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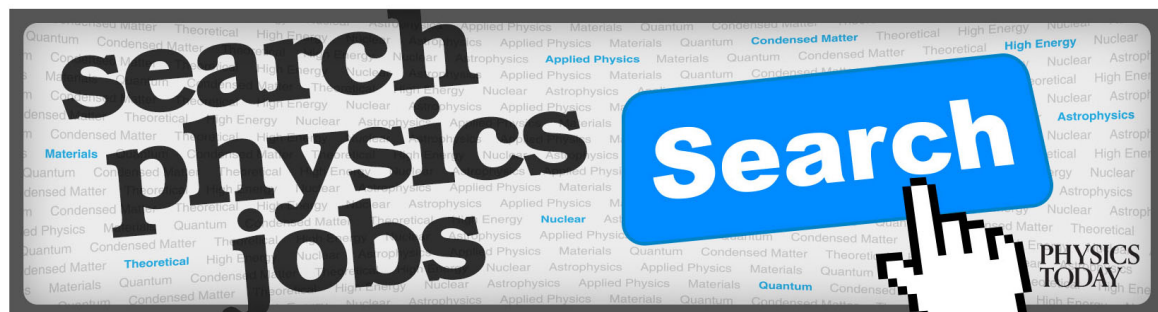
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On covariant Poisson brackets in classical field theory

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How to give a natural geometric definition of a covariant Poisson bracket in classical field theory has for a long time been an open problem—as testified by the extensive literature on “multisymplectic Poisson brackets,” together with the fact that all these proposals suffer from serious defects. On the other hand, the functional approach does provide a good candidate which has come to be known as the Peierls–De Witt bracket and whose construction in a geometrical setting is now well understood. Here, we show how the basic “multisymplectic Poisson bracket” already proposed in the 1970s can be derived from the Peierls–De Witt bracket, applied to a special class of functionals. This relation allows to trace back most (if not all) of the problems encountered in the past to ambiguities (the relation between differential forms on multiphase space and the functionals they define is not one-to-one) and also to the fact that this class of functionals does not form a Poisson subalgebra. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4932011>]

I. INTRODUCTION

The quest for a fully covariant hamiltonian formulation of classical field theory has a long history which can be traced back to the work of Carathéodory,³ DeDonder,⁷ and Weyl³⁴ on the calculus of variations. From a modern point of view, one of the main motivations is the issue of quantization which, in traditional versions like canonical quantization as well as more recent ones such as deformation quantization, starts by bringing the classical theory into hamiltonian form. In the context of mechanics, where one is dealing with systems with a finite number of degrees of freedom, this has led mathematicians to develop entire new areas of differential geometry, namely, symplectic geometry and then Poisson geometry, whereas physicists have been motivated to embark on a more profound analysis of basic physical concepts such as those of states and observables. In the context of (relativistic) field theory, however, this is not sufficient since, besides facing the formidable mathematical problem of handling systems with an infinite number of degrees of freedom, we have to cope with new physical principles, most notably those of covariance and of locality. The principle of covariance states that meaningful laws of physics do not depend on the choice of (local) coordinates in space-time employed in their formulation: extending the axiom of Lorentz invariance in special relativity, it is one of the cornerstones of general relativity and underlies the modern geometrical approach to field theory as a whole. Equally important is the principle of locality, stating that events (including measurements of physical quantities) localized in regions of space-time that are spacelike separated cannot exert any influence on each other. Clearly, a mathematically and physically correct hamiltonian formalism for classical field theory should respect these principles: it should be manifestly covariant and local, as is the modern algebraic approach to quantum field theory; see, e.g., Ref. 2.

As an example of a method that does not meet these requirements, we may quote the standard hamiltonian formulation of classical field theory, based on a functional formalism in terms of Cauchy data: there, the mere necessity of choosing some Cauchy surface spoils covariance from the very beginning! To avoid that, a different approach is needed.

Over the last few decades, attempts to construct such a different approach have produced a variety of proposals that, roughly speaking, can be assembled into two groups.

One of these extends the geometrical tools that were so successful in mechanics to the situation encountered in field theory by treating spatial derivatives of fields on the same footing as time derivatives: in the context of a first order formalism, as in mechanics, this requires associating to each field component, say φ^i , not just one canonically conjugate momentum $\pi_i = \partial L / \partial \dot{\varphi}^i$ but rather n canonically conjugate momenta $\pi_i^\mu = \partial L / \partial \partial_\mu \varphi^i$, where n is the dimension of space-time. (In mechanics, time is the only independent variable, so $n = 1$.) Identifying the appropriate geometrical context has led to the introduction of new geometrical entities now commonly referred to as “multisymplectic” and/or “polysymplectic” structures, and although their correct mathematical definition has only recently been completely elucidated,¹¹ the entire circle of ideas surrounding them is already reasonably well established, forming a new area of differential geometry; see Refs. 4, 16–21, and 27–31 for early references.

A different line of thought is centered around the concept of “covariant phase space,”^{5,6,35} defined as the space of solutions of the equations of motion: using this space to substitute the corresponding space of Cauchy data eliminates the need to refer to a specific choice of Cauchy surface and has the additional benefit of providing an embedding into the larger space of all field configurations, allowing us to classify statements as valid “off shell” (i.e., on the entire space of field configurations) or “on shell” (i.e., only on the subspace of solutions of the equations of motion).

Each of the two methods, the *multisymplectic formalism* as well as the *covariant functional formalism*, has its own merits and its own drawbacks, and experience has shown that best results are obtained by appropriately combining them.

As an example to demonstrate how useful the combination of these two approaches can become, we shall in the present paper discuss the problem of giving an appropriate definition of the Poisson bracket, or better, the *covariant Poisson bracket*. From the point of view of quantization, this is a question of fundamental importance, given the fact that the Poisson bracket is expected to be the classical limit of the commutator in quantum field theory. Moreover, quantum field theory provides compelling motivation for discussing this limit in a covariant setting, taking into account that the (non-covariant) equal-time Poisson brackets of the standard hamiltonian formulation of classical field theory would correspond, in the sense of a classical limit, to the (non-covariant) equal-time commutators of quantum field theory, which are known not to exist in interacting quantum field theories, due to Haag’s theorem.

Unfortunately, in the context of the multisymplectic formalism, the status of covariant Poisson brackets is highly unsatisfactory. This may come as a bit of a surprise, given the beautiful and conceptually simple situation prevailing in mechanics, where the existence of a Poisson bracket on the algebra $C^\infty(P)$ of smooth functions on a manifold P is equivalent to the statement that P is a Poisson manifold and, as such, qualifies as a candidate for the phase space of a classical hamiltonian system: for any such system, the algebra of observables is just the Poisson algebra $C^\infty(P)$ itself or, possibly, an appropriate subalgebra thereof, and the space of pure states is just the Poisson manifold P itself. In particular, this is true in the special case when P is a *symplectic manifold*, with symplectic form ω , say, and where the Poisson bracket of two functions $f, g \in C^\infty(P)$ is the function $\{f, g\} \in C^\infty(P)$ defined by

$$\{f, g\} = i_{X_g} i_{X_f} \omega = \omega(X_f, X_g), \quad (1)$$

where $X_f \in \mathfrak{X}(P)$ denotes the hamiltonian vector field associated with $f \in C^\infty(P)$, uniquely determined by the formula

$$i_{X_f} \omega = df. \quad (2)$$

This situation changes considerably, and for the worse, when we pass to the multisymplectic setting, where ω is no longer a 2-form but rather an $(n + 1)$ -form and the hamiltonian vector field X_f is no longer associated with a function f but rather with an $(n - 1)$ -form f , n being the dimension of space-time. It can then be shown that Equation (2) imposes restrictions not only on the type of vector field that is allowed on its lhs but also on the type of differential form that is allowed on its rhs. Indeed, the validity of an equation of the form $i_X \omega = df$ implies that the vector field X must be

locally hamiltonian, i.e., we have $L_X\omega = 0$, but it also implies that the form f must be hamiltonian, which by definition means that its exterior derivative df must vanish on all multivectors of degree n whose contraction with the $(n + 1)$ -form ω is zero, and this is a non-trivial condition as soon as $n > 1$. (It is trivial for $n = 1$ since ω is assumed to be non-degenerate.) Thus it is only on the space $\Omega_H^{n-1}(P)$ of hamiltonian $(n - 1)$ -forms that Equation (1) provides a reasonable candidate for a Poisson bracket. And even with this restriction, we get a Lie algebra $\Omega_H^{n-1}(P)$ which, in contrast to the situation in mechanics, has a huge center, containing the entire space $Z^{n-1}(P)$ of closed $(n - 1)$ -forms on P , since the linear map from $\Omega_H^{n-1}(P)$ to $\mathfrak{X}_{LH}(P)$ that takes f to X_f is far from being one-to-one: its kernel is precisely $Z^{n-1}(P)$. Still, the argument suggests that the transition from mechanics to field theory should somehow involve a replacement of functions by differential forms of degree $n - 1$ — which is not completely unreasonable when we consider the fact that, in field theory, conservation laws are formulated in terms of conserved currents, which are closed $(n - 1)$ -forms.

Unfortunately, this replacement leads to a whole bunch of serious problems, some of which seem insurmountable. First and foremost, there is no reasonable candidate for an associative product on the space $\Omega_H^{n-1}(P)$ which would provide even a starting point for defining a Poisson algebra. Second, as has been observed repeatedly in the literature,^{16,22–27} the condition of being a locally hamiltonian vector field or a hamiltonian $(n - 1)$ -form forces these objects to depend at most linearly on the multimomentum variables, and moreover we can easily think of observables that are associated to forms of other degree (such as a scalar field, corresponding to a 0-form, or the electromagnetic field strength tensor, corresponding to a 2-form): this by itself provides enough evidence to conclude that hamiltonian $(n - 1)$ -forms constitute an extremely restricted class of observables and that setting up an adequate framework for general observables will require going beyond this domain. And finally, as has already also been noted long ago,^{16,17,20,27–29} the multisymplectic Poisson bracket defined by Equation (1) fails to satisfy the Jacobi identity. In the case of an exact multisymplectic form (i.e., when $\omega = -d\theta$), this last problem can be cured by modifying the defining equation (1) through the addition of an exact (hence closed) term, as follows¹²:

$$\{f, g\} = i_{X_g}i_{X_f}\omega + d(i_{X_g}f - i_{X_f}g - i_{X_g}i_{X_f}\theta). \quad (3)$$

However, this does not settle any of the other two issues, namely,

- the lack of an associative product to construct a Poisson algebra;
- the restriction to hamiltonian forms and forms of degree $n - 1$, which leads to unreasonable constraints on the observables that are allowed, excluding some that appear naturally in physicists' calculations.

It should be mentioned here that these are long-standing problems: they have been recognized since the early stages of development of the subject (see, e.g., Refs. 16 and 27 and also Refs. 25 and 26) but have so far remained unsolved.

A simple idea in this direction that has already been exploited is based on the observation that differential forms do admit a natural associative product, namely, the wedge product, so one may ask what happens if, in the above construction, vector fields are replaced by multivector fields and $(n - 1)$ -forms by forms of arbitrary degree. As it turns out, this leads to a modified super-Poisson bracket, defined by a formula analogous to Equation (3).^{13,14} But it does not help to overcome either of the aforementioned other two issues.

On the other hand, in the context of the covariant functional formalism, there is an obvious associative and commutative product, namely, just the pointwise product of functionals, and apart from that, there also exists a natural and completely general definition of a covariant Poisson bracket such that, when both are taken together, all the properties required of a Poisson algebra are satisfied: this bracket is known as the Peierls–De Witt bracket.^{8–10,15,32}

Thus the question arises as to what might be the relation, if any, between the covariant functional Poisson bracket, or Peierls–De Witt bracket, and the various candidates for multisymplectic Poisson brackets that have been discussed in the literature, among them the ones written down in Equations (1) and (3). That is the question we shall address in this paper.

