JOURNAL OF MATHEMATICAL PHYSICS **46**, 1 (2005)

Hamiltonian multivector fields and Poisson forms in multisymplectic field theory

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(Received 4 August 2005; accepted 16 September 2005)

We present a general classification of Hamiltonian multivector fields and of Poisson forms on the extended multiphase space appearing in the geometric formulation of first order classical field theories. This is a prerequisite for computing explicit expressions for the Poisson bracket between two Poisson forms. © *2005 American Institute of Physics.* [DOI: 10.1063/1.2116320]

I. INTRODUCTION AND GENERAL SETUP

The present paper is a continuation of previous work on Poisson brackets of differential forms in the multiphase space approach to classical field theory.[1,2] Our aim is to specialize the general constructions of Ref. 2 from abstract (exact) multisymplectic manifolds to the extended multiphase spaces of field theory, which at present seem to be the only known examples of multisymplectic manifolds, to clarify the structure of Hamiltonian multivector fields, of Hamiltonian forms and of Poisson forms on these spaces and to give explicit formulas for the Poisson bracket between the latter introduced in Refs. 1 and 2.

The structure of this paper is as follows. In the remainder of this introduction, we briefly review the geometric constructions needed in the paper. We put particular emphasis on the consequences that arise from the existence of a certain vector field, the scaling or Euler vector field. Also, we fix the notation to be used in what follows. In Sec. II, we present an explicit classification of locally Hamiltonian multivector fields on extended multiphase space in terms of adapted local coordinates and, following the logical inclusion from locally Hamiltonian to (globally) Hamiltonian to exact Hamiltonian multivector fields, show how the last two are situated within the first. Sec. III is devoted to the study of Hamiltonian forms and Poisson forms that are associated with (globally) Hamiltonian multivector fields. In Sec. IV, we use the outcome of our previous analysis to derive expressions for the Poisson bracket between two Poisson forms. In Sec. V, we summarize our main conclusions and comment on the relation of our results to other approaches, as well as on perspectives for future research. Finally, in order to make the paper self-contained, we include in an appendix a proposition that is not new but is needed in some of the proofs.

We begin with a few comments on the construction of the extended multiphase space of field theory,[3–7] which starts out from a given general fiber bundle over space-time, with base space M ($\dim M = n$), total space E, bundle projection $\pi : E \to M$ and typical fiber Q ($\dim Q = N$). It is

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usually referred to as the configuration bundle since its sections constitute the possible field configurations of the system. (Of course, the manifold M represents space-time, whereas the manifold Q plays the role of a configuration space.) The extended multiphase space, which we shall simply denote by P, is then the total space of a larger fiber bundle over M and in fact the total space of a vector bundle over E which can be defined in several equivalent ways, e.g., by taking the twisted affine dual $J^{\circledast}E$ of the first order jet bundle JE of E or by taking the bundle $\Lambda^n_{n-1}T^*E$ of $(n-1)$-horizontal n-forms on E; see Refs. 2, 5, and 7 for details. Therefore, there is a natural class of local coordinate systems on P, namely those that arise from combining fiber bundle charts of E over M with vector bundle charts of P over E: these so-called adapted local coordinates (x^μ, q^i, p_i^μ, p) are completely fixed by specifying local coordinates x^μ for M (the space-time coordinates), local coordinates q^i for Q (the position variables) and a local trivialization of E over M, and are such that the induced local coordinates p_i^μ (the multimomentum variables) and p (the energy variable) are linear along the fibers of P over E. For details, we refer to Ref. 2, where one can also find the explicit transformation law for the multimomentum variables and the energy variable induced by a change of the space-time coordinates, of the position variables and of the local trivialization.

A first important feature of the extended multiphase space P is that it carries a naturally defined multicanonical form θ whose exterior derivative is, up to a sign, the multisymplectic form ω,

$$\omega = -\,\mathrm{d}\theta. \tag{1}$$

The global construction can be found in Refs. 2, 5, and 7, so we shall just state their explicit form in adapted local coordinates,

$$\theta = p_i^\mu\, \mathrm{d}q^i \wedge \mathrm{d}^n x_\mu + p\mathrm{d}^n x. \tag{2}$$

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i^\mu \wedge \mathrm{d}^n x_\mu - \mathrm{d}p \wedge \mathrm{d}^n x. \tag{3}$$

Here, we have already employed part of the following conventions concerning local differential forms defined by a system of adapted local coordinates, which will be used systematically throughout this paper,

$$\mathrm{d}^n x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \quad \mathrm{d}^n x_{\mu_1\ldots\mu_r} = i_{\partial_{\mu_r}} \cdots i_{\partial_{\mu_1}} \mathrm{d}^n x.$$

For later use, we also recall the definition of the Lie derivative of a differential form α along an r-multivector field X,

$$L_X\alpha = \mathrm{d}i_X\alpha - (-1)^r i_X\, \mathrm{d}\alpha, \tag{4}$$

which leads to the following relations, valid for any differential form α and any two multivector fields X and Y of tensor degrees r and s, respectively,

$$\mathrm{d}L_X\alpha = (-1)^{r-1} L_X\, \mathrm{d}\alpha, \tag{5}$$

$$i_{[X,Y]}\alpha = (-1)^{(r-1)s} L_X i_Y\alpha - i_Y L_X\alpha, \tag{6}$$

$$L_{[X,Y]}\alpha = (-1)^{(r-1)(s-1)} L_X L_Y\alpha - L_Y L_X\alpha, \tag{7}$$

$$L_{X\wedge Y}\alpha = (-1)^s i_Y L_X\alpha + L_Y i_X\alpha, \tag{8}$$

where $[X,Y]$ denotes the Schouten bracket of X and Y. For decomposable multivector fields $X = X_1 \wedge \cdots \wedge X_r$ and $Y = Y_1 \wedge \cdots \wedge Y_s$, it can be defined in terms of the Lie bracket of vector fields according to the formula

$$[X,Y] = \sum_{i=1}^{r} \sum_{j=1}^{s} (-1)^{i+j}[X_i,X_j] \wedge X_1 \wedge \cdots \hat{X}_i \cdots \wedge X_r \wedge Y_1 \wedge \cdots \hat{Y}_j \cdots \wedge Y_s,$$

where as usual the hat over a symbol denotes its omission. We shall also write

$$L_X Y = [X,Y],$$

for any two multivector fields X and Y. For properties of the Schouten bracket, we refer to Ref. 8. A proof of the above identities relating the Schouten bracket and the Lie derivative of forms along multivector fields can be found in the appendix of Ref. 2.

