as opposed to a functional approach), is that of Lie group bundles and their actions on fiber bundles over space-time M (whose sections constitute the fields of the model at hand). This general statement applies to internal symmetries, global as well as local, allowing to view the passage from the former to the latter – generally known as the procedure of "gauging a symmetry" – simply as the transition from the original Lie group bundle to its jet bundle. It also allows for a conceptually transparent and natural formulation of various procedures and statements that play an important role in gauge theories, such as the prescription of "minimal coupling" and Utiyama's theorem on the possible form of gauge invariant lagrangians for the pure gauge field sector. The entire approach is a generalization of a corresponding appraoch to classical mechanics [6], to which it reduces when one takes $M = \mathbb{R}$, which implies that the principal G-bundle P over M is trivial (and hence so are all other bundles involved), and when one supposes that a fixed trivialization has been chosen: then the product in $P \times_G G \cong \mathbb{R} \times G$ and its action on $P \times_G Q \cong \mathbb{R} \times Q$ do not depend on the base point, or in other words, they reduce to an ordinary Lie group product in G and an ordinary action of G on the manifold Q. Similarly, the induced product in $J(P \times_G G) \cong \mathbb{R} \times TG$ and its induced action on $J(P \times_G Q) \cong \mathbb{R} \times TQ$ also do not depend on the base point and reduce to the ordinary induced Lie group product in TG and the ordinary induced action of TG on TQ. These reductions explain why in the case of mechanics, the need for using Lie group bundles, rather than ordinary Lie groups, was not properly appreciated.

Finally, the extension of this approach to achieve unification of internal symmetries with space-time symmetries requires the transition from Lie group bundles to Lie groupoids – a problem which is presently under investigation. Another issue is how to correctly formulate invariance of geometric objects represented by certain prescribed tensor fields (such as pseudo-riemannian metrics or symplectic forms, for instance) under actions of Lie group bundles or, more generally, Lie groupoids and/or their infinitesimal counterparts, that is, Lie algebra bundles or, more generally, Lie algebroids. These and similar questions will have to be answered before one can hope to really understand what is the field theoretical analogue of the momentum map of classical mechanics.

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References

[1] H. Weyl: *Gravitation und Elektrizität*, Sitzungsber. Preuss. Akad. Wiss. Berlin **26** (1918) 465-480.

[2] H. Weyl: *Elektron und Gravitation I*, Z. Phys. **56** (1929) 330-352.

[3] C.N. Yang & R.L. Mills: *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, Phys. Rev. **96** (1954) 191-195.

[4] R. Utiyama: *Invariant Theoretical Interpretation of Interaction*, Phys. Rev. **101** (1956) 1597-1607.

[5] S. Hawking & G.F.R. Ellis: *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge 1973.

[6] M. Forger & J. Kellendonk: *Classical BRST Cohomology and Invariant Functions on Constraint Manifolds I*, Commun. Math. Phys. **143** (1992) 235-251.

[7] B.L. Soares: *Simetrias Globais e Locais em Teorias de Calibre*, PhD thesis, IME-USP, March 2007.

[8] M. Daniel & C.M. Viallet: *The Geometrical Setting of Gauge Theories of the Yang-Mills Type*, Rev. Mod. Phys. **52** (1980) 175-197.

[9] D. Bleecker: *Gauge Theory and Variational Principles*, Addison-Wesley, Reading 1981.

[10] G. Naber: *Topology, Geometry and Gauge Fields - Interactions*, Springer, Berlin 2000.

[11] P.L. Garcia: *Gauge Algebras, Curvature and Symplectic Geometry*, J. Diff. Geom. **12** (1977) 209-227.

[12] L. Mangiarotti & M. Modugno: *On the Geometric Structure of Gauge Theories*, J. Math. Phys. **26** (1985) 1373-1379.

[13] D. Betounes: *The Geometry of Gauge Particle Field Interaction: A Generalization of Utiyama's Theorem*, J. Geom. Phys. **6** (1989) 107-125.

[14] K.C.H, Mackenzie: *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press, Cambridge 2005.

[15] M.E. Mayer: *Groupoids and Lie Bigebras in Gauge and String Theories*, in: Differential Geometric Methods in Theoretical Physics (Proceedings of the NATO Advanced Research Workshop and the 16th International Conference, Como, Italy 1987), pp. 149-164, eds: K. Bleuler & M. Werner, Kluwer, Dordrecht 1988.

[16] M. Keyl: *About the Geometric Structure of Symmetry Breaking*, J. Math. Phys. **32** (1991) 1065-1071.

[17] D.J. Saunders: *The Geometry of Jet Bundles*, Cambridge University Press, Cambridge 1989.

[18] M. Forger & S.V. Romero: *Covariant Poisson Brackets in Geometric Field Theory*, Commun. Math. Phys. **256** (2005) 375-410, `arXiv:math-ph/0408008`.