In the remainder of this introduction, we want to briefly sketch the answer proposed here: details will be filled in later on. Starting out from the paradigm that, mathematically, classical fields are to be described by sections of fiber bundles, suppose we are given a fiber bundle P over a base manifold M , where M represents space-time, with projection $\rho : P \rightarrow M$, and suppose that the classical fields appearing in the field theoretical model under study are sections $\phi : M \rightarrow P$ of P (i.e., maps ϕ from M to P satisfying $\rho \circ \phi = \text{id}_M$), subject to appropriate regularity conditions: for the sake of definiteness, we shall assume here that all manifolds and bundles are “regular” in the sense of being smooth, while the regularity of sections may vary between smooth (C^∞) and distributional ($C^{-\infty}$), but whenever it is left unspecified, we will tacitly be assuming that we are dealing with smooth sections. To fix terminology, we define, for any section f of any vector bundle V over P , its *base support* or *space-time support*, denoted here by $\text{supp} f$, to be the closure of the set of points in M such that the restriction of f to the corresponding fibers of P does not vanish identically, i.e.,

$$\text{supp} f = \overline{\{x \in M \mid f|_{P_x} \neq 0\}}. \quad (4)$$

(Using the abbreviation “supp” for the base support rather than the ordinary support, which would be a subset of P , constitutes a certain abuse of language, but will do no harm since the ordinary support will play no role in this paper.) Now suppose that f is a differential form on P of degree p , say, and that Σ is a closed p -dimensional submanifold of M , possibly with boundary, subject to the restriction that Σ and $\text{supp} f$ should have compact intersection, so as to guarantee that the following integral is well defined, providing a functional $\mathcal{F}_{\Sigma, f}$ on the space of sections of P ,

$$\mathcal{F}_{\Sigma, f}[\phi] = \int_{\Sigma} \phi^* f, \quad (5)$$

where $\phi^* f$ is of course the pull-back of f to M via ϕ . (Note that the aforementioned restriction is automatically satisfied if Σ is compact and also if f has compact base support; moreover, given an arbitrary differential form f on P , we can always construct one with compact base support by multiplying with a “cutoff function,” i.e., the pull-back to P of a function of compact support on M .) Regarding boundary conditions, we shall usually require that if Σ has a boundary, it should not intersect the base support of f ,

$$\partial \Sigma \cap \text{supp} f = \emptyset. \quad (6)$$

This simple construction provides an especially interesting class of functionals for various reasons, the most important of them being the fact that they are *local*, since they are simple integrals, over regions or more general submanifolds of space-time, of local densities such as, e.g., polynomials of the basic fields and their derivatives, up to a certain order. (More specifically, derivatives up to order r , say, of fields that are sections of some fiber bundle E over M are incorporated by taking P to be the r th order jet bundle $J^r E$ of E .) This is an intuitive notion of locality for functionals of classical fields, but as has been shown recently, it can also be formulated in mathematically rigorous terms.¹ Either way, it is clear that the product of two local functionals of the form (5) is no longer a local functional of the same form: rather, we get a “bilocal” functional associated with a submanifold of $M \times M$ and a differential form on $P \times P$. Therefore, a mathematically interesting object to study would be the algebra of “multilocal” functionals which is generated by the local ones, much in the same way as, on an ordinary vector space, the algebra of polynomials is generated by the monomials.

But the point of main interest for our work appears when we assume P to be a multisymplectic fiber bundle¹¹ and M to be a Lorentz manifold, usually satisfying some additional hypotheses regarding its causal structure: more specifically, we shall assume M to be globally hyperbolic since this is the property that allows us to speak of Cauchy surfaces. In fact, as is now well known, M will in this case admit a foliation by Cauchy surfaces defined as the level sets of some smooth time function. However, it is often convenient not to fix any metric on M “a priori” since, in the context of general relativity, the space-time metric itself is a dynamical entity and not a fixed background field. Within this context, and for the case of a regular first-order hamiltonian system where fields are sections of a given configuration bundle E over M and the dynamics is obtained from a regular

first-order lagrangian via Legendre transform, it has been shown in Ref. 15 that, using this extra structure, one can define the Peierls–De Witt bracket as a functional Poisson bracket on covariant phase space. Here, we want to show how, in the same context, *multisymplectic Poisson brackets between forms*, such as in Equations (1) and (3), *can be derived from the Peierls–De Witt bracket between the corresponding functionals*. For the sake of simplicity, this will be done for the case of $(n - 1)$ -forms, but we expect similar arguments to work in any degree.

Concretely, we shall prove that given a fixed hypersurface Σ in M (typically, a Cauchy surface) and two hamiltonian $(n - 1)$ -forms f and g , we have

$$\{\mathcal{F}_{\Sigma,f}, \mathcal{F}_{\Sigma,g}\} = \mathcal{F}_{\Sigma,\{f,g\}}, \tag{7}$$

where the bracket on the lhs is the Peierls–De Witt bracket of functionals and the bracket $\{f, g\}$ that appears on the rhs is a “multisymplectic pseudo-bracket” or “multisymplectic bracket” given by a formula analogous to Equation (1) or to Equation (3). A more detailed explanation of this result will be deferred to the main body of the paper—mainly because of several technical issues that appear when trying to formulate it with the required amount of mathematical rigor. In fact, the construction turns out to involve both types of multiphase space that appear in field theory and that we refer to as “ordinary multiphase space” and “extended multiphase space,” respectively: they differ in that the latter is a one-dimensional extension of the former, obtained by including an additional scalar “energy type” variable. Geometrically, extended multiphase space is an affine line bundle over ordinary multiphase space, and the hamiltonian \mathcal{H} of any theory with this type of “field content” is a section of this affine line bundle. Moreover, each of these two multiphase spaces comes equipped with a multisymplectic structure which is exact (i.e., the multisymplectic form is, up to a sign introduced merely for convenience, the exterior derivative of a multicanonical form), naturally defined as follows. First, one constructs the multisymplectic form ω and the multicanonical form θ on the extended multiphase space by means of a procedure that can be thought of as a generalization of the construction of the symplectic structure on the cotangent bundle of an arbitrary manifold. Then, the corresponding forms on the ordinary multiphase space are obtained from the previous ones by pull-back via the hamiltonian \mathcal{H} : therefore, they will in what follows be denoted by $\omega_{\mathcal{H}}$ and by $\theta_{\mathcal{H}}$ to indicate their dependence on the choice of hamiltonian. We can express this by saying that the multisymplectic structure on extended multiphase space is “kinematical,” whereas that on ordinary multiphase space is “dynamical.” Correspondingly, we shall often refer to the brackets on extended multiphase space, defined by Equation (1), together with Equation (2), or by Equation (3), as “kinematical multisymplectic (Poisson) brackets,” and to the brackets on ordinary multiphase space, defined by the analogous equations

$$\{f, g\} = i_{X_g} i_{X_f} \omega_{\mathcal{H}} = \omega_{\mathcal{H}}(X_f, X_g), \tag{8}$$

together with

$$i_{X_f} \omega_{\mathcal{H}} = df \tag{9}$$

or

$$\{f, g\} = i_{X_g} i_{X_f} \omega_{\mathcal{H}} + d(i_{X_g} f - i_{X_f} g - i_{X_g} i_{X_f} \theta_{\mathcal{H}}) \tag{10}$$

as “dynamical multisymplectic (Poisson) brackets.” In both cases, the brackets defined by the simpler formulas (1) and (8) are really only “pseudo-brackets” because they fail to satisfy the Jacobi identity, and the correction terms that appear in Equations (3) and (10) are introduced to cure this defect. As we shall see, what appears on the rhs of Equation (7) above is the dynamical bracket on ordinary multiphase space and not the kinematical bracket on extended multiphase space—in accordance with the fact that the Peierls–De Witt bracket itself is dynamical.

We conclude this introduction with a few comments about the organization of the paper. In Section II, we set up the geometric context for the functional calculus in classical field theory, introduce the class of local functionals to be investigated and give an explicit formula for their first functional derivative. In Section III, we present a few elementary concepts from multisymplectic geometry, which is the adequate mathematical background for the covariant hamiltonian formulation of classical field theory. In Section IV, we combine the two previous sections to formulate, in

this context, the variational principle that provides the dynamics and derive not only the equations of motion (De Donder–Weyl equations) but also their linearization around a given solution (linearized De Donder–Weyl equations), with emphasis on a correct treatment of boundary conditions. Although much of this material can be found in the literature, it is included here to set the stage, explain important parts of the necessary background and fix the notation, so as to make the paper reasonably self-contained. In Section V, we present the classification of locally hamiltonian and exact hamiltonian vector fields on both types of multiphase space; it is stated in Theorem 1 for extended multiphase space and in Theorem 2 for ordinary multiphase space. We would like to point out that to our knowledge, Theorem 2 is new (and as explained above, it is this version that covers the case of interest here), whereas Theorem 1 has been known for some time; in fact, an even more general classification, namely, of (locally or exact) hamiltonian multivector fields—rather than just vector fields—on extended multiphase space has been established in Ref. 14. Still, we found it worthwhile to include both cases here, mainly to facilitate comparison between the two situations and to illustrate to what extent the statements are parallel and where deviations occur. In Section VI, we complete our discussion of the background by briefly reviewing the central theorems of Ref. 15, and then pass on to present the main result of this paper, specifying a precise formulation of the connection between multisymplectic Poisson brackets and the functional Poisson bracket of Peierls and De Witt. Finally, Section VII provides further discussion of this result, its implications and perspectives for future investigations.

The paper presents a substantially revised and expanded version of the main results contained in the PhD thesis of the second author,³³ which was elaborated under the supervision of the first author.

II. GEOMETRIC SETUP FOR THE FUNCTIONAL CALCULUS

We begin by collecting some concepts and notations that we use throughout the article. As already mentioned in the introduction, classical fields are sections of fiber bundles over space-time, so our starting point will be to fix a fiber bundle P over the space-time manifold M (not necessarily endowed with a fixed metric, as mentioned before), with projection $\rho : P \rightarrow M$. The space of field configurations \mathcal{C} is then the space of (smooth) sections of P , or an appropriate subspace thereof,

$$\mathcal{C} \subset \Gamma^\infty(P), \quad (11)$$

whose elements will, typically, be denoted by ϕ . Formally, we can view this space as a manifold which, at each point ϕ , has a tangent space $T_\phi\mathcal{C}$ and, similarly, a cotangent space $T_\phi^*\mathcal{C}$. Explicitly, denoting by V_ϕ the vertical bundle of P , pulled back to M via ϕ ,

$$V_\phi = \phi^*(\text{Ver}P), \quad (12)$$

and by V_ϕ^\otimes its twisted dual, defined by taking the tensor product of its ordinary dual with the line bundle of volume forms over the base space,

$$V_\phi^\otimes = V_\phi^* \otimes \wedge^n T^*M, \quad (13)$$

we have that, according to the principles of the variational calculus, $T_\phi\mathcal{C}$ is the space of smooth sections of V_ϕ , or an appropriate subspace thereof,

$$T_\phi\mathcal{C} \subset \Gamma^\infty(V_\phi), \quad (14)$$

whose elements will, typically, be denoted by $\delta\phi$ and called variations of ϕ , whereas $T_\phi^*\mathcal{C}$ is the space of distributional sections of V_ϕ^\otimes , or an appropriate subspace thereof,

$$T_\phi^*\mathcal{C} \subset \Gamma^{-\infty}(V_\phi^\otimes). \quad (15)$$

The reader will note that in Equations (11), (14), and (15), we have required only inclusion, rather than equality. One reason is that the system may be subject to constraints on the fields which cannot be reduced to the simple statement that they should take values in some appropriate subbundle of the original bundle (this case could be handled by simply changing the choice of the bundle P).

But even for unconstrained systems, which are the only ones that we shall be dealing with in this paper, there is another reason, namely, that we have not yet specified the support properties to be employed. One obvious possibility is to set

$$T_\phi \mathcal{C} = \Gamma_c^\infty(V_\phi), \quad T_\phi^* \mathcal{C} = \Gamma^{-\infty}(V_\phi^\otimes), \tag{16}$$

which amounts to allowing only variations with compact support. At the other extreme, we may set

$$T_\phi \mathcal{C} = \Gamma^\infty(V_\phi), \quad T_\phi^* \mathcal{C} = \Gamma_c^{-\infty}(V_\phi^\otimes). \tag{17}$$

And finally, there is a third option, specifically adapted to the situation where the base space is a globally hyperbolic lorentzian manifold and adopted in Ref. 15, which is to take

$$T_\phi \mathcal{C} = \Gamma_{sc}^\infty(V_\phi), \quad T_\phi^* \mathcal{C} = \Gamma_{tc}^{-\infty}(V_\phi^\otimes), \tag{18}$$

where the symbols “sc” and “tc” indicate that the sections are required to have *spatially compact support* and *temporally compact support*, respectively. These options correspond to different choices for the support properties of the functionals that will be allowed.

Generally speaking, given a functional F on \mathcal{C} , we define its *base support* or *space-time support*, denoted here simply by $\text{supp } F$, as follows¹:

$$\begin{aligned} &\text{There exists an open neighborhood } U_x \text{ of } x \text{ in } M \\ x \notin \text{supp } F &\iff \text{ such that for any two field configurations } \phi_1, \phi_2 \in \mathcal{C} \\ &\text{ satisfying } \phi_1 = \phi_2 \text{ on } M \setminus U_x, F[\phi_1] = F[\phi_2]. \end{aligned} \tag{19}$$

This definition implies that $\text{supp } F$ is a closed subset of M since its complement is open: it is the largest open subset of M such that, intuitively speaking, F is *insensitive to variations of its argument localized within that open subset*. It also implies that the functional derivative of F (if it exists) satisfies

$$F'[\phi] \cdot \delta\phi = 0 \quad \text{if } \text{supp } F \cap \text{supp } \delta\phi = \emptyset. \tag{20}$$

For later use, we note that the functional derivative will often be expressed in terms of a (formal) variational derivative,

$$F'[\phi] \cdot \delta\phi = \int_M d^n x \frac{\delta F}{\delta \phi^i}[\phi](x) \delta \phi^i(x). \tag{21}$$

Typically, as always in distribution theory, the functional derivative will be well defined on variations $\delta\phi$ such that $\text{supp } F \cap \text{supp } \delta\phi$ is compact. Thus if no restrictions on the space-time support of F are imposed, we must adopt the choice made in Equation (16). At the other extreme, if the space-time support of F is supposed to be compact, we may adopt the choice made in Equation (17). And finally, the choice made in Equation (18) is the adequate one for dealing with functionals that have *temporally compact support*, i.e., space-time support contained in the inverse image of a bounded interval in \mathbb{R} under some global time function: the typical example is that of a local functional of the form given by Equation (5) when Σ is some Cauchy surface. More generally, note that for local functionals of the form given by Equation (5), we have

$$\text{supp } \mathcal{F}_{\Sigma, f} = \Sigma \cap \text{supp } f. \tag{22}$$

However, it should not be left unnoticed that the equality in Equations (16)-(18) and, possibly, in Equation (22), can only be guaranteed to hold for non-degenerate systems, since in the case of degenerate systems, there will be additional constraints implying that we must return to the option of replacing equalities by inclusions, as before.