A second property of the extended multiphase space P which provides additional structures for tensor calculus on this manifold is that it is the total space of a fiber bundle, which implies that we may speak of vertical vectors and horizontal covectors. In fact, it is so in no less than three different ways. Namely, P is the total space of a fiber bundle over M (with respect to the so-called source projection), the total space of a vector bundle over E (with respect to the so-called target projection) and the total space of an affine line bundle over the ordinary multiphase space P_0.[2] Therefore, the notions of verticality for multivector fields and of horizontality for differential forms on P admit different interpretations, depending on which projection is used. In any case, one starts by defining tangent vectors to the total space of a fiber bundle to be vertical if they are annihilated by the tangent map to the bundle projection, or what amounts to the same thing, if they are tangent to the fibers. Dually, a k-form on the total space of a fiber bundle is said to be l-horizontal if it vanishes whenever one inserts at least $k-l+1$ vertical tangent vectors; the standard horizontal forms are obtained by taking $l=k$. Finally, an r-multivector on the total space of a fiber bundle is said to be s-vertical if its contraction with any $(r-s+1)$-horizontal form vanishes. It is not difficult to show that these definitions are equivalent to requiring that, locally, an l-horizontal k-form should be a sum of exterior products of k one-forms, among which there are at least l horizontal ones, and that an s-vertical r-multivector field should be a sum of exterior products of r tangent vectors, among which there are at least s vertical ones. Using this rule, properties of verticality for multivectors or horizontality for forms are easily derived from the corresponding properties for vectors or one-forms, respectively. In what follows, the terms "vertical" and "horizontal" will usually refer to the source projection, except when explicitly stated otherwise.

A third important feature of the extended multiphase space P is that it carries a naturally defined vector field Σ, the scaling vector field or Euler vector field, which exists on any manifold that is the total space of a vector bundle. In adapted local coordinates,

$$\Sigma = p_i^{\mu} \frac{\partial}{\partial p_i^{\mu}} + p \frac{\partial}{\partial p}.$$

It is then easy to verify the following relations (see Proposition 2.1 of Ref. 2):

$$i_\Sigma \theta = 0, \quad i_\Sigma \omega = -\theta, \quad L_\Sigma \theta = \theta, \quad L_\Sigma \omega = \omega. \tag{9}$$

The main utility of Σ is that taking the Lie derivative L_Σ along Σ provides a device for controlling the dependence of functions and, more generally, of tensor fields on P on the multimomentum variables and the energy variable, that is, along the fibers of P over E: L_Σ has only integer eigenvalues, and eigenfunctions of L_Σ with eigenvalue k are homogeneous polynomials of degree k in these variables.

As we shall see soon, homogeneity under L_Σ plays a central role in the analysis of various classes of multivector fields and differential forms on P.

Let us recall a few definitions. An r-multivector field X on P is called *locally Hamiltonian* if $i_X\omega$ is closed, or equivalently, if

$$L_X \omega = 0. \tag{10}$$

It is called *globally Hamiltonian* if $i_X\omega$ is exact, that is, if there exists an $(n\text{-}r)$-form f on P such that

$$i_X\omega = \mathrm{d}f. \tag{11}$$

In this case, f is said to be a *Hamiltonian form associated with* X. Finally, it is called *exact Hamiltonian* if

$$L_X\theta = 0. \tag{12}$$

Of course, exact Hamiltonian multivector fields are globally Hamiltonian [to show this, set $f=(-1)^{r-1}i_X\theta$ and apply Eqs. (4) and (1)], and globally Hamiltonian multivector fields are obviously locally Hamiltonian. Conversely, an $(n\text{-}r)$-form f on P is called a *Hamiltonian form* if there exists an r-multivector field X on P such that Eq. (11) holds; in this case, X is said to be a *Hamiltonian multivector field associated with* f. Moreover, f is called a *Poisson form* if in addition, it vanishes on the kernel of ω, that is, if for any multivector field Z, we have

$$i_Z\omega = 0 \Rightarrow i_Zf = 0. \tag{13}$$

A trivial example of a Poisson form is the multisymplectic form ω itself. Another example is provided by the multicanonical form θ, since it can be written as $\theta=-i_\Sigma\omega$.

Concerning stability under the Lie derivative along the scaling vector field Σ, we have the following.

Proposition 1.1: The space $\mathfrak{X}^\wedge_{LH}(P)$ of locally Hamiltonian multivector fields, the space $\mathfrak{X}^\wedge_{H}(P)$ of globally Hamiltonian multivector fields, the space $\mathfrak{X}^\wedge_{EH}(P)$ of exact Hamiltonian multivector fields and the space $\mathfrak{X}^\wedge_{0}(P)$ of multivector fields taking values in the kernel of ω are all invariant under the Lie derivative along the scaling vector field Σ,

$$L_X\omega = 0 \;\Rightarrow\; L_{[\Sigma,X]}\omega = 0, \tag{14}$$

$$i_X\omega = \mathrm{d}f \;\Rightarrow\; i_{[\Sigma,X]}\omega = \mathrm{d}(L_\Sigma f - f), \tag{15}$$

$$L_X\theta = 0 \;\Rightarrow\; L_{[\Sigma,X]}\theta = 0, \tag{16}$$

$$i_\xi\omega = 0 \;\Rightarrow\; i_{[\Sigma,\xi]}\omega = 0. \tag{17}$$

Proof: All these relations can be shown by direct calculation. □

Dually, we have the following.

Proposition 1.2: The space $\Omega_H(P)$ of Hamiltonian forms, the space $\Omega_0(P)$ of forms that vanish on the kernel of ω and the space $\Omega_P(P)$ of Poisson forms are all invariant under the Lie derivative along the scaling vector field Σ,

$$\mathrm{d}f = i_X\omega \;\Rightarrow\; \mathrm{d}(L_\Sigma f) = i_{X+[\Sigma,X]}\omega. \tag{18}$$

Proof: The first statement is a consequence of Eq. (18), which follows directly from combining Eqs. (5) and (6) with Eq. (9). For the second statement, assume that f vanishes on the kernel of ω. Then if ξ is any multivector field ϑ taking values in the kernel of ω, the multivector field $[\Sigma,\xi]$ takes values in the kernel of ω as well [cf. Eq. (17)], so that according to Eq. (6),

$$i_\xi(L_\Sigma f) = L_\Sigma i_\xi f - i_{[\Sigma,\xi]}f = 0.$$

But this means that $L_\Sigma f$ vanishes on the kernel of ω. Finally, the third statement follows by combining the first two. □