In what follows, we shall make extensive use of the fact that variations of sections can always be written as compositions with projectable vector fields, or even with vertical vector fields, on the total space P . To explain this, recall that a vector field X on the total space of a fiber bundle is called *projectable* if the tangent map to the bundle projection takes the values of X at any two points in the same fiber to the same tangent vector at the corresponding base point, i.e.,

$$T_{p_1} \rho \cdot X(p_1) = T_{p_2} \rho \cdot X(p_2) \quad \text{for } p_1, p_2 \in P \text{ such that } \rho(p_1) = \rho(p_2). \tag{23}$$

This is equivalent to requiring that there exists a vector field X_M on the base space which is ρ -related to X ,

$$X_M(m) = T_p \rho \cdot X(p) \quad \text{for } p \in P \text{ such that } \rho(p) = m. \tag{24}$$

In particular, X is called *vertical* if X_M vanishes. Now note that given any projectable vector field X on P , we obtain a functional vector field X on \mathcal{C} whose value at each point $\phi \in \mathcal{C}$ is the functional tangent vector $X[\phi] \in T_\phi \mathcal{C}$, denoted in what follows by $\delta_X \phi$, defined as

$$\delta_X \phi = X(\phi) - T\phi(X_M), \tag{25}$$

or more explicitly,

$$\delta_X \phi(m) = X(\phi(m)) - T_m \phi(X_M(m)) \quad \text{for } m \in M. \tag{26}$$

Conversely, it can be shown that every functional tangent vector can be obtained in this way from a vertical vector field X on P , i.e., given a section $\delta\phi$ of $\phi^*(\text{Ver}P)$, there exists a vertical vector field X on P representing it in the sense that $\delta\phi$ is equal to $\delta_X \phi$. (To do so, we can apply the implicit function theorem to construct, for any point m in M , a system of local coordinates (x^μ, y^α) for P around $\phi(m)$ in which ρ corresponds to the projection onto the first factor, $(x^\mu, y^\alpha) \mapsto x^\mu$, and ϕ corresponds to the embedding in the first factor, $x^\mu \mapsto (x^\mu, 0)$. Moreover, in these coordinates, $\delta\phi$ is given by functions $\delta\phi^\alpha(x^\mu)$, whereas X is given by functions $X^\alpha(x^\mu, y^\beta)$, so we may simply define an extension of the former to the latter by requiring the X^α to be independent of the y^β , setting $X^\alpha(x^\mu, y^\beta) = \delta\phi^\alpha(x^\mu)$ in a neighborhood of the origin in y -space and then using a smooth cutoff function in y -space.)

Of course, the reader may wonder why, in this context, we bother to allow for projectable vector fields rather than just vertical ones. The point is that although vertical vector fields are entirely sufficient to represent variations of sections, we shall often encounter the converse situation in which we consider variations of sections induced by vector fields which are *not* vertical but only projectable, such as the hamiltonian vector fields appearing in Equations (1)-(3) and (8)-(10).

Regarding notation, we shall often think of a projectable vector field as a pair $X = (X_P, X_M)$ consisting of a vector field X_P on the total space P and a vector field X_M on the base space M , related to one another by the bundle projection: then Equations (25) and (26) should be written as

$$\delta_X \phi = X_P(\phi) - T\phi(X_M), \tag{27}$$

and

$$\delta_X \phi(m) = X_P(\phi(m)) - T_m \phi(X_M(m)) \quad \text{for } m \in M, \tag{28}$$

respectively. The same argument as in the previous paragraph can then be used to prove the following.

Lemma 1. Let ϕ be a section of a fiber bundle P over a base manifold M . Given any vector field X_M on M , there exists a projectable vector field X_P^ϕ on P which is ϕ -related to X_M , i.e., satisfies $X_P^\phi(\phi) = T\phi(X_M)$, and then we have $\phi^*(i_{X_P^\phi} \alpha) = i_{X_M}(\phi^* \alpha)$, for any differential form α on P .

As an example of how useful the representation of variations of sections of a fiber bundle by composition with vertical vector fields or even projectable vector fields can be, we present explicit formulas for the first and second functional derivatives of a local functional of the type considered above—beginning with a more detailed definition of this class of functionals.

Definition 1. Given a fiber bundle P over an n -dimensional base manifold M , let Σ be a p -dimensional submanifold of M , possibly with boundary $\partial\Sigma$, and f be a p -form on the total space P such that the intersection of Σ with the base support of f is compact. The **local functional** associated to Σ and f is the functional $\mathcal{F}_{\Sigma, f} : \mathcal{C} \rightarrow \mathbb{R}$ on the space $\mathcal{C} \subset \Gamma^\infty(P)$ of field configurations defined by

$$\mathcal{F}_{\Sigma, f}[\phi] = \int_\Sigma \phi^* f \quad \text{for } \phi \in \mathcal{C}. \tag{29}$$

These functionals are differentiable, and their derivative is given by a completely explicit formula.

Proposition 1. Given a fiber bundle P over an n -dimensional base manifold M , let Σ be a p -dimensional submanifold of M , possibly with boundary $\partial\Sigma$, and f be a p -form on the total space P such that the intersection of Σ with the base support of f is compact. Then the local functional $\mathcal{F}_{\Sigma,f}$ associated to Σ and f is differentiable, and representing variations of sections of P in the form $\delta_X\phi$ where $X = (X_P, X_M)$ is a projectable vector field, its functional derivative is given by the formula

$$\mathcal{F}'_{\Sigma,f}[\phi] \cdot \delta_X\phi = \int_{\Sigma} (\phi^*(L_{X_P}f) - L_{X_M}(\phi^*f)) \quad \text{for } \phi \in \mathcal{C}, \delta_X\phi \in T_{\phi}\mathcal{C}, \tag{30}$$

where L_Z denotes the Lie derivative along the vector field Z .

Remark 1. Under the boundary condition that the intersection of $\partial\Sigma$ with the base support of f is empty, Equation (30) can be rewritten as follows:

$$\mathcal{F}'_{\Sigma,f}[\phi] \cdot \delta_X\phi = \int_{\Sigma} (\phi^*(i_{X_P}df) - i_{X_M}(\phi^*df)) \quad \text{for } \phi \in \mathcal{C}, \delta_X\phi \in T_{\phi}\mathcal{C}. \tag{31}$$

The same equation holds when this boundary condition is replaced by the requirement that $\delta_X\phi$ should vanish on $\partial\Sigma$.

Proof. Recall first that for any functional F on \mathcal{C} , its functional derivative at $\phi \in \mathcal{C}$ along $\delta\phi \in T_{\phi}\mathcal{C}$ is defined by

$$F'[\phi] \cdot \delta\phi = \left. \frac{d}{d\lambda} F[\phi_{\lambda}] \right|_{\lambda=0},$$

where the $\phi_{\lambda} \in \mathcal{C}$ constitute a smooth one-parameter family of sections of P such that

$$\phi = \phi_{\lambda}|_{\lambda=0}, \quad \delta\phi = \left. \frac{d}{d\lambda} \phi_{\lambda} \right|_{\lambda=0}.$$

Fixing ϕ and $\delta\phi$ and choosing a projectable vector field $X = (X_P, X_M)$ that represents $\delta\phi$ as $\delta_X\phi$, according to Equation (25), consider its flow, which is a (local) one-parameter group of (local) automorphisms $\Phi_{\lambda} = (\Phi_{P,\lambda}, \Phi_{M,\lambda})$ such that

$$X_P = \left. \frac{d}{d\lambda} \Phi_{P,\lambda} \right|_{\lambda=0} \quad \text{and} \quad X_M = \left. \frac{d}{d\lambda} \Phi_{M,\lambda} \right|_{\lambda=0}.$$

This allows us to take the one-parameter family of sections ϕ_{λ} of P to be given by the one-parameter group of automorphisms Φ_{λ} , according to

$$\phi_{\lambda} = \Phi_{P,\lambda} \circ \phi \circ \Phi_{M,\lambda}^{-1}.$$

Now we are ready to calculate

$$\begin{aligned} \mathcal{F}'_{\Sigma,f}[\phi] \cdot \delta_X\phi &= \left. \frac{d}{d\lambda} \left(\int_{\Sigma} \phi_{\lambda}^* f \right) \right|_{\lambda=0} \\ &= \int_{\Sigma} \left. \frac{d}{d\lambda} \phi_{\lambda}^* f \right|_{\lambda=0} = \int_{\Sigma} \left. \frac{d}{d\lambda} (\Phi_{P,\lambda} \circ \phi \circ \Phi_{M,\lambda}^{-1})^* f \right|_{\lambda=0} \\ &= \int_{\Sigma} \left(\phi^* \left(\left. \frac{d}{d\lambda} (\Phi_{P,\lambda}^* f) \right|_{\lambda=0} \right) + \left. \frac{d}{d\lambda} (\Phi_{M,\lambda}^{-1})^* (\phi^* f) \right|_{\lambda=0} \right) \\ &= \int_{\Sigma} (\phi^*(L_{X_P}f) - L_{X_M}(\phi^*f)). \end{aligned}$$

To derive Equation (31) from Equation (30), it suffices to apply standard formulas such as $L_Z = di_Z + i_Z d$ and the fact that d commutes with pull-backs, together with Stokes' theorem, and use the boundary condition (6) to kill the resulting two integrals over $\partial\Sigma$. The same argument works when $\delta_X\phi$ is supposed to vanish on $\partial\Sigma$, since we may then arrange X_M to vanish on $\partial\Sigma$ and X_P to vanish on $P|_{\partial\Sigma}$. □

III. MULTIPHASE SPACES AND MULTISYMPLECTIC STRUCTURE

As has already been stated before, the bundle P appearing in Secs. I and II, representing the *multiphase space* of the system under consideration, will be required to carry additional structure, namely, a *multisymplectic form*. There has been much debate and even some confusion in the literature on what should be the “right” definition of the concept of a multisymplectic structure, but all proposals made so far can be subsumed under the following.

Definition 2. A **multisymplectic form** on a manifold P is a differential form ω on P of degree $n + 1$, say, which is closed,

$$d\omega = 0 \quad (32)$$

and satisfies certain additional algebraic constraints—among them that of being non-degenerate, in the sense that for any vector field X on P , we have

$$i_X\omega = 0 \implies X = 0. \quad (33)$$

Of course, this definition is somewhat vague since it leaves open what other algebraic constraints should be imposed besides non-degeneracy. One rather natural criterion is that they should be sufficient to guarantee the validity of a Darboux type theorem. Clearly, when $n = 1$, the above definition reduces to that of a symplectic form, and no additional constraints are needed. But when $n > 1$, which is the case pertaining to field theory rather than to mechanics, this is no longer so. An important aspect here is that P is not simply a manifold but rather the total space of a fiber bundle over the space-time manifold M , which is supposed to be n -dimensional, so one restriction is that the degree of the form ω is linked to the space-time dimension. Another restriction is that ω should be $(n - 1)$ -horizontal, which means that its contraction with three vertical vector fields vanishes,

$$i_X i_Y i_Z \omega = 0 \quad \text{for } X, Y, Z \text{ vertical.} \quad (34)$$

And finally, there is a restriction that, roughly speaking, guarantees existence of a “sufficiently high-dimensional” lagrangian subbundle of the tangent bundle of P , but since we shall not need it here, we omit the details: they can be found in Ref. 11.

The main advantage of the definition of a multisymplectic form as given above is that we can proceed to discuss a number of concepts which do not depend on the precise nature of the additional algebraic constraints to be imposed. For example, a vector field X on P is said to be **locally hamiltonian** if

$$L_X\omega = 0, \quad (35)$$

which according to Cartan’s formula $L_X = di_X + i_X d$ (to be used without further mention throughout this paper) is equivalent to the condition that $i_X\omega$ is closed, and is said to be **globally hamiltonian**, or simply **hamiltonian**, if $i_X\omega$ is exact, that is, if there exists an $(n - 1)$ -form f on P such that

$$i_X\omega = df. \quad (36)$$

Reciprocally, an $(n - 1)$ -form f on P is said to be **hamiltonian** if there exists a vector field X on P such that Equation (36) holds: this condition is trivially satisfied when $n = 1$ but not when $n > 1$. Note that due to non-degeneracy of ω , X is uniquely determined by f and will therefore often be denoted by X_f , whereas f is determined by X only up to addition of a closed form: despite this (partial) ambiguity, we shall say that X is associated with f and f is associated with X . In the special case when ω is exact, i.e., we have

$$\omega = -d\theta, \quad (37)$$

where θ is an appropriate n -form on P called the **multicanonical form**, a vector field X on P is said to be **exact hamiltonian** if

$$L_X\theta = 0. \quad (38)$$

In this case, of course, the associated hamiltonian form can be simply chosen to be $i_X\theta$, since $di_X\theta = L_X\theta - i_X d\theta = i_X\omega$. In particular, this happens when P is the total space of a vector bundle

(over some base space E , say, which in turn will be the total space of a fiber bundle over the space-time manifold M), provided that ω is homogeneous of degree one with respect to the corresponding **Euler vector field** or **scaling vector field** Σ , i.e.,

$$L_\Sigma \omega = \omega, \tag{39}$$

since we may then define θ by

$$\theta = -i_\Sigma \omega. \tag{40}$$

Moreover, we can then employ Σ to decompose vector fields and differential forms on P according to their scaling degree, and as we shall see below, this turns out to be extremely useful for the classification of hamiltonian vector fields (whether locally or globally or exact) and of hamiltonian forms.

The standard example of this kind of structure is provided by any (first order) hamiltonian system obtained from a (first order) lagrangian system via a *non-degenerate* covariant Legendre transformation. In this approach, one starts out from another fiber bundle over M , denoted here by E and called the *configuration bundle*: the relation between the two bundles E and P is then established by taking appropriate duals of first order jet bundles. Namely, consider the first order jet bundle of E , denoted simply by JE , which is both a fiber bundle over M (with respect to the source projection) and an affine bundle over E (with respect to the target projection), together with its difference vector bundle, called the linearized first order jet bundle of E and denoted here by $\vec{J}E$, which is both a fiber bundle over M (with respect to the source projection) and a vector bundle over E (with respect to the target projection), and introduce the corresponding duals: the affine dual J^*E of JE and the usual linear dual \vec{J}^*E of $\vec{J}E$. In what follows, we shall refer to the latter as the *ordinary multiphase space* and to the former as the *extended multiphase space* of the theory. As it turns out and has been emphasized since the beginning of the “modern phase” of the development of the subject in the early 1990s (see, e.g., Ref. 4), both of these play an important role since not only are both of them fiber bundles over M and vector bundles over E , but J^*E is also an affine line bundle over \vec{J}^*E , and the dynamics of the theory is given by the choice of a *hamiltonian*, which is a section $\mathcal{H} : \vec{J}^*E \rightarrow J^*E$ of this affine line bundle. Moreover, and this is of central importance, both of these multiphase spaces carry a multisymplectic structure. Namely, J^*E comes with a naturally defined *multisymplectic form* of degree $n + 1$, denoted here by ω , which (up to sign) is the exterior derivative of an equally naturally defined *multicanonical form* of degree n , denoted here by θ , i.e., $\omega = -d\theta$, and if we choose a hamiltonian $\mathcal{H} : \vec{J}^*E \rightarrow J^*E$, we can pull them back to obtain a corresponding multicanonical form

$$\theta_{\mathcal{H}} = \mathcal{H}^* \theta \tag{41}$$

and a corresponding multisymplectic form

$$\omega_{\mathcal{H}} = \mathcal{H}^* \omega \tag{42}$$

on \vec{J}^*E , so again, $\omega_{\mathcal{H}} = -d\theta_{\mathcal{H}}$. The main difference between the extended and the ordinary multiphase space is that θ and ω are universal and “kinematical,” whereas $\theta_{\mathcal{H}}$ and $\omega_{\mathcal{H}}$ are “dynamical.” In terms of local Darboux coordinates (x^μ, q^i, p_i^μ) for \vec{J}^*E and (x^μ, q^i, p_i^μ, p) for J^*E , induced by the choice of local coordinates x^μ for M , q^i for the typical fiber Q of E and a local trivialization of E over M , we have

$$\theta = p_i^\mu dq^i \wedge d^n x_\mu + p d^n x, \tag{43}$$

and

$$\omega = dq^i \wedge dp_i^\mu \wedge d^n x_\mu - dp \wedge d^n x, \tag{44}$$

so that writing $\mathcal{H} = -H d^n x$,

$$\theta_{\mathcal{H}} = p_i^\mu dq^i \wedge d^n x_\mu - H d^n x, \tag{45}$$

and

$$\omega_{\mathcal{H}} = dq^i \wedge dp_i^\mu \wedge d^n x_\mu + dH \wedge d^n x, \tag{46}$$

or more explicitly,

$$\omega_{\mathcal{H}} = dq^i \wedge dp_i^\mu \wedge d^n x_\mu + \frac{\partial H}{\partial q^i} dq^i \wedge d^n x + \frac{\partial H}{\partial p_i^\mu} dp_i^\mu \wedge d^n x, \tag{47}$$

where $d^n x_\mu$ is the (local) $(n - 1)$ -form obtained by contracting the (local) volume form $d^n x$ with the local vector field $\partial_\mu \equiv \partial/\partial x^\mu$; for more details, including a global definition of θ that does not depend on any of these choices, we refer to Refs. 4, 15, and 19. (It may be worthwhile to note that whereas the form ω is always non-degenerate, the form $\omega_{\mathcal{H}}$ is degenerate in mechanics, i.e., for $n = 1$, but is non-degenerate in field theory, i.e., for $n > 1$.)

IV. VARIATIONAL PRINCIPLE AND EQUATIONS OF MOTION

The fundamental link that merges the functional and multisymplectic formalisms discussed in Secs. II and III into one common picture becomes apparent when the construction of functionals of fields from forms on multiphase space outlined in the introduction is applied to the multicanonical n -form $\theta_{\mathcal{H}}$ on (ordinary) multiphase space: this provides the *action functional* \mathcal{S} of the theory, defining the variational principle whose stationary points are the solutions of the equations of motion. Indeed, the action functional is really an entire family of functionals \mathcal{S}_K on the space \mathcal{C} of field configurations ϕ (see Equation (11)), given by

$$\mathcal{S}_K[\phi] = \int_K \phi^* \theta_{\mathcal{H}}, \tag{48}$$

where K runs through the compact submanifolds of M which are the closure of their interior in M and have smooth boundary ∂K .

Within this setup, a section ϕ in \mathcal{C} is said to be a *stationary point of the action* if, for any compact submanifold K of M which is the closure of its interior in M and has smooth boundary ∂K , ϕ becomes a critical point of the functional \mathcal{S}_K restricted to the (formal) submanifold

$$\mathcal{C}_{K,\phi} = \{ \tilde{\phi} \in \mathcal{C} \mid \tilde{\phi}|_{\partial K} = \phi|_{\partial K} \} \tag{49}$$

of \mathcal{C} , or equivalently, if the functional derivative $\mathcal{S}'_K[\phi]$ of \mathcal{S}_K at ϕ vanishes on the subspace

$$T_\phi \mathcal{C}_{K,\phi} = \{ \delta\phi \in T_\phi \mathcal{C} \mid \delta\phi = 0 \text{ on } \partial K \} \tag{50}$$

of $T_\phi \mathcal{C}$. As is well known, this is the case if and only if ϕ satisfies the corresponding equations of motion, which in the present case are the De Donder–Weyl equations; see, e.g., Refs. 15, 18, and 19. Globally, these can be cast in the form

$$\phi^*(i_X \omega_{\mathcal{H}}) = 0 \tag{51}$$

for any vertical vector field X on P ,

or even

$$\phi^*(i_{X_P} \omega_{\mathcal{H}}) = 0 \tag{52}$$

for any projectable vector field $X = (X_P, X_M)$,

whereas, when written in terms of local Darboux coordinates (x^μ, q^i, p_i^μ) as before, they read

$$\partial_\mu \phi^i = \frac{\partial H}{\partial p_i^\mu}(\varphi, \pi), \quad \partial_\mu \pi_i^\mu = - \frac{\partial H}{\partial q^i}(\varphi, \pi), \tag{53}$$

where $P = \vec{J}^{\otimes} E$, $\phi = (\varphi, \pi)$ and $\mathcal{H} = -H d^n x$. Similarly, given such a solution ϕ , we shall say that a section $\delta\phi$ in $T_\phi \mathcal{C}$ is an *infinitesimal stationary point of the action* if it is formally tangent to the (formal) submanifold of solutions, or equivalently, if it satisfies the corresponding linearized equations of motion, which in the present case are the linearized De Donder–Weyl equations. Globally, representing $\delta\phi$ in the form $\delta_X \phi$ where $X = (X_P, X_M)$ is a projectable vector field, these can be cast

in the form

$$\phi^*(i_Y L_{X_P} \omega_{\mathcal{H}}) = 0 \tag{54}$$

for any vertical vector field Y on P ,

or even

$$\phi^*(i_{Y_P} L_{X_P} \omega_{\mathcal{H}}) = 0 \tag{55}$$

for any projectable vector field $Y = (Y_P, Y_M)$,

whereas, when written in terms of local Darboux coordinates (x^μ, q^i, p_i^μ) as before, they read

$$\begin{aligned} \partial_\mu \delta \varphi^i &= + \frac{\partial^2 H}{\partial q^j \partial p_i^\mu}(\varphi, \pi) \delta \varphi^j + \frac{\partial^2 H}{\partial p_j^\nu \partial p_i^\mu}(\varphi, \pi) \delta \pi_j^\nu, \\ \partial_\mu \delta \pi_i^\mu &= - \frac{\partial^2 H}{\partial q^j \partial q^i}(\varphi, \pi) \delta \varphi^j - \frac{\partial^2 H}{\partial p_j^\nu \partial q^i}(\varphi, \pi) \delta \pi_j^\nu, \end{aligned} \tag{56}$$

where $P = \overset{\circ}{J}E$, $\phi = (\varphi, \pi)$, $\delta \phi = (\delta \varphi, \delta \pi)$, and $\mathcal{H} = -H d^n x$.

Proof. For the first part (concerning the derivation of the full equations of motion from the variational principle), we begin by specializing Equation (30) to vertical vector fields X on P (i.e., setting $X_M = 0$ and replacing X_P by X), with $f = \theta_{\mathcal{H}}$, and using standard facts such as the formula $L_Z = d i_Z + i_Z d$ or that d commutes with Lie derivatives and pull-backs, together with Stokes' theorem, to obtain that, for any vertical vector field X on P ,

$$S'_K[\phi] \cdot \delta_X \phi = - \int_K \phi^*(i_X \omega_{\mathcal{H}}) + \int_{\partial K} \phi^*(i_X \theta_{\mathcal{H}}).$$

Obviously, condition (51) implies that this expression will be equal to zero for all vertical vector fields X on P which vanish on $P|_{\partial K}$. Conversely, it follows from Lemma 2 below that if this is the case, then condition (51) holds. Moreover, it is easily seen that this condition is really equivalent to the condition

$$\phi^*(i_{X_P} \omega_{\mathcal{H}}) - i_{X_M}(\phi^* \omega_{\mathcal{H}}) = 0$$

for any projectable vector field $X = (X_P, X_M)$,

but it so happens that the form $\phi^* \omega_{\mathcal{H}}$ is identically zero, for dimensional reasons. Finally, a simple calculation shows that Equation (51), when written out explicitly in local Darboux coordinates, can be reduced to the system (53).

For the second part (concerning the linearization of the full equations of motion around a given solution ϕ), we proceed as in the proof of Proposition 1: fixing ϕ and $\delta \phi$, suppose we are given a smooth one-parameter family of sections $\phi_\lambda \in \mathcal{C}$ of P such that

$$\phi = \phi_\lambda|_{\lambda=0}, \quad \delta \phi = \frac{d}{d\lambda} \phi_\lambda|_{\lambda=0},$$

as well as a projectable vector field $X = (X_P, X_M)$ that represents $\delta \phi$ as $\delta_X \phi$, according to Equation (25), together with its flow, which is a (local) one-parameter group of (local) automorphisms $\Phi_\lambda = (\Phi_{P,\lambda}, \Phi_{M,\lambda})$ such that

$$X_P = \frac{d}{d\lambda} \Phi_{P,\lambda}|_{\lambda=0} \quad \text{and} \quad X_M = \frac{d}{d\lambda} \Phi_{M,\lambda}|_{\lambda=0},$$

allowing us to take the one-parameter family of sections ϕ_λ of P to be given by the one-parameter group of automorphisms Φ_λ , according to

$$\phi_\lambda = \Phi_{P,\lambda} \circ \phi \circ \Phi_{M,\lambda}^{-1}.$$

Then for any vertical vector field Y on P , we have

$$\begin{aligned} \frac{d}{d\lambda} \phi^*(i_Y \omega_{\mathcal{H}})|_{\lambda=0} &= \frac{d}{d\lambda} (\Phi_{P,\lambda} \circ \phi \circ \Phi_{M,\lambda}^{-1})^*(i_Y \omega_{\mathcal{H}})|_{\lambda=0} \\ &= \frac{d}{d\lambda} \phi^*(\Phi_{P,\lambda}^*(i_Y \omega_{\mathcal{H}}))|_{\lambda=0} + \frac{d}{d\lambda} (\Phi_{M,\lambda}^{-1})^*(\phi^*(i_Y \omega_{\mathcal{H}}))|_{\lambda=0} \\ &= \phi^*(L_{X_P} i_Y \omega_{\mathcal{H}}) - L_{X_M}(\phi^*(i_Y \omega_{\mathcal{H}})) \\ &= \phi^*(i_Y L_{X_P} \omega_{\mathcal{H}}) + \phi^*(i_{[X_P, Y]} \omega_{\mathcal{H}}) - L_{X_M}(\phi^*(i_Y \omega_{\mathcal{H}})), \end{aligned}$$

where the last two terms vanish according to Equation (51), and the same argument holds with Y replaced by Y_P where $Y = (Y_P, Y_M)$ is any projectable vector field. Finally, an elementary but somewhat lengthy calculation shows that Equation (54), when written out explicitly in local Darboux coordinates, can be reduced to the system (56), provided ϕ satisfies the system (53). \square

The lemma we have used in the course of the argument is the following.