A special class of multivector fields and of differential forms on P which will be of particular importance in what follows is that of *fiberwise polynomial multivector fields* and of *fiberwise polynomial differential forms* on P: their coefficients are polynomials along the fibers of P over E, or in other words, polynomials in the multimomentum variables and the energy variable. The main advantage of working with tensor fields on the total space of a vector bundle which are fiberwise polynomial is that they allow a unique and globally defined (or in other words, coordinate independent) decomposition into homogeneous components, according to the different eigenspaces of the Lie derivative L_Σ along Σ; the corresponding eigenvalue will in what follows be called the *scaling degree* (to distinguish it from the ordinary tensor degree). In doing so, it must be borne in mind that, in an expansion with respect to an adapted local coordinate system, the scaling degree receives contributions not only from the coefficient functions but also from some of the coordinate vector fields and differentials since the vector fields $\partial/\partial x^\mu$, $\partial/\partial q^i$, $\partial/\partial p_i^\mu$, and $\partial/\partial p$ carry scaling degree 0, 0, -1, and -1, respectively, while the differentials $\mathrm{d}x^\mu$, $\mathrm{d}q^i$, $\mathrm{d}p_i^\mu$, and $\mathrm{d}p$ carry scaling degree 0, 0, $+1$, and $+1$, respectively; moreover, the scaling degree is additive under the exterior product, since L_Σ is a derivation. Therefore, a fiberwise polynomial r-multivector field on P admits a globally defined decomposition into a finite sum

$$X = \sum_{s \geqslant -r} X_s,$$

where X_s is its homogeneous component of scaling degree s,

$$L_\Sigma X_s = s X_s.$$

Each X_s can be obtained from X by applying a projector which is itself a polynomial in L_Σ,

$$X_s = \prod_{\substack{s' \geqslant -r \\ s' \neq s}} \frac{1}{s - s'} (L_\Sigma - s') X.$$

Similarly, a fiberwise polynomial $(n$-$r)$-form f on P admits a globally defined decomposition into a finite sum

$$f = \sum_{s \geqslant 0} f_s,$$

where f_s is its homogeneous component of scaling degree s,

$$L_\Sigma f_s = s f_s.$$

Again, the f_s can be obtained from f,

$$f_s = \prod_{\substack{s' \geqslant 0 \\ s' \neq s}} \frac{1}{s - s'} (L_\Sigma - s') f.$$

The relevance of these decompositions for locally Hamiltonian multivector fields and for Hamiltonian forms on the extended multiphase space P stems from the following theorem, whose proof will follow from statements to be derived in the course of the next two sections, by means of explicit calculations in adapted local coordinates.

Theorem 1.3: *For* $0 < \tau < n$ *and up to trivial contributions (*τ*-multivector fields taking values in the kernel of* ω *and closed* $(n-r)$*-forms, respectively), locally Hamiltonian* τ*-multivector fields and Hamiltonian* $(n-\tau)$*-forms on* P *are fiberwise polynomial and have non-trivial homogeneous components of scaling degree* s *only for* $s = -1, 0, \ldots, r-1$ *and for* $s = 0, 1, \ldots, r$*, respectively. More precisely, we have*

(i) *Every fiberwise polynomial locally Hamiltonian (Hamiltonian, exact Hamiltonian)* r*-multivector field* X *on* P*, admits a unique, globally defined decomposition into homogeneous components with respect to scaling degree, which can be written in the form (we abbreviate* X_{-1} *as* X_-*)*

$$X = X_- + X_+ + \xi \quad \textit{with } X_+ = \sum_{s=0}^{r-1} X_s, \tag{19}$$

where each X_s is locally Hamiltonian (Hamiltonian, exact Hamiltonian) and

$$\xi = \sum_{-r \leqslant s \leqslant -2} \xi_s + \sum_{s \geqslant r} \xi_s \tag{20}$$

is a fiberwise polynomial r-multivector field on P taking values in the kernel of ω.

(ii) *Every fiberwise polynomial Hamiltonian form (Poisson form) f of degree $n-r$ on P, admits a unique, globally defined decomposition into homogeneous components with respect to scaling degree, which can be written in the form*

$$f = f_0 + f_+ + f_c \quad \textit{with } f_+ = \sum_{s=1}^{r} f_s, \tag{21}$$

where each f_s is Hamiltonian (Poisson) and

$$f_c = \sum_{s \geqslant r+1} (f_c)_s \tag{22}$$

is a fiberwise polynomial closed $(n-r)$-form on P.

The cases $r=0$ and $r=n$ are exceptional and must be dealt with separately; see Propositions 2.2 and 3.2 for $r=0$ and Propositions 2.3 and 3.1 for $r=n$.

In view of this theorem, it is sufficient to study locally Hamiltonian multivector fields and Hamiltonian forms which are homogeneous under the Lie derivative along the scaling vector field Σ. This condition of homogeneity is also compatible with the correspondence between globally Hamiltonian multivector fields X and Hamiltonian forms f established by the fundamental relation (11), because ω itself is homogeneous: according to Eq. (9), ω has scaling degree 1. Indeed, except for the ambiguity inherent in this correspondence (f determines X only up to a multivector field taking values in the kernel of ω and X determines f only up to a closed form), Eq. (11) preserves the scaling degree, up to a shift by 1: X is homogeneous with scaling degree $s-1$ if and only if f is homogeneous with scaling degree s,

$$\begin{array}{c} L_\Sigma X = (s-1)X \\ \text{modulo multivector fields} \\ \text{taking values in the kernel of } \omega \end{array} \quad \Leftrightarrow \quad \begin{array}{c} L_\Sigma f = sf \\ \text{modulo closed forms} \end{array}. \tag{23}$$

For a proof, note that the condition on the left-hand side (lhs) amounts to requiring that $i_{[\Sigma,X]}\omega = (s-1) i_X \omega$, while the condition on the right-hand side (rhs) amounts to requiring that $\mathrm{d}L_\Sigma f = s\,\mathrm{d}f$, so the equivalence stated in Eq. (23) is an immediate consequence of Eq. (18). A particular case occurs when $s=1$, since the locally Hamiltonian multivector fields which are homogeneous of scaling degree 0 are precisely the exact Hamiltonian multivector fields: for $L_X\omega = 0$,

$$\begin{array}{c} L_\Sigma X = 0 \\ \text{modulo multivector fields} \\ \text{taking values in the kernel of } \omega \end{array} \quad \Leftrightarrow \quad L_X \theta = 0. \tag{24}$$

Indeed, the properties of θ and ω give

$$L_X \theta = -L_X i_\Sigma \omega = (-1)^r (i_{[X,\Sigma]}\omega - i_\Sigma L_X \omega) = (-1)^{r-1} i_{[\Sigma,X]}\omega. \tag{25}$$

More generally, the fundamental relation (11) preserves the property of being fiberwise polynomial, in the following sense: If X is a fiberwise polynomial Hamiltonian r-multivector field and f is a Hamiltonian $(n-r)$-form associated with X, then modifying f by addition of an appropriate closed $(n-r)$-form if necessary, we may always assume, without loss of generality, that f is fiberwise polynomial as well. Conversely, if f is a fiberwise polynomial Hamiltonian $(n-r)$-form and X is a Hamiltonian r-multivector field associated with f, then modifying X by addition of an appropriate r-multivector field taking values in the kernel of ω if necessary, we may always assume, without loss of generality, that X is fiberwise polynomial as well.