Lemma 2. Given a fiber bundle P over an n -dimensional base manifold M , let ϕ be a section of P and α be an $(n + 1)$ -form on P such that, for any compact submanifold K of M which is the closure of its interior in M and has smooth boundary ∂K , and for any vertical vector field X on P that vanishes on $P|_{\partial K}$ together with all its derivatives, the integral

$$\int_K \phi^*(i_X \alpha)$$

vanishes. Then the form $\phi^*(i_X \alpha)$ vanishes identically, for any vertical vector field X on P (not subject to any boundary conditions).

Proof. Suppose X is any vertical vector field on P and m is a point in M where $\phi^*(i_X \alpha)$ does not vanish. Choosing an appropriately oriented system of local coordinates x^μ in M around m , we may write $\phi^*(i_X \alpha) = a d^n x$ where a is a function that is strictly positive at the coordinate origin (corresponding to the point m) and hence, for an appropriate choice of sufficiently small positive numbers δ and ϵ , will be $\geq \epsilon$ on \bar{B}_δ (here we denote by B_r and by \bar{B}_r the open and closed ball of radius r around the coordinate origin, respectively). Choosing a test function χ on M such that $0 \leq \chi \leq 1$, $\text{supp } \chi \subset B_\delta$ and $\chi = 1$ on $\bar{B}_{\delta/2}$, lifted to P by pull-back and multiplied by X to give a new vertical vector field χX on P that vanishes on $P|_{\partial \bar{B}_\delta}$ together with all its derivatives, we get

$$\int_{\bar{B}_\delta} \phi^*(i_{\chi X} \alpha) = \int_{\bar{B}_\delta} d^n x \chi(x) a(x) \geq \int_{\bar{B}_{\delta/2}} d^n x a(x) \geq \epsilon \text{vol}(\bar{B}_{\delta/2}) > 0,$$

which is a contradiction. \square

To summarize, the space of stationary points of the action, denoted here by \mathcal{S} and considered as a (formal) submanifold of the space of field configurations \mathcal{C} , can be described in several equivalent ways: we have

$$\mathcal{S} = \{ \phi \in \mathcal{C} \mid \phi \text{ is stationary point of the action} \}, \tag{57}$$

or

$$\mathcal{S} = \{ \phi \in \mathcal{C} \mid \phi \text{ satisfies the equations of motion (53)} \}, \tag{58}$$

or

$$\mathcal{S} = \{ \phi \in \mathcal{C} \mid \phi^*(i_{X_P} \omega_{\mathcal{H}}) = 0 \text{ for any projectable vector field } X = (X_P, X_M) \}, \tag{59}$$

or

$$\mathcal{S} = \{ \phi \in \mathcal{C} \mid \phi^*(i_X \omega_{\mathcal{H}}) = 0 \text{ for any vertical vector field } X \text{ on } P \}. \tag{60}$$

This space \mathcal{S} plays a central role in the functional approach: it is widely known under the name of *covariant phase space*. Moreover, given ϕ in \mathcal{S} and allowing $X = (X_P, X_M)$ to run through the projectable vector fields, the (formal) tangent space $T_\phi \mathcal{S}$ to \mathcal{S} at ϕ is

$$T_\phi \mathcal{S} = \{ \delta \phi \in T_\phi \mathcal{C} \mid \delta \phi \text{ is infinitesimal stationary point of the action} \}, \tag{61}$$

or

$$T_\phi \mathcal{S} = \{ \delta\phi \in T_\phi \mathcal{C} \mid \delta\phi \text{ satisfies the linearized equations of motion (56)} \}, \tag{62}$$

or

$$T_\phi \mathcal{S} = \{ \delta_X \phi \in T_\phi \mathcal{C} \mid \begin{array}{l} \phi^*(i_{Y_P} L_{X_P} \omega_{\mathcal{H}}) = 0 \\ \text{for any projectable vector field } Y = (Y_P, Y_M) \end{array} \}, \tag{63}$$

or

$$T_\phi \mathcal{S} = \{ \delta_X \phi \in T_\phi \mathcal{C} \mid \begin{array}{l} \phi^*(i_Y L_{X_P} \omega_{\mathcal{H}}) = 0 \\ \text{for any vertical vector field } Y \text{ on } P \end{array} \}. \tag{64}$$

V. LOCALLY HAMILTONIAN VECTOR FIELDS

The results of Sec. IV, in particular, Equations (63) and (64), provide strong motivation for studying projectable vector fields $X = (X_P, X_M)$ on (ordinary) multiphase space which are locally hamiltonian (that is, such that X_P is locally hamiltonian), since they imply that each of these provides a functional vector field X on covariant phase space defined by a simple algebraic composition rule:

$$X[\phi] = \delta_X \phi \quad \text{for } \phi \in \mathcal{S}. \tag{65}$$

As we shall see in Sec. VI, this functional vector field is also hamiltonian with respect to the natural symplectic form on covariant phase space to be presented there. But before doing so, we want to address the problem of classifying the locally hamiltonian vector fields and, along the way, also the globally hamiltonian and exact hamiltonian vector fields on multiphase space. For the case of extended multiphase space, using the forms ω and θ , this classification has been available in the literature for some time,^{13,14} even for the general case of multivector fields. But what is relevant here is the corresponding result for the case of ordinary multiphase space, using the forms $\omega_{\mathcal{H}}$ and $\theta_{\mathcal{H}}$, for a given hamiltonian \mathcal{H} . As we shall see, there is one basic phenomenon common to both cases, namely, that when $n > 1$, the condition of a vector field to be locally hamiltonian imposes severe restrictions on the momentum dependence of its components, forcing them to be at most affine (linear + constant): this appears to be a characteristic feature distinguishing mechanics ($n = 1$) from field theory ($n > 1$) and has been noticed early by various authors (see, e.g., Refs. 16 and 27) and repeatedly rediscovered later on (see, e.g., Refs. 12–14, 25, and 26). However, despite all similarities, some of the details depend on which of the two types of multiphase space we are working with, so it seems worthwhile to give a full statement for both cases, for the sake of comparison.

We begin with an explicit computation in local Darboux coordinates (x^μ, q^i, p_i^μ) for $J^{\otimes}E$ and (x^μ, q^i, p_i^μ, p) for $J^{\otimes}E$ induced by the choice of local coordinates x^μ for M , q^i for the typical fiber Q of E and a local trivialization of E over M , supposing that we are given a hamiltonian $\mathcal{H} : J^{\otimes}E \rightarrow J^{\otimes}E$ which we represent in the form $\mathcal{H} = -H d^n x$, as usual. Starting out from Equations (43)-(47), we distinguish two cases.

A. Extended multiphase space

Using Equations (43) and (44) and writing an arbitrary vector field X on $J^{\otimes}E$ as

$$X = X^\mu \frac{\partial}{\partial x^\mu} + X^i \frac{\partial}{\partial q^i} + X_i^\mu \frac{\partial}{\partial p_i^\mu} + X_0 \frac{\partial}{\partial p} \tag{66}$$

we first note that X will be projectable to E if and only if the coefficients X^μ and X^i do not depend on the energy variable p nor on the multimomentum variables p_k^κ and will be projectable to M if and only if the coefficients X^μ depend neither on the energy variable p nor on the multimomentum variables p_k^κ nor on the position variables q^k . Next, we compute

$$i_X \omega = X^\nu dq^i \wedge dp_i^\mu \wedge d^n x_{\mu\nu} - X_i^\mu dq^i \wedge d^n x_\mu + X^i dp_i^\mu \wedge d^n x_\mu + X^\mu dp \wedge d^n x_\mu - X_0 d^n x \tag{67}$$

and

$$i_X \theta = (p_i^\mu X^i + p X^\mu) d^n x_\mu - p_i^\mu X^\nu dq^i \wedge d^n x_{\mu\nu}. \tag{68}$$

Then we have

Theorem 1. Any locally hamiltonian vector field on $J^{\otimes}E$ is projectable to E . Moreover, given any vector field X on $J^{\otimes}E$ which is projectable to E , X is locally hamiltonian if and only if

1. If $N > 1$, X is also projectable to M , i.e., the coefficients X^μ do not depend on the position variables q^k .
2. The coefficients X_i^μ and X_0 can be expressed in terms of the previous ones and of new coefficients X_-^μ which, once again, depend neither on the energy variable p nor on the multimomentum variables p_k^μ , according to

$$X_i^\mu = -p \frac{\partial X^\mu}{\partial q^i} - p_j^\mu \frac{\partial X^j}{\partial q^i} + p_i^\nu \frac{\partial X^\mu}{\partial x^\nu} - p_i^\mu \frac{\partial X^\nu}{\partial x^\nu} + \frac{\partial X_-^\mu}{\partial q^i}, \tag{69}$$

$$X_0 = -p \frac{\partial X^\mu}{\partial x^\mu} - p_i^\mu \frac{\partial X^i}{\partial x^\mu} + \frac{\partial X_-^\mu}{\partial x^\mu}. \tag{70}$$

Finally, X is exact hamiltonian if and only if, in addition, the coefficients X_-^μ vanish.

Proof. The proof is carried out by “brute force” computation and a term by term analysis of the coefficients that appear in the Lie derivative of ω along X : it will be omitted since an explicit proof of a more general statement along these lines can be found in Ref. 14. \square

B. Ordinary multiphase space

Using Equations (45)-(47) and writing an arbitrary vector field X on $\vec{J}^{\otimes}E$ as

$$X = X^\mu \frac{\partial}{\partial x^\mu} + X^i \frac{\partial}{\partial q^i} + X_i^\mu \frac{\partial}{\partial p_i^\mu}, \tag{71}$$

we first note that X will be projectable to E if and only if the coefficients X^μ and X^i do not depend on the multimomentum variables p_k^μ and will be projectable to M if and only if the coefficients X^μ depend neither on the multimomentum variables p_k^μ nor on the position variables q^k . Next, we compute

$$i_X \omega_{\mathcal{H}} = X^\nu dq^i \wedge dp_i^\mu \wedge d^n x_{\mu\nu} - \left(X_i^\mu + X^\mu \frac{\partial H}{\partial q^i} \right) dq^i \wedge d^n x_\mu + \left(X^i \delta_\mu^\nu - X^\nu \frac{\partial H}{\partial p_i^\mu} \right) dp_i^\mu \wedge d^n x_\nu + \left(X^i \frac{\partial H}{\partial q^i} + X_i^\mu \frac{\partial H}{\partial p_i^\mu} \right) d^n x, \tag{72}$$

and

$$i_X \theta_{\mathcal{H}} = (p_i^\mu X^i - H X^\mu) d^n x_\mu - p_i^\mu X^\nu dq^i \wedge d^n x_{\mu\nu}. \tag{73}$$

Then we have

Theorem 2. Any locally hamiltonian vector field on $\vec{J}^{\otimes}E$ is projectable to E , except possibly when $N = 1$ and $n = 2$, and any exact hamiltonian vector field on $\vec{J}^{\otimes}E$ is projectable to E . Moreover, given any vector field X on $\vec{J}^{\otimes}E$ which is projectable to E , X is locally hamiltonian if and only if

1. If $N > 1$, X is also projectable to M , i.e., the coefficients X^μ do not depend on the position variables q^k .

2. The coefficients X_i^μ can be expressed in terms of the previous ones and of new coefficients X_-^μ which, once again, do not depend on the multimomentum variables p_k^μ , according to

$$X_i^\mu = H \frac{\partial X^\mu}{\partial q^i} - p_j^\mu \frac{\partial X^j}{\partial q^i} + p_i^\nu \frac{\partial X^\mu}{\partial x^\nu} - p_i^\mu \frac{\partial X^\nu}{\partial x^\nu} + \frac{\partial X_-^\mu}{\partial q^i}, \tag{74}$$

where the first term is absent as soon as $N > 1$.

3. The coefficients X^μ , X^i and X_-^μ satisfy the compatibility condition

$$\frac{\partial H}{\partial x^\mu} X^\mu + \frac{\partial H}{\partial q^i} X^i + \frac{\partial H}{\partial p_i^\mu} X_i^\mu = -H \frac{\partial X^\mu}{\partial x^\mu} + p_i^\mu \frac{\partial X^i}{\partial x^\mu} - \frac{\partial X_-^\mu}{\partial x^\mu}. \tag{75}$$

Finally, X is exact hamiltonian if and only if, in addition, the coefficients X_-^μ vanish.

Maybe somewhat surprisingly, the case where $N = 1$ and $n = 2$, concerning the theory of a single real scalar field in two space-time dimensions, is somewhat exceptional in that it allows for locally hamiltonian vector fields which fail to be projectable and are not covered by the above classification theorem; we shall address this question in Remark 2 below.

Proof. As in the case of Theorem 1, the proof is carried out by “brute force” computation. First, we apply the exterior derivative to Equation (72) and collect terms to get

$$\begin{aligned} L_X \omega_{\mathcal{H}} = & \left(\frac{\partial}{\partial x^\mu} \left(X_i^\mu + X^\mu \frac{\partial H}{\partial q^i} \right) + \frac{\partial}{\partial q^i} \left(X^j \frac{\partial H}{\partial q^j} + X_j^\nu \frac{\partial H}{\partial p_j^\nu} \right) \right) dq^i \wedge d^n x \\ & - \left(\frac{\partial}{\partial x^\nu} \left(\delta_\mu^\nu X^i - X^\nu \frac{\partial H}{\partial p_i^\mu} \right) - \frac{\partial}{\partial p_i^\mu} \left(X^j \frac{\partial H}{\partial q^j} + X_j^\nu \frac{\partial H}{\partial p_j^\nu} \right) \right) dp_i^\mu \wedge d^n x \\ & + \left(\delta_l^k \left(\delta_\kappa^\sigma \frac{\partial X^\tau}{\partial x^\tau} - \frac{\partial X^\sigma}{\partial x^\kappa} \right) \right. \\ & \quad \left. + \frac{\partial}{\partial p_k^\kappa} \left(X_l^\sigma + X^\sigma \frac{\partial H}{\partial q^l} \right) + \frac{\partial}{\partial q^l} \left(\delta_\kappa^\sigma X^k - X^\sigma \frac{\partial H}{\partial p_k^\kappa} \right) \right) dq^l \wedge dp_k^\kappa \wedge d^n x_\sigma \\ & - \frac{\partial}{\partial q^j} \left(X_i^\mu + X^\mu \frac{\partial H}{\partial q^i} \right) dq^j \wedge dq^i \wedge d^n x_\mu \\ & + \frac{\partial}{\partial p_k^\kappa} \left(\delta_\lambda^\sigma X^l - X^\sigma \frac{\partial H}{\partial p_l^\lambda} \right) dp_k^\kappa \wedge dp_l^\lambda \wedge d^n x_\sigma \\ & + \frac{\partial X^\sigma}{\partial q^k} dq^k \wedge dq^l \wedge dp_l^\lambda \wedge d^n x_{\lambda\sigma} \\ & - \frac{\partial X^\sigma}{\partial p_k^\kappa} dq^l \wedge dp_k^\kappa \wedge dp_l^\lambda \wedge d^n x_{\lambda\sigma}. \end{aligned}$$

Numbering the terms in this equation from 1 to 7, we begin by analyzing terms no. 7, 5, and 6. Obviously, when X is projectable to E and the coefficients X^μ satisfy the condition stated in item 1. of the theorem, these terms vanish identically (term no. 6 is absent when $N = 1$), so what we need to analyze is the converse statement.

- Term No. 7: For any choice of indices i, j, m, μ, ν and mutually different indices $\rho_1, \dots, \rho_{n-2}$, contracting $L_X \omega_{\mathcal{H}}$ with the multivector field $\partial_m \wedge \partial_\mu^i \wedge \partial_\nu^j \wedge \partial_{\rho_1} \wedge \dots \wedge \partial_{\rho_{n-2}}$ gives the relation

$$\delta_m^j \frac{\partial X^\sigma}{\partial p_i^\mu} \epsilon_{\nu\sigma\rho_1 \dots \rho_{n-2}} = \delta_m^i \frac{\partial X^\sigma}{\partial p_j^\nu} \epsilon_{\mu\sigma\rho_1 \dots \rho_{n-2}}. \tag{76}$$

Now if to begin with, we fix only the indices i and μ , together with some other index ρ , we can always choose the remaining free indices in this equation to be such that $m = j$ and $\nu, \rho_1, \dots, \rho_{n-2}$ are all mutually different and $\neq \rho$: this reduces the lhs to the expression $\pm \partial X^\rho / \partial p_i^\mu$, while the rhs vanishes if we take $m \neq i$, which is possible as soon as $N > 1$. Thus we conclude that

$$\frac{\partial X^\rho}{\partial p_i^\mu} = 0,$$

except perhaps when $N = 1$. But even when $N = 1$, where Equation (76) reduces to

$$\frac{\partial X^\sigma}{\partial p_i^\mu} \epsilon_{\nu\sigma\rho_1\dots\rho_{n-2}} = \frac{\partial X^\sigma}{\partial p_i^\nu} \epsilon_{\mu\sigma\rho_1\dots\rho_{n-2}}, \tag{77}$$

this conclusion remains valid as long as $n > 2$. Indeed, if we choose $\nu, \rho_1, \dots, \rho_{n-2}$ as above (all mutually different and $\neq \rho$), then if $\rho \neq \mu$, the lhs reduces to the expression $\pm \partial X^\rho / \partial p_i^\mu$, as before, while the rhs vanishes since in that case, μ must appear among the indices $\rho_1, \dots, \rho_{n-2}$, whereas if $\rho = \mu$, the equation assumes the form

$$\frac{\partial X^\mu}{\partial p_i^\mu} \epsilon_{\nu\mu\rho_1\dots\rho_{n-2}} = \frac{\partial X^\nu}{\partial p_i^\nu} \epsilon_{\mu\nu\rho_1\dots\rho_{n-2}} \quad (\text{no sum over } \mu \text{ and } \nu),$$

which implies

$$\frac{\partial X^\mu}{\partial p_i^\mu} = - \frac{\partial X^\nu}{\partial p_i^\nu} = \frac{\partial X^\kappa}{\partial p_i^\kappa} = - \frac{\partial X^\mu}{\partial p_i^\mu} \quad (\text{no sum over } \mu, \nu \text{ and } \kappa),$$

for mutually different μ, ν, κ . On the other hand, when $N = 1$ and $n = 2$, these arguments fail, and the only conclusion that can be drawn from Equation (77) is that the following divergence must vanish:

$$\frac{\partial X^\mu}{\partial p_i^\mu} = 0. \tag{78}$$

- Term No. 5: For any choice of indices i, j, μ, ν and mutually different indices $\rho_1, \dots, \rho_{n-1}$, contracting $L_X \omega_{\mathcal{H}}$ with the multivector field $\partial_\mu^i \wedge \partial_\nu^j \wedge \partial_{\rho_1} \wedge \dots \wedge \partial_{\rho_{n-1}}$ gives the relation

$$\frac{\partial}{\partial p_i^\mu} \left(\delta_\nu^\sigma X^j - X^\sigma \frac{\partial H}{\partial p_j^\nu} \right) \epsilon_{\sigma\rho_1\dots\rho_{n-1}} = \frac{\partial}{\partial p_j^\nu} \left(\delta_\mu^\sigma X^i - X^\sigma \frac{\partial H}{\partial p_i^\mu} \right) \epsilon_{\sigma\rho_1\dots\rho_{n-1}},$$

so that, for any choice of indices i, j, μ, ν, ρ , taking $\rho_1, \dots, \rho_{n-1}$ to be $\neq \rho$ shows that

$$\delta_\nu^\rho \frac{\partial X^j}{\partial p_i^\mu} - \frac{\partial X^\rho}{\partial p_i^\mu} \frac{\partial H}{\partial p_j^\nu} = \delta_\mu^\rho \frac{\partial X^i}{\partial p_j^\nu} - \frac{\partial X^\rho}{\partial p_j^\nu} \frac{\partial H}{\partial p_i^\mu}. \tag{79}$$

When $N > 1$ or $n > 2$, we can use the result of the previous item to conclude that

$$\delta_\nu^\rho \frac{\partial X^j}{\partial p_i^\mu} = \delta_\mu^\rho \frac{\partial X^i}{\partial p_j^\nu}.$$

Now if to begin with, we fix only the indices i, j , and μ , we can always choose the other free indices ν and ρ in this equation to be equal and $\neq \mu$: this reduces the lhs to the expression $\pm \partial X^j / \partial p_i^\mu$, while the rhs vanishes. Thus we conclude that

$$\frac{\partial X^j}{\partial p_i^\mu} = 0.$$

On the other hand, when $N = 1$ and $n = 2$, Equation (79) reduces to a simple statement of symmetry,

$$\epsilon^{\mu\nu} \left(\delta_\nu^\rho \frac{\partial X^i}{\partial p_i^\mu} - \frac{\partial X^\rho}{\partial p_i^\mu} \frac{\partial H}{\partial p_i^\nu} \right) = 0. \tag{80}$$

- Term No. 6: For any choice of indices i, j, m, ν and mutually different indices $\rho_1, \dots, \rho_{n-2}$, contracting $L_X \omega_{\mathcal{H}}$ with the multivector field $\partial_i \wedge \partial_j \wedge \partial_\nu^m \wedge \partial_{\rho_1} \wedge \dots \wedge \partial_{\rho_{n-2}}$ gives the relation

$$\delta_j^m \frac{\partial X^\sigma}{\partial q^i} \epsilon_{\nu\sigma\rho_1\dots\rho_{n-2}} = \delta_i^m \frac{\partial X^\sigma}{\partial q^j} \epsilon_{\nu\sigma\rho_1\dots\rho_{n-2}}.$$

As before, this implies

$$\frac{\partial X^\rho}{\partial q^i} = 0,$$

except when $N = 1$: in this case, the whole term vanishes identically and no conclusion can be drawn.

Note that when $N = 1$, the “internal” index on the position and multimomentum variables can only assume a single fixed value, say i , and so in equations such as (78) or (80), we could in principle just omit it, or else repeat it as often as we like.

To proceed further, we write down the equations obtained from the remaining terms:

- Term No. 1:

$$\frac{\partial}{\partial q^i} \left(X^j \frac{\partial H}{\partial q^j} + X_j^\nu \frac{\partial H}{\partial p_j^\nu} \right) = - \frac{\partial}{\partial x^\mu} \left(X_i^\mu + X^\mu \frac{\partial H}{\partial q^i} \right). \tag{81}$$

- Term No. 2:

$$\frac{\partial}{\partial p_i^\mu} \left(X^j \frac{\partial H}{\partial q^j} + X_j^\nu \frac{\partial H}{\partial p_j^\nu} \right) = \frac{\partial}{\partial x^\nu} \left(\delta_\mu^\nu X^i - X^\nu \frac{\partial H}{\partial p_i^\mu} \right). \tag{82}$$

- Term No. 3:

$$\frac{\partial}{\partial p_j^\nu} \left(X_i^\mu + X^\mu \frac{\partial H}{\partial q^i} \right) = - \frac{\partial}{\partial q^i} \left(\delta_\nu^\mu X^j - X^\mu \frac{\partial H}{\partial p_j^\nu} \right) + \delta_i^j \frac{\partial X^\mu}{\partial x^\nu} - \delta_\nu^\mu \delta_i^j \frac{\partial X^\kappa}{\partial x^\kappa}. \tag{83}$$

- Term No. 4:

$$\frac{\partial X_i^\mu}{\partial q^j} = \frac{\partial X_j^\mu}{\partial q^i}. \tag{84}$$

Assuming that X is projectable to E , we observe that Equation (83) can be integrated directly to conclude that

$$X_i^\mu = H \frac{\partial X^\mu}{\partial q^i} - p_j^\mu \frac{\partial X^j}{\partial q^i} + p_i^\nu \frac{\partial X^\mu}{\partial x^\nu} - p_i^\mu \frac{\partial X^\kappa}{\partial x^\kappa} + Y_i^\mu,$$

where the Y_i^μ are local functions on E which, once again, are independent of the multimomentum variables p_k^ν . Substituting this relation into Equation (84), we get

$$\frac{\partial Y_i^\mu}{\partial q^j} = \frac{\partial Y_j^\mu}{\partial q^i},$$

which can be solved by setting

$$Y_i^\mu = \frac{\partial Y_-^\mu}{\partial q^i},$$

where the Y_-^μ are local functions on E which, as before, are independent of the multimomentum variables p_k^ν . Finally, substituting this expression into Equations (81) and (82), we get

$$\frac{\partial}{\partial q^i} \left(H \frac{\partial X^\nu}{\partial x^\nu} - p_j^\nu \frac{\partial X^j}{\partial x^\nu} + \frac{\partial Y_-^\nu}{\partial x^\nu} + \frac{\partial H}{\partial x^\nu} X^\nu + \frac{\partial H}{\partial q^j} X^j + \frac{\partial H}{\partial p_j^\nu} X_j^\nu \right) = 0,$$

and

$$\frac{\partial}{\partial p_i^\mu} \left(H \frac{\partial X^\nu}{\partial x^\nu} - p_j^\nu \frac{\partial X^j}{\partial x^\nu} + \frac{\partial Y_-^\nu}{\partial x^\nu} + \frac{\partial H}{\partial x^\nu} X^\nu + \frac{\partial H}{\partial q^j} X^j + \frac{\partial H}{\partial p_j^\nu} X_j^\nu \right) = 0,$$

showing that

$$H \frac{\partial X^\nu}{\partial x^\nu} - p_j^\nu \frac{\partial X^j}{\partial x^\nu} + \frac{\partial Y_-^\nu}{\partial x^\nu} + \frac{\partial H}{\partial x^\nu} X^\nu + \frac{\partial H}{\partial q^j} X^j + \frac{\partial H}{\partial p_j^\nu} X_j^\nu = Y_-,$$

where Y_- is a local function on M which is independent of the position variables q^k and multimomentum variables p_k^κ . Writing Y_- as a divergence,

$$Y_- = \frac{\partial Y'_\mu}{\partial x^\mu},$$

and putting $X^\mu = Y^\mu - Y'_\mu$, we arrive at Equations (74) and (75).