II. HAMILTONIAN MULTIVECTOR FIELDS

Our aim in this section is to determine the explicit form, in adapted local coordinates, of locally Hamiltonian r-multivector fields on the extended multiphase space P, where $0 \leqslant r \leqslant n+1$. (Multivector fields of tensor degree $>n+1$ are uninteresting since they always take their values in the kernel of ω.)

As a first step towards this goal, we shall determine the explicit form, in adapted local coordinates, of the multivector fields on P taking values in the kernel of ω; this will also serve to identify, in the next section, the content of the kernel condition (13) that characterizes Poisson forms. To this end, note first that ω being a homogeneous differential form (of degree $n+1$), its kernel is graded, that is, if an inhomogeneous multivector field takes values in the kernel of ω, so do all its homogeneous components.

Proposition 2.1: Every r-multivector field X on P admits, in adapted local coordinates, a unique decomposition of the form

$$\begin{aligned} X = {} & \frac{1}{r!} X^{\mu_1\cdots\mu_r} \frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} + \frac{1}{(r-1)!} X^{i,\mu_2\cdots\mu_r} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} \\ & + \frac{1}{r!} X_i^{\mu_1\cdots\mu_r} \frac{\partial}{\partial p_i^{\mu_1}} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} + \frac{1}{(r-1)!} \tilde{X}^{\mu_2\cdots\mu_r} \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} + \xi, \end{aligned} \tag{26}$$

where all coefficients are totally antisymmetric in their space-time indices and ξ *takes values in the kernel of* ω.

Proof: This is an immediate consequence of the particular form of ω in adapted local coordinates, Eq. (3). For more details, see Ref. 9. □

With this local coordinate representation at hand, we are in a position to analyze the restrictions imposed on the coefficients $X^{\mu_1\cdots\mu_r}$, $X^{i,\mu_2\cdots\mu_r}$, $X_i^{\mu_1\cdots\mu_r}$, and $\tilde{X}^{\mu_2\cdots\mu_r}$ by requiring X to be locally Hamiltonian. (Of course, it makes no sense to discuss the question which locally Hamiltonian multivector fields are also globally Hamiltonian when working in local coordinates.) As a warm-up exercise, we shall settle the extreme cases of tensor degree 0 and $n+1$.

Proposition 2.2: A function on P, regarded as a 0*-multivector field, is locally Hamiltonian if and only if it is constant; it is then also exact Hamiltonian. Similarly, an* $(n+1)$*-multivector field on P, with standard local coordinate representation*

$$X = \tilde{X} \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n} + \xi, \tag{27}$$

where ξ *takes values in the kernel of* ω, *is locally Hamiltonian if and only if the coefficient function* $\tilde{X}$ *is constant and is exact Hamiltonian if and only if it vanishes.*

Proof: For functions, we use the fact that the operator i_1 corresponding to the constant function 1 on a manifold is defined to be the identity, so that the operator i_f corresponding to an arbitrary function f on a manifold is simply multiplication by f. Therefore, we have for any differential form α

$$L_f \alpha = \mathrm{d}(i_f \alpha) - i_f \, \mathrm{d}\alpha = \mathrm{d}(f\alpha) - f \, \mathrm{d}\alpha = \mathrm{d}f \wedge \alpha,$$

implying that if f is constant, $L_f\alpha=0$ no matter what α one chooses. On the other hand, an explicit calculation in adapted local coordinates shows that the condition $L_f\omega=0$ forces all partial derivatives of f to vanish; see Ref. 9. Similarly, for multivector fields of degree $n+1$, it is clear that, as r equals $n+1$, the first three terms in Eq. (26) also take values in the kernel of ω and can thus be incorporated into ξ. Therefore, by setting $\tilde{X}^{\mu_1 \cdots \mu_n} = \epsilon^{\mu_1 \cdots \mu_n} \tilde{X}$, we see that $i_X \omega = -\tilde{X}$. But $L_X \omega = \mathrm{d}(i_X \omega) - (-1)^n i_X \, \mathrm{d}\omega = \mathrm{d}(i_X \omega)$ and $L_X \theta = \mathrm{d}(i_X \theta) - (-1)^{n+1} i_X \, \mathrm{d}\theta = (-1)^{n+1} i_X \omega$, so the proposition follows. ☐

The intermediate cases $(0 < r \leq n)$ are much more interesting. However, the situation for tensor degree n is substantially different from that for tensor degree $<n$ and hence will be dealt with first. To simplify the notation, we write

$$X^{\mu_1 \cdots \mu_n} = \epsilon^{\mu_1 \cdots \mu_n} \tilde{X}, \quad X^{i,\mu_2 \cdots \mu_n} = \epsilon^{\mu_2 \cdots \mu_n \mu} X^i_{\mu},$$

$$X_i^{\mu_1 \cdots \mu_n} = \epsilon^{\mu_1 \cdots \mu_n} X_i, \quad \tilde{X}^{\mu_2 \cdots \mu_n} = \epsilon^{\mu_2 \cdots \mu_n \mu} X_{\mu},$$

so that we obtain

$$i_X \omega = (-1)^{n-1} \tilde{X} \, \mathrm{d}p + X^i_{\mu} \, \mathrm{d}p_i^{\mu} - (-1)^{n-1} X_i \, \mathrm{d}q^i - X_{\mu} \, \mathrm{d}x^{\mu}, \tag{28}$$

and

$$i_X \theta = p \tilde{X} + (-1)^{n-1} p_i^{\mu} X^i_{\mu},$$

respectively.