All that remains to be shown now is the final statement concerning exact hamiltonian vector fields. To this end, we apply the exterior derivative to Equation (73) and subtract the expression in Equation (72); then collecting terms, we get

$$\begin{aligned} L_X \theta_{\mathcal{H}} &= d(i_X \theta_{\mathcal{H}}) - i_X \omega_{\mathcal{H}} \\ &= \left(- \frac{\partial X^\mu}{\partial x^\mu} H + \frac{\partial X^i}{\partial x^\mu} p_i^\mu - \left(\frac{\partial H}{\partial x^\mu} X^\mu + \frac{\partial H}{\partial q^i} X^i + \frac{\partial H}{\partial p_i^\mu} X_i^\mu \right) \right) d^n x \\ &\quad + \left(\frac{\partial X^j}{\partial q^i} p_j^\mu - \frac{\partial X^\mu}{\partial x^\nu} p_i^\nu + \frac{\partial X^\nu}{\partial x^\nu} p_i^\mu - \frac{\partial X^\mu}{\partial q^i} H + X_i^\mu \right) dq^i \wedge d^n x_\mu \\ &\quad + \left(\frac{\partial X^i}{\partial p_j^\nu} p_i^\mu - \frac{\partial X^\mu}{\partial p_j^\nu} H \right) dp_j^\nu \wedge d^n x_\mu \\ &\quad + \frac{\partial X^\nu}{\partial q^j} p_i^\mu dq^i \wedge dq^j \wedge d^n x_{\mu\nu} \\ &\quad + \frac{\partial X^\nu}{\partial p_k^\kappa} p_i^\mu dq^i \wedge dp_k^\kappa \wedge d^n x_{\mu\nu}. \end{aligned}$$

Numbering the terms in this equation from 1 to 5, we see that the conditions imposed by the fact that X should be exact hamiltonian are the following:

- Term No. 5: This term vanishes if and only if the coefficients X^μ do not depend on the variables p_k^κ .
- Term No. 3: Due to the previous condition, this term vanishes if and only the coefficients X^i do not depend on the variables p_k^κ .
- Term No. 4: This term vanishes if and only if the coefficients X^μ do not depend on the variables q^k , except when $N = 1$: in this case the whole term vanishes identically and no conclusion can be drawn.
- Term No. 2: This term vanishes if and only if the coefficients X_i^μ are defined in terms of the coefficients X^μ and X^i according to Equation (74), with $X^\mu = 0$.
- Term No. 1: This term vanishes if and only if Equation (75) is required to hold, with $X^\mu = 0$.

□

Remark 2. The classification of locally hamiltonian vector fields provided by Theorem 2 is not quite complete since it does not cover non-projectable locally hamiltonian vector fields. This may not be a reason for great concern since such vector fields are pathological in the sense that their flows do not respect any of the bundle structures involved and, perhaps more importantly, since such vector fields can only exist in one very special and exceptional case, namely, when $N = 1$ and $n = 2$. Still, it is somewhat annoying that they do not seem to admit any reasonable classification. To give an idea of what is involved, consider first the more general case of one degree of freedom in any space-time dimension ($N = 1, n > 1$), where it is common practice to omit the “internal” index i on the variables q^i and p_i^μ ; it is then appropriate to redefine the components of X in Equation (71), say by writing

$$X = X^\mu \frac{\partial}{\partial x^\mu} + \tilde{X} \frac{\partial}{\partial q} + \tilde{X}^\mu \frac{\partial}{\partial p^\mu}. \tag{85}$$

Then when $n = 2$, the arguments presented in the proof above do not allow to conclude that the coefficients X^μ and \tilde{X} are independent of the multimomentum variables p^κ , but only that

$$\frac{\partial X^\mu}{\partial p^\mu} = 0, \tag{86}$$

as stated in Equation (78), and

$$\frac{\partial \tilde{X}}{\partial p^\mu} - \frac{\partial X^\nu}{\partial p^\mu} \frac{\partial H}{\partial p^\nu} = 0, \tag{87}$$

which, taking into account Equation (78) and using the relation $\pm \epsilon_{\mu\nu} \epsilon^{\kappa\lambda} = \delta_\mu^\kappa \delta_\nu^\lambda - \delta_\nu^\kappa \delta_\mu^\lambda$ (with any fixed sign convention for \pm), is easily shown to be equivalent to Equation (80). Introducing a new function F defined as

$$F = \tilde{X} - X^\mu \frac{\partial H}{\partial p^\mu}, \tag{88}$$

together with the Hessian matrix $H_{\mu\nu}$ of the hamiltonian function H ,

$$H_{\mu\nu} = \frac{\partial^2 H}{\partial p^\mu \partial p^\nu}, \tag{89}$$

and its inverse $H^{\mu\nu}$, it is possible to express all coefficients of X in terms of F and H and their partial derivatives up to first order (for F) or second order (for H),

$$X^\mu = -H^{\mu\nu} \frac{\partial F}{\partial p^\nu}, \tag{90}$$

$$\tilde{X} = F - H^{\mu\nu} \frac{\partial H}{\partial p^\mu} \frac{\partial F}{\partial p^\nu}, \tag{91}$$

$$\begin{aligned} \tilde{X}^\mu = & H^{\mu\nu} \left(\frac{\partial F}{\partial x^\nu} + \frac{\partial H}{\partial p^\nu} \frac{\partial F}{\partial q} - \frac{\partial^2 H}{\partial q \partial p^\nu} F \right) \\ & - H^{\mu\kappa} \left(\frac{\partial^2 H}{\partial x^\kappa \partial p^\lambda} - \frac{\partial^2 H}{\partial x^\lambda \partial p^\kappa} + \frac{\partial H}{\partial p^\kappa} \frac{\partial^2 H}{\partial q \partial p^\lambda} - \frac{\partial H}{\partial p^\lambda} \frac{\partial^2 H}{\partial q \partial p^\kappa} \right) H^{\lambda\nu} \frac{\partial F}{\partial p^\nu}. \end{aligned} \tag{92}$$

However, finding the general solution of the entire system seems to be an exceedingly difficult task, except if one makes some simplifying assumptions on the hamiltonian H . One obvious choice would be to take

$$H = \frac{1}{2} g_{\mu\nu}(x) p^\mu p^\nu + A_\mu(x) p^\mu + V(x, q), \tag{93}$$

where g represents a Lorentz metric, A is a gauge potential and V is some scalar potential, but even in this situation we have not come to a definite conclusion. The only case in which a complete solution has been found is in the absence of external fields, i.e., when the metric tensor g and the scalar potential V are both independent of x , whereas the gauge potential A vanishes, so M is two-dimensional Minkowski space \mathbb{R}^2 and g is the standard Minkowski metric η ; see the Appendix of Ref. 27.

VI. COVARIANT PHASE SPACE

A. Symplectic structure on covariant phase space

One of the most important properties of the covariant phase space \mathcal{S} introduced above (see Equations (57)-(60)) is that it carries a naturally defined symplectic structure^{5,6,35} which can in fact be derived immediately from the multisymplectic structure on multiphase space.¹⁵ Namely, generalizing the prescription of Equation (5) in the sense of using ordinary differential forms on multiphase space to produce functional differential forms, rather than just functionals, we can define functional canonical 1-forms Θ_{K_Σ} and 2-forms Ω_{K_Σ} on \mathcal{C} , where Σ is a hypersurface in M (typically, when a Lorentz metric is given, a Cauchy surface) and K_Σ runs through the compact submanifolds of Σ which are the closure of their interior in Σ and have smooth boundary ∂K_Σ , by setting

$$(\Theta_{K_\Sigma})_\phi(\delta_X \phi) = \int_{K_\Sigma} \phi^*(i_X \theta_{\mathcal{H}}), \tag{94}$$

for $\phi \in \mathcal{C}$ and $\delta_X \phi \in T_\phi \mathcal{C}$ with X vertical, and

$$(\Omega_{K_\Sigma})_\phi(\delta_{X_1}\phi, \delta_{X_2}\phi) = \int_{K_\Sigma} \phi^*(i_{X_2}i_{X_1}\omega_{\mathcal{H}}), \tag{95}$$

for $\phi \in \mathcal{C}$ and $\delta_{X_1}\phi, \delta_{X_2}\phi \in T_\phi\mathcal{C}$ with X_1, X_2 vertical. (The same formulas continue to hold if we require the vector fields X, X_1, X_2 to be only vertical on the image of ϕ .) As observed, e.g., in Refs. 5, 6, and 35 and, in the present context, in Ref. 15, the restriction of the form Ω_{K_Σ} to \mathcal{S} does not depend on the submanifold K_Σ , provided that appropriate boundary conditions are imposed: this happens because when $\phi \in \mathcal{S}$ and $\delta_{X_1}\phi, \delta_{X_2}\phi \in T_\phi\mathcal{S}$, the expression under the integral in Equation (95) is a closed form (called the ‘‘symplectic current’’ in Ref. 5), so that according to Stokes’ theorem, its integral over any compact submanifold without boundary vanishes. (For example, when we compare the integral in Equation (95) over two compact submanifolds K_{1,Σ_1} and K_{2,Σ_2} of hypersurfaces Σ_1 and Σ_2 in M whose union $K_{1,\Sigma_1} \cup K_{2,\Sigma_2}$ is the boundary ∂K of a compact submanifold K of M , we will get the same result. This also happens when $K_{1,\Sigma_1} \cup K_{2,\Sigma_2}$ is just part of the boundary of a compact submanifold K of M but the remainder, $\partial K \setminus (K_{1,\Sigma_1} \cup K_{2,\Sigma_2})$, has empty intersection with the intersection of the base supports of $\delta_{X_1}\phi$ and $\delta_{X_2}\phi$.) Thus, at least formally, covariant phase space becomes a symplectic manifold—albeit an infinite-dimensional one; its symplectic form will in what follows be simply denoted by Ω and is explicitly given by the formula

$$\Omega_\phi(\delta_{X_1}\phi, \delta_{X_2}\phi) = \int_\Sigma \phi^*(i_{X_2}i_{X_1}\omega_{\mathcal{H}}), \tag{96}$$

where Σ is any Cauchy surface in M and where $\phi \in \mathcal{S}$ and $\delta_{X_1}\phi, \delta_{X_2}\phi \in T_\phi\mathcal{S}$, with X_1, X_2 vertical (or possibly just vertical on the image of ϕ) and such that $\text{supp } \delta_{X_1}\phi \cap \text{supp } \delta_{X_2}\phi \cap \Sigma$ is compact.

B. Functional Hamiltonian vector fields and Poisson brackets

The central result obtained in Ref. 15 can be summarized in the form of two theorems which we state explicitly because they form the background for the work reported here. The basic object that appears there is the *Jacobi operator* $\mathcal{J}[\phi]$, obtained by linearizing the De Donder–Weyl operator around a solution $\phi \in \mathcal{S}$ and whose kernel is precisely the space $T_\phi\mathcal{S}$ of solutions of the linearized equations of motion, and its causal Green function G_ϕ .

Theorem 3. *Given a functional F with temporally compact support on covariant phase space \mathcal{S} , the functional hamiltonian vector field X_F on \mathcal{S} associated to F , as defined by the formula*

$$\Omega_\phi(X_F[\phi], \delta\phi) = F'[\phi] \cdot \delta\phi \quad \text{for } \phi \in \mathcal{S}, \delta\phi \in T_\phi\mathcal{S}, \tag{97}$$

is given by ‘‘convolution’’ of the variational derivative of F (see Equation (21)) with the causal Green function of the corresponding Jacobi operator,

$$X_F^i[\phi](x) = \int_M d^n y G_\phi^{ij}(x, y) \frac{\delta F}{\delta \phi^j}[\phi](y) \quad \text{for } \phi \in \mathcal{S}. \tag{98}$$

Note that the condition that F should have temporally compact support will guarantee that both sides of Equation (97) make sense provided we interpret $T_\phi\mathcal{S}$ as being the space of solutions of the linearized equations of motion of spatially compact support, i.e., we regard it as the subspace of the space $T_\phi\mathcal{C}$ given by Equations (61)–(64) where the latter is defined according to Equation (18).