Proposition 2.3: An n-multivector field X on P is locally Hamiltonian if and only if, locally and modulo terms taking values in the kernel of ω, it can be written in terms of a single function f, as follows:

$$X = -\frac{1}{(n-1)!} \epsilon^{\mu_2 \cdots \mu_n \mu} \left(\frac{\partial f}{\partial x^{\mu}} \frac{\partial}{\partial p} - \frac{1}{n} \frac{\partial f}{\partial p} \frac{\partial}{\partial x^{\mu}} \right) \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_n}}$$
$$+ \frac{1}{(n-1)!} \epsilon^{\mu_2 \cdots \mu_n \mu} \left(\frac{\partial f}{\partial p_i^{\mu}} \frac{\partial}{\partial q^i} - \frac{1}{n} \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i^{\mu}} \right) \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_n}}. \tag{29}$$

Moreover, X is exact Hamiltonian if and only if f is a linear function of the multimomentum variables p_r^{ρ} *and the energy variable p.*

Proof: Obviously, X is locally Hamiltonian if and only if, locally, $i_X \omega = \mathrm{d}f$ for some function f, which in view of Eq. (28) leads to the following system of equations for the coefficients $\tilde{X}$, X^i_{μ}, X_i, and X_{μ} of X:

$$\tilde{X} = (-1)^{n-1} \frac{\partial f}{\partial p}, \quad X^i_{\mu} = \frac{\partial f}{\partial p_i^{\mu}}, \quad X_i = (-1)^n \frac{\partial f}{\partial q^i}, \quad X_{\mu} = -\frac{\partial f}{\partial x^{\mu}}.$$

Inserting this back into X, we arrive at Eq. (29). Note also that then,

$$i_X \theta = (-1)^{n-1} p \frac{\partial f}{\partial p} + (-1)^{n-1} p_i^{\mu} \frac{\partial f}{\partial p_i^{\mu}} = (-1)^{n-1} L_{\Sigma} f.$$

Next, X will be exact Hamiltonian if and only if, in addition,

$$f = (-1)^{n-1} i_X \theta,$$

which in view of the preceding equation means that f must be an eigenfunction of the scaling operator L_{Σ} with eigenvalue 1, this is well known to be the case if and only if f is linear in the multimomentum variables p_r^{ρ} and the energy variable p. ☐

Now we turn to multivector fields of tensor degree $<n$. Here, the main result is the following.

Theorem 2.4: *An r-multivector field X on P, with* $0 < r < n$, *is locally Hamiltonian if and only* *if the coefficients* $X^{\mu_1\cdots\mu_r}$, $X^{i,\mu_2\cdots\mu_r}$, $X_i^{\mu_1\cdots\mu_r}$, *and* $\widetilde{X}^{\mu_2\cdots\mu_r}$ *in its standard local coordinate representation (26) satisfy the following conditions:*

(1) *The coefficients* $X^{\mu_1\cdots\mu_r}$ *depend only on the local coordinates* x^ρ *for* M *and, in the special case* $N=1$, *also on the local fiber coordinates* q^r *for* E,

(2) *The coefficients* $X^{i,\mu_2\cdots\mu_r}$ *are "antisymmetric polynomials in the multimomentum variables" of degree* $r-1$, *i.e., they can be written in the form*

$$X^{i,\mu_2\cdots\mu_r} = \sum_{s=1}^{r} X_{s-1}^{i,\mu_2\cdots\mu_r}, \tag{30}$$

with

$$X_{s-1}^{i,\mu_2\cdots\mu_r} = \frac{1}{(s-1)!}\frac{1}{(r-s)!}\sum_{\pi\in S_{r-1}} (-1)^\pi p_{i_2}^{\mu_{\pi(2)}}\cdots p_{i_s}^{\mu_{\pi(s)}} Y_{s-1}^{ii_2\cdots i_s,\mu_{\pi(s+1)}\cdots\mu_{\pi(r)}}, \tag{31}$$

where S_{r-1} *denotes the permutation group of* $\{2,\ldots,r\}$ *and the coefficients* $Y_{s-1}^{ii_2\cdots i_s,\mu_{s+1}\cdots\mu_r}$ *depend only on the local coordinates* x^ρ *for* M *as well as the local fiber coordinates* q^r *for* E *and are totally antisymmetric in* $i, i_2, \ldots, i_s$ *as well as in* $\mu_{s+1}, \ldots, \mu_r$.

(3) *The remaining coefficients* $X_i^{\mu_1\cdots\mu_r}$ *and* $\widetilde{X}^{\mu_2\cdots\mu_r}$ *can be expressed in terms of the previous ones and of new coefficients* $X_-^{\mu_1\cdots\mu_r}$ *depending only on the local coordinates* x^ρ *for* M *as well as the local fiber coordinates* q^r *for* E *and are totally antisymmetric in* $\mu_1, \ldots, \mu_r$, *according to*

$$\begin{aligned} X_i^{\mu_1\cdots\mu_r} = {} & -p\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial q^i} + p_i^\mu \frac{\partial X^{\mu_1\cdots\mu_r}}{\partial x^\mu} - \sum_{s=1}^{r} p_i^{\mu_s}\frac{\partial X^{\mu_1\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^\nu} \\ & - \Sigma^{-1}\left(\sum_{s=1}^{r}(-1)^{s-1} p_j^{\mu_s}\frac{\partial X^{j,\mu_1\cdots\mu_{s-1}\mu_{s+1}\cdots\mu_r}}{\partial q^i}\right) + \frac{\partial X_-^{\mu_1\cdots\mu_r}}{\partial q^i}, \end{aligned} \tag{32}$$

(the first term being absent as soon as $N>1$*) and*

$$\begin{aligned} \widetilde{X}^{\mu_2\cdots\mu_r} = {} & (-1)^r p\frac{\partial X^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu} - \Sigma^{-1}\left(p_i^\mu\frac{\partial X^{i,\mu_2\cdots\mu_r}}{\partial x^\mu} - \sum_{s=2}^{r} p_i^{\mu_s}\frac{\partial X^{i,\mu_2\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^\nu}\right) \\ & - (-1)^r\frac{\partial X_-^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu}. \end{aligned} \tag{33}$$

It is exact Hamiltonian if and only if, in addition, the coefficents $X^{i,\mu_2\cdots\mu_r}$ *depend only on the local coordinates* x^ρ *for* M *as well as the local fiber coordinates* q^r *for* E *and the coefficients* $X_-^{\mu_1\cdots\mu_r}$ *vanish.*

Proof: The proof will be carried out by "brute force" computation.[9] We obtain for the Lie derivative of ω along X the expression

$$\begin{aligned} L_X\omega = {} & -\frac{1}{(r-2)!}\frac{\partial \widetilde{X}^{\mu_3\cdots\mu_r\nu}}{\partial x^\nu}\mathrm{d}^n x_{\mu_3\cdots\mu_r} - \frac{1}{(r-1)!}\left(\frac{\partial \widetilde{X}^{\mu_2\cdots\mu_r}}{\partial q^i} - (-1)^{r-1}\frac{\partial X_i^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu}\right)\mathrm{d}q^i \wedge \mathrm{d}^n x_{\mu_2\cdots\mu_r} \\ & - \frac{1}{(r-1)!}\left(\frac{\partial \widetilde{X}^{\mu_2\cdots\mu_r}}{\partial p_i^\mu} + \frac{\partial X^{i,\mu_2\cdots\mu_r}}{\partial x^\mu} - \sum_{s=2}^{r}\delta_\mu^{\mu_s}\frac{\partial X^{i,\mu_2\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^\nu}\right)\mathrm{d}p_i^\mu \wedge \mathrm{d}^n x_{\mu_2\cdots\mu_r} \\ & - \frac{1}{(r-1)!}\left(\frac{\partial \widetilde{X}^{\mu_2\cdots\mu_r}}{\partial p} + (-1)^{r-1}\frac{\partial X^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu}\right)\mathrm{d}p \wedge \mathrm{d}^n x_{\mu_2\cdots\mu_r} \\ & - \frac{(-1)^r}{r!}\left(\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial q^i} + \frac{\partial X_i^{\mu_1\cdots\mu_r}}{\partial p}\right)\mathrm{d}q^i \wedge \mathrm{d}p \wedge \mathrm{d}^n x_{\mu_1\cdots\mu_r} \end{aligned}$$

$$
\begin{aligned}
&-\frac{(-1)^r}{r!}\Bigg(\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial p_i^\mu}-\sum_{s=1}^{r}(-1)^{s-1}\delta_\mu^{\mu_s}\frac{\partial X^{i,\mu_1\cdots\mu_{s-1}\mu_{s+1}\cdots\mu_r}}{\partial p}\Bigg)\mathrm{d}p_i^\mu\wedge\mathrm{d}p\wedge\mathrm{d}^n x_{\mu_1\cdots\mu_r}\\
&+\frac{(-1)^r}{r!}\Bigg(\delta_i^k\delta_\kappa^\mu\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial x^\mu}-\sum_{s=1}^{r}\delta_i^k\delta_\kappa^{\mu_s}\frac{\partial X^{\mu_1\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^\nu}\\
&-\sum_{s=1}^{r}(-1)^{s-1}\delta_\kappa^{\mu_s}\frac{\partial X^{k,\mu_1\cdots\mu_{s-1}\mu_{s+1}\cdots\mu_r}}{\partial q^i}-\frac{\partial X_i^{\mu_1\cdots\mu_r}}{\partial p_k^\kappa}\Bigg)\mathrm{d}q^i\wedge\mathrm{d}p_k^\kappa\wedge\mathrm{d}^n x_{\mu_1\cdots\mu_r}\\
&-\frac{(-1)^r}{r!}\frac{\partial X_i^{\mu_1\cdots\mu_r}}{\partial q^j}\mathrm{d}q^i\wedge\mathrm{d}q^j\wedge\mathrm{d}^n x_{\mu_1\cdots\mu_r}+\frac{(-1)^{r-1}}{(r-1)!}\frac{\partial X^{l,\mu_2\cdots\mu_r}}{\partial p_k^\kappa}\mathrm{d}p_k^\kappa\wedge\mathrm{d}p_l^\lambda\wedge\mathrm{d}^n x_{\lambda\mu_2\cdots\mu_r}\\
&-\frac{1}{r!}\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial q^j}\mathrm{d}q^i\wedge\mathrm{d}q^j\wedge\mathrm{d}p_i^\mu\wedge\mathrm{d}^n x_{\mu\mu_1\cdots\mu_r}-\frac{1}{r!}\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial p_k^\kappa}\mathrm{d}q^l\wedge\mathrm{d}p_k^\kappa\wedge\mathrm{d}p_l^\lambda\wedge\mathrm{d}^n x_{\lambda\mu_1\cdots\mu_r}\\
&+\frac{1}{r!}\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial p}\mathrm{d}q^i\wedge\mathrm{d}p_i^\mu\wedge\mathrm{d}p\wedge\mathrm{d}^n x_{\mu\mu_1\cdots\mu_r}.
\end{aligned}
$$

(Note that the last three terms would have to be omitted if $r=n$.)

Let us number the terms in this equation from 1 to 12. As we shall see, each of these terms must vanish separately.

Term No. 12: After contraction with a suitably chosen $(n-r+2)$-multivector field, we see that $X^{\kappa_1\cdots\kappa_r}$ cannot depend on p.

Term No. 11: Given indices i, μ and mutually different indices $\kappa_1,\ldots,\kappa_r$, we choose indices j and $\nu\notin\{\kappa_1,\ldots,\kappa_r\}$ (here we use the hypothesis that $r<n$) such that either $j\neq i$ or $\nu\neq\mu$ and, when $r<n-1$, a complementary set of indices $\nu_1,\ldots,\nu_{n-r-1}$ to contract this term with the multivector field $\partial_j\wedge\partial_\mu^i\wedge\partial_\nu^j\wedge\partial_{\nu_1}\wedge\cdots\wedge\partial_{\nu_{n-r-1}}$ (no sum over j), concluding that $X^{\kappa_1\cdots\kappa_r}$ cannot depend on p_i^μ. Obviously, there is one case where this argument does not work, namely when $N=1$, $r=n-1$, and $\mu\notin\{\kappa_1,\ldots,\kappa_r\}$. This situation will however be covered in the next item.

Term No. 6: Given indices k, κ and mutually different indices $\kappa_1,\ldots,\kappa_r$ such that $\kappa\notin\{\kappa_1,\ldots,\kappa_r\}$, we choose a complementary set of indices $\nu_1,\ldots,\nu_{n-r-1}$ to contract this term with the multivector field $\partial_\kappa^k\wedge\partial_0\wedge\partial_\kappa\wedge\partial_{\nu_1}\wedge\cdots\wedge\partial_{\nu_{n-r-1}}$, concluding that $X^{\kappa_1\cdots\kappa_r}$ cannot depend on p_k^κ, since in this case the second term in the bracket gives no contribution. In particular, this settles the remaining case of the previous item.