With this statement at hand, it is easy to write down the Poisson bracket of two functionals F and G on \mathcal{S} : in complete analogy with the formula $\{f, g\} = i_{X_g}i_{X_f}\omega = -dg(X_f)$ from mechanics, it can be defined by

$$\{F, G\}[\phi] = -G'[\phi](X_F[\phi]) \quad \text{for } \phi \in \mathcal{S}, \tag{99}$$

or more explicitly,

$$\{F, G\}[\phi] = - \int_M d^n x \frac{\delta G}{\delta \phi^k}[\phi](x) X_F^k[\phi](x) \quad \text{for } \phi \in \mathcal{S}. \tag{100}$$

Combining this expression with that given in Theorem 3, we arrive at the second main conclusion.

Theorem 4. *Given two functionals F and G with temporally compact support on covariant phase space \mathcal{S} , their Poisson bracket $\{F, G\}$, with respect to the symplectic form Ω introduced above, is precisely their Peierls–De Witt bracket, given by*

$$\{F, G\}[\phi] = \int_M d^n x \int_M d^n y \frac{\delta F}{\delta \phi^k}[\phi](x) G_\phi^{kl}(x, y) \frac{\delta G}{\delta \phi^l}[\phi](y) \quad \text{for } \phi \in \mathcal{S}. \quad (101)$$

Note that in view of the regularity conditions imposed to arrive at these results, the previous constructions do not apply directly to degenerate systems such as gauge theories: these require a separate treatment.

C. The main theorems

In this subsection, we present the two main theorems of the present paper which, for local functionals generated by locally hamiltonian forms according to Equation (5), provide a simple algebraic construction of the functional hamiltonian vector field associated with such a functional and, as a corollary, a simple algebraic formula for the Poisson bracket of two such functionals. That such formulas should exist is not at all obvious, taking into account that the corresponding formulas for general functionals, as given in Theorem 3 and Theorem 4 above, are essentially analytic: to apply them in concrete examples, one needs to solve a system of (linear) partial differential equations in order to calculate the corresponding causal Green function. Surprisingly, for local functionals of the form given by Equation (5), the “convolution type” integral operator which has this Green function as its kernel collapses.

Theorem 5. *Suppose we are given a fiber bundle E (the field configuration bundle) over an n -dimensional globally hyperbolic space-time manifold M and a hamiltonian $\mathcal{H} : \tilde{J}^{\otimes} E \rightarrow J^{\otimes} E$, which is a section of extended multiphase space $J^{\otimes} E$ over ordinary multiphase space $\tilde{J}^{\otimes} E$, together with a hamiltonian $(n - 1)$ -form f on $\tilde{J}^{\otimes} E$ of spatially compact support such that the corresponding hamiltonian vector field X_f on $\tilde{J}^{\otimes} E$, defined by the formula $i_{X_f} \omega_{\mathcal{H}} = df$, is projectable to M . (Recall that according to Theorem 2, this is automatic if $N > 1$.) Then given any Cauchy surface Σ in M and writing $\mathcal{F}_{\Sigma, f}$ for the local functional on \mathcal{S} associated to Σ and f , as in Definition 1, and $X_{\mathcal{F}_{\Sigma, f}}$ for the functional hamiltonian vector field on \mathcal{S} associated to $\mathcal{F}_{\Sigma, f}$, as in Theorem 3, we have*

$$X_{\mathcal{F}_{\Sigma, f}}[\phi] = \delta_{X_f} \phi \quad \text{for } \phi \in \mathcal{S}. \quad (102)$$

Note that this is a version of Noether’s theorem, i.e., we are dealing with a conservation law, for the lhs of Equation (102) appears to depend on Σ and the theorem states that it is equal to the rhs, which does not!

Proof. Rather than analyzing the integral formula (98), we shall show directly that the algebraic formula (102) satisfies all required conditions. First of all, we note that $\delta_{X_f} \phi$ has spatially compact support because f does. Moreover, it is clear from the results of Section IV that Equation (102) does provide a functional vector field not only on \mathcal{C} but also on \mathcal{S} since, according to Equation (63), X_f being locally hamiltonian with respect to $\omega_{\mathcal{H}}$ and projectable to M implies that $\delta_{X_f} \phi \in T_\phi \mathcal{S}$ when $\phi \in \mathcal{S}$. Therefore, all that needs to be verified is that the expression given in Equation (102) satisfies the condition (97). To this end, let $X_{f, M}$ denote the projection of X_f to M and, for any given $\phi \in \mathcal{S}$, apply Lemma 1 to construct some projectable vector field \tilde{X}_f^ϕ on $\tilde{J}^{\otimes} E$ which is ϕ -related to $X_{f, M}$; then the difference $X_f - \tilde{X}_f^\phi$ will be vertical on the image of ϕ , and according to Equation (25), Equation (102) becomes

$$X_{\mathcal{F}_{\Sigma, f}}[\phi] = (X_f - \tilde{X}_f^\phi)(\phi) = \delta_{(X_f - \tilde{X}_f^\phi)} \phi.$$

Inserting this expression into Equation (96), we get, for any $\delta_X\phi \in T_\phi\mathcal{S}$ where X is some vertical vector field on $\vec{J}^\otimes E$ of spatially compact support,

$$\Omega_\phi(X_{\mathcal{F}_{\Sigma,f}}[\phi], \delta_X\phi) = \Omega_\phi(\delta_{(X_f - \vec{X}_f)\phi}\phi, \delta_X\phi) = \int_\Sigma \phi^*(i_X i_{(X_f - \vec{X}_f)\phi} \omega_{\mathcal{H}}).$$

Now observe that the expression

$$\phi^*(i_X i_{\vec{X}_f} \omega_{\mathcal{H}}) = -\phi^*(i_{\vec{X}_f} i_X \omega_{\mathcal{H}}) = -i_{X_{f,M}}(\phi^*(i_X \omega_{\mathcal{H}}))$$

vanishes due to the assumption that ϕ is a solution of the equations of motion. Therefore,

$$\Omega_\phi(X_{\mathcal{F}_{\Sigma,f}}[\phi], \delta_X\phi) = \int_\Sigma \phi^*(i_X i_{X_f} \omega_{\mathcal{H}}),$$

which, according to Proposition 1, is equal to

$$\mathcal{F}'_{\Sigma,f}[\phi] \cdot \delta_X\phi = \int_\Sigma \phi^*(i_X df) = \int_\Sigma \phi^*(i_X i_{X_f} \omega_{\mathcal{H}}).$$

□

An immediate corollary of this theorem is that we can express the Peierls–De Witt bracket between two local functionals associated to hamiltonian $(n - 1)$ -forms directly in terms of their “multisymplectic Poisson bracket.”

Theorem 6. *Suppose we are given a fiber bundle E (the field configuration bundle) over an n -dimensional globally hyperbolic space-time manifold M and a hamiltonian $\mathcal{H} : \vec{J}^\otimes E \rightarrow J^\otimes E$, which is a section of extended multiphase space $J^\otimes E$ over ordinary multiphase space $\vec{J}^\otimes E$, together with two hamiltonian $(n - 1)$ -forms f and g on $\vec{J}^\otimes E$ of spatially compact support such that the corresponding hamiltonian vector fields X_f and X_g on $\vec{J}^\otimes E$ are projectable to M . (Recall that according to Theorem 2, this is automatic if $N > 1$.) Then given any Cauchy surface Σ in M , the Peierls–De Witt bracket between the local functionals $\mathcal{F}_{\Sigma,f}$ and $\mathcal{F}_{\Sigma,g}$ on \mathcal{S} associated to Σ and f and to Σ and g , as in Definition 1, is the local functional associated to Σ and any of the multisymplectic Poisson brackets $\{f, g\}$ on $\vec{J}^\otimes E$ that can be found in the literature, among them the simple “pseudo-bracket” defined by Equation (8) as well as the modified bracket defined by Equation (10). In other words, with any one of these choices, we have*

$$\{\mathcal{F}_{\Sigma,f}, \mathcal{F}_{\Sigma,g}\} = \mathcal{F}_{\Sigma,\{f,g\}}. \tag{103}$$

Proof. Combining Equation (99) with Equation (102) from the previous theorem and applying Equation (31) from Proposition 1 (which is applicable because the relevant integral need only be extended over a compact subset of Σ such that the base supports of f and g are contained in its interior), we obtain, for any $\phi \in \mathcal{S}$,

$$\begin{aligned} \{\mathcal{F}_{\Sigma,f}, \mathcal{F}_{\Sigma,g}\}[\phi] &= -\mathcal{F}'_{\Sigma,g}[\phi](X_{\mathcal{F}_{\Sigma,f}}[\phi]) = -\mathcal{F}'_{\Sigma,g}[\phi](\delta_{X_f}\phi) \\ &= -\int_\Sigma (\phi^*(i_{X_f} dg) - i_{X_{f,M}}(\phi^* dg)). \end{aligned}$$

Using that $dg = i_{X_g} \omega_{\mathcal{H}}$, we see that the second integral vanishes due to the equations of motion (52), and we get

$$\{\mathcal{F}_{\Sigma,f}, \mathcal{F}_{\Sigma,g}\}[\phi] = \int_\Sigma \phi^*(i_{X_g} i_{X_f} \omega_{\mathcal{H}}) = \int_\Sigma \phi^*\{f, g\} = \mathcal{F}_{\Sigma,\{f,g\}}[\phi],$$

where the second equality is obvious if we employ the “pseudo-bracket” of Equation (8) but holds equally well if we employ the modified bracket of Equation (10) since, once again, the relevant integral need only be extended over a compact subset of Σ such that the base supports of f and g are contained in its interior and then the integral over the additional term can, by Stokes’s theorem, be converted to an integral over the boundary of that compact subset and hence vanishes. □

VII. CONCLUSIONS AND OUTLOOK

In this paper we have established a link between the multisymplectic Poisson brackets that have been studied by geometers over more than four decades and the covariant functional Poisson bracket of classical field theory, commonly known as the Peierls–De Witt bracket. This link is based on associating to each differential form on the pertinent multiphase space a certain local functional obtained by pulling that form back to space-time via a solution of the equations of motion and integrating over some fixed submanifold of space-time of the appropriate dimension, considering the result as a functional on covariant phase space (the space of solutions of the equations of motion). Here, we have restricted attention to forms of degree $n - 1$, where n is the dimension of space-time, which have to be integrated over submanifolds of codimension 1 (hypersurfaces), but we cannot see any obvious obstruction to extending this kind of analysis to forms of other degree. This would be of considerable interest since in physics there appear many functionals that are localized on submanifolds of space-time of other dimensions, such as values of observable fields at space-time points (dimension 0), Wilson loops (traces of parallel transport operators around loops) in gauge theories (dimension 1), electromagnetic field strength tensors and curvature tensors (dimension 2), etc.

The overall picture that emerges is that the correct approach to the concept of observables in classical field theory is to regard them as smooth functionals on covariant phase space. As is well known, covariant phase space is, at least formally and for nondegenerate systems, an (infinite-dimensional) symplectic manifold, so the space of all such functionals constitutes a Poisson algebra, and that is what we are referring to when we speak about the “algebra of observables” in classical field theory. Of course, this algebra is huge, and the construction of local functionals on covariant phase space from differential forms on multiphase space, as employed in this paper, is merely a device for producing special (and quite small) classes of such observables. But the reduction of the algebraic structure at the level of such functionals to some corresponding algebraic structure for the generating differential forms is highly problematic: in fact, there is no reason to expect that there might exist any product or bracket between differential forms on multiphase space capable of reproducing the standard product or bracket between the corresponding functionals on covariant phase space. One possible obstacle is that any such prescription would most likely be highly ambiguous since a crucial piece of information is missing: after all, the functional does not only depend on the differential form which (after pull-back) is being integrated but also on the submanifold over which one integrates! One way out would be to restrict to functionals defined by some fixed submanifold and hope that the resulting algebraic structure does not depend on which submanifold (within a certain given class) is chosen: that is what we have done in this paper when reducing the covariant Peierls–De Witt bracket to a multisymplectic bracket. But the fact that this actually works is to a certain extent a miracle which cannot be expected to happen in general, since we will very likely be forced into admitting functionals defined by integration over different submanifolds, including submanifolds of different dimension. For example, this happens as soon as we want to include differential forms of different degrees and/or explore the existence of a relation between the product of functionals and the exterior product of forms, or some modified form thereof. In particular, these arguments show why the notorious absence of a decent associative product in the multisymplectic formalism should come as no surprise: it merely expresses the fact that functionals of the form given by Equation (5) do not form a subalgebra.

At any rate, it is a highly interesting question what kind of algebraic structure on what kind of spaces of differential forms (or pairs of submanifolds and differential forms) will ultimately result from the functional approach advocated in this paper. These and similar questions are presently under investigation and will be reported in a future publication.

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