Term No. 10: After contraction with a suitably chosen $(n-r+2)$ - multivector field, we see that $X^{\kappa_1\cdots\kappa_r}$ cannot depend on q^l if $N>1$. For $N=1$, the whole term vanishes identically, and no conclusion can be drawn.

This proves the statements in item (1) of the theorem. Moreover, it allows to simplify *term No. 6*, as follows:

$$
-\frac{(-1)^{r-1}}{(r-1)!}\frac{\partial X^{i,\mu_2\cdots\mu_r}}{\partial p}\mathrm{d}p_i^\mu\wedge\mathrm{d}p\wedge\mathrm{d}^n x_{\mu\mu_2\cdots\mu_r}.
$$

As before, contraction with a suitably chosen $(n-r+2)$-multivector field shows that $X^{k,\kappa_2\cdots\kappa_r}$ cannot depend on p.

Next we analyze *term No. 9*, which will give an important restriction on the coefficients $X^{i,\mu_2\cdots\mu_r}$. Given indices i,j,μ,ν and mutually different indices $\kappa_2,\ldots,\kappa_r$, we choose a set of indices $\nu_1,\ldots,\nu_{n-r}$ such that $\{\kappa_2,\ldots,\kappa_r\}\cap\{\nu_1,\ldots,\nu_{n-r}\}=\varnothing$ to contract this term with the multivector field $\partial_\mu^i\wedge\partial_\nu^j\wedge\partial_{\nu_1}\wedge\cdots\wedge\partial_{\nu_{n-r}}$, obtaining

$$\frac{\partial X^{j,\mu_2\cdots\mu_r}}{\partial p_i^{\mu}}\epsilon_{\nu\mu_2\cdots\mu_r\nu_1\cdots\nu_{n-r}} = \frac{\partial X^{i,\mu_2\cdots\mu_r}}{\partial p_j^{\nu}}\epsilon_{\mu\mu_2\cdots\mu_r\nu_1\cdots\nu_{n-r}}. \tag{34}$$

Now assume the index ν to be chosen such that $\nu \notin \{\kappa_2,\ldots,\kappa_r,\nu_1,\ldots,\nu_{n-r}\}$. Then if $\mu \notin \{\kappa_2,\ldots,\kappa_r\}$, we can take $\mu=\nu_1$, say, to conclude that $X^{j,\kappa_2\cdots\kappa_r}$ cannot depend on p_i^{μ},

$$\frac{\partial X^{j,\mu_2\cdots\mu_r}}{\partial p_i^{\mu}} = 0 \quad \text{if } \mu \notin \{\mu_2,\ldots,\mu_r\}. \tag{35}$$

Moreover, if $\mu \in \{\kappa_2,\ldots,\kappa_r\}$, this result implies that applying an operator $\partial^{i'}_{\mu}$ (with arbitrary i') to Eq. (34) gives zero since on the right-hand side (rhs), the ϵ-tensor kills all terms in the sum over the indices $\mu_2,\ldots,\mu_r$ in which the index μ appears among them,

$$\frac{\partial^2 X^{j,\mu_2\cdots\mu_r}}{\partial p_{i_1}^{\mu}\partial p_{i_2}^{\mu}} = 0 \quad \text{if } \mu \in \{\mu_2,\ldots,\mu_r\} \quad (\text{no sum over } \mu). \tag{36}$$

The general solution to Eqs. (35) and (36) can be written in the form

$$X^{j,\mu_2\cdots\mu_r} = \sum_{s=1}^{r}\frac{1}{(s-1)!}\frac{1}{(r-s)!}\sum_{\pi\in S_{r-1}}(-1)^{\pi}p_{j_2}^{\mu_{\pi(2)}}\cdots p_{j_s}^{\mu_{\pi(s)}}Y_{s-1}^{j,j_2\cdots j_s,\mu_{\pi(s+1)}\cdots\mu_{\pi(r)}},$$

where S_{r-1} denotes the permutation group of $\{2,\ldots,r\}$ and the newly introduced coefficients $Y_{s-1}^{i,j_2\cdots j_s,\mu_{s+1}\cdots\mu_r}$ are local functions on E, they do not depend on the multimomentum variables p_k^{κ} or the energy variable p and are totally antisymmetric both in $j_2,\ldots,j_s$ and in $\mu_{s+1},\ldots,\mu_r$. Differentiating this expression with respect to p_i^{μ} with $\mu=\mu_2$ gives

$$\frac{\partial X^{j,\mu\mu_3\cdots\mu_r}}{\partial p_i^{\mu}}\epsilon_{\nu\mu\mu_3\cdots\mu_r\nu_1\cdots\nu_{n-r}} \quad (\text{no sum over } \mu)$$
$$= \sum_{s=2}^{r}\frac{1}{(s-2)!}\frac{1}{(r-s)!}\sum_{\pi\in S_{r-2}}(-1)^{\pi}p_{j_3}^{\mu_{\pi(3)}}\cdots p_{j_s}^{\mu_{\pi(s)}}Y_{s-1}^{j,ij_3\cdots j_s,\mu_{\pi(s+1)}\cdots\mu_{\pi(r)}}\epsilon_{\nu\mu\mu_3\cdots\mu_r\nu_1\cdots\nu_{n-r}},$$

where S_{r-2} denotes the permutation group of $\{3,\ldots,r\}$, which shows that Eq. (34) will hold provided that

$$Y_{s-1}^{j,ij_3\cdots j_s,\mu_{\pi(s+1)}\cdots\mu_{\pi(r)}} = -\,Y_{s-1}^{i,jj_3\cdots j_s,\mu_{\pi(s+1)}\cdots\mu_{\pi(r)}}.$$

This proves the statements in item (2) of the theorem.

We proceed with *terms Nos. 4 and 5* which imply

$$\frac{\partial \tilde{X}^{\mu_2\cdots\mu_r}}{\partial p} = (-1)^r\frac{\partial X^{\mu_2\cdots\mu_r\nu}}{\partial x^{\nu}}, \qquad \frac{\partial X_i^{\mu_1\cdots\mu_r}}{\partial p} = -\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial q^i}. \tag{37}$$

We observe first of all that the rhs of both equations does not depend on the energy variable, so they can be immediately integrated with respect to p.

From *term No. 3* we infer

$$\frac{\partial \tilde{X}^{\mu_2\cdots\mu_r}}{\partial p_i^{\mu}} = -\frac{\partial X^{i,\mu_2\cdots\mu_r}}{\partial x^{\mu}} + \sum_{s=2}^{r}\delta_{\mu}^{\mu_s}\frac{\partial X^{i,\mu_2\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^{\nu}}. \tag{38}$$

An explicit calculation shows that the rhs of this equation does not depend on the p_j^{μ}, not only when $\mu \notin \{\mu_2,\ldots,\mu_r\}$ but even when $\mu \in \{\mu_2,\ldots,\mu_r\}$. (Of course, it also does not depend on p.) Thus, according to Lemma A.2 formulated in the appendix, we can integrate Eq. (38) explicitly to obtain (recall that Σ^{-1} is the operator that acts on polynomials in the multimomentum variables and the energy variable without constant term by multiplying the homogeneous component of degree s by $1/s$)

$$\tilde{X}^{\mu_2\cdots\mu_r} = (-1)^r p \frac{\partial X^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu} - \Sigma^{-1}\left(p_i^\mu \frac{\partial X^{i,\mu_2\cdots\mu_r}}{\partial x^\mu} - \sum_{s=2}^{r} p_i^{\mu_s} \frac{\partial X^{i,\mu_2\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^\nu} \right) + \tilde{Y}^{\mu_2\cdots\mu_r}, \tag{39}$$

where the $\tilde{Y}^{\mu_2\cdots\mu_r}$ are local functions on E: they do not depend on the multimomentum variables or on the energy variable.

The same procedure works for *term No. 7*. There, we are left with

$$\frac{\partial X_i^{\mu_1\cdots\mu_r}}{\partial p_k^\kappa} = \delta_i^k \delta_\kappa^\mu \frac{\partial X^{\mu_1\cdots\mu_r}}{\partial x^\mu} - \sum_{s=1}^{r} \delta_i^k \delta_\kappa^{\mu_s} \frac{\partial X^{\mu_1\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^\nu} - \sum_{s=1}^{r} (-1)^{s-1} \delta_\kappa^{\mu_s} \frac{\partial X^{k,\mu_1\cdots\mu_{s-1}\mu_{s+1}\cdots\mu_r}}{\partial q^i}. \tag{40}$$

Using the same argument as before, we show that the rhs does not depend on the p_l^κ, not only when $\kappa \notin \{\mu_1, \ldots, \mu_r\}$ but even when $\kappa \in \{\mu_1, \ldots, \mu_r\}$, and neither does it depend on p. Therefore, we can integrate Eq. (40) explicitly to obtain

$$\begin{aligned} X_i^{\mu_1\cdots\mu_r} = {} & -p \frac{\partial X^{\mu_1\cdots\mu_r}}{\partial q^i} + p_i^\mu \frac{\partial X^{\mu_1\cdots\mu_r}}{\partial x^\mu} - \sum_{s=1}^{r} p_i^{\mu_s} \frac{\partial X^{\mu_1\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^\nu} \\ & - \Sigma^{-1}\left(\sum_{s=1}^{r} (-1)^{s-1} p_k^{\mu_s} \frac{\partial X^{k,\mu_1\cdots\mu_{s-1}\mu_{s+1}\cdots\mu_r}}{\partial q^i} \right) + Y_i^{\mu_1\cdots\mu_r}, \end{aligned} \tag{41}$$

where the $Y_i^{\mu_1\cdots\mu_r}$ are local functions on E: they do not depend on the multimomentum variables or on the energy variable.

Finally, we turn to *terms Nos. 1, 2, and 8*. They imply

$$\frac{\partial \tilde{X}^{\mu_3\cdots\mu_r\nu}}{\partial x^\nu} = 0, \quad \frac{\partial \tilde{X}^{\mu_2\cdots\mu_r}}{\partial q^i} = (-1)^{r-1} \frac{\partial X_i^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu}, \quad \frac{\partial X_i^{\mu_1\cdots\mu_r}}{\partial q^j} = \frac{\partial X_j^{\mu_1\cdots\mu_r}}{\partial q^i},$$

respectively. With the help of (39), these reduce to

$$\frac{\partial \tilde{Y}^{\mu_3\cdots\mu_r\nu}}{\partial x^\nu} = 0, \quad \frac{\partial \tilde{Y}^{\mu_2\cdots\mu_r}}{\partial q^i} = (-1)^{r-1} \frac{\partial Y_i^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu}, \quad \frac{\partial Y_i^{\mu_1\cdots\mu_r}}{\partial q^j} = \frac{\partial Y_j^{\mu_1\cdots\mu_r}}{\partial q^i},$$

which is easily solved by setting

$$\tilde{Y}^{\mu_2\cdots\mu_r} = (-1)^{r-1} \frac{\partial X_{-}^{\mu_2\cdots\mu_r\mu}}{\partial x^\mu}, \quad Y_i^{\mu_1\cdots\mu_r} = \frac{\partial X_{-}^{\mu_1\cdots\mu_r}}{\partial q^i}. \tag{42}$$

Here, the $X_{-}^{\mu_1\cdots\mu_r}$ are local functions on E: they do not depend on the multimomentum variables or on the energy variable. This completes the proof of the statements in item (3) of the theorem.

All that remains to be shown are the final statements concerning exact Hamiltonian multivector fields. To this end, we calculate

$$
\begin{aligned}
L_X\theta = {} & \frac{1}{(r-1)!}\left(\frac{\partial X^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu}p - (-1)^r\frac{\partial X^{i,\mu_2\cdots\mu_r}}{\partial x^\mu}p_i^\mu\right. \\
& \left. + (-1)^r\sum_{s=2}^{r}\frac{\partial X^{i,\mu_2\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^\nu}p_i^{\mu_s} - (-1)^r\widetilde{X}^{\mu_2\cdots\mu_r}\right)\mathrm{d}^n x_{\mu_2\cdots\mu_r} \\
& - \frac{1}{r!}\left(\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial x^\mu}p_i^\mu - \sum_{s=1}^{r}\frac{\partial X^{\mu_1\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^\nu}p_i^{\mu_s} - \frac{\partial X^{\mu_1\cdots\mu_r}}{\partial q^i}p\right. \\
& \left. - \sum_{s=1}^{r}(-1)^{s-1}\frac{\partial X^{j,\mu_1\cdots\mu_{s-1}\mu_{s+1}\cdots\mu_r}}{\partial q^i}p_j^{\mu_s} - X_i^{\mu_1\cdots\mu_r}\right)\mathrm{d}q^i\wedge\mathrm{d}^n x_{\mu_1\cdots\mu_r} \\
& + \frac{1}{r!}\left(\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial p_j^\nu}p + \sum_{s=1}^{r}(-1)^{s-1}\frac{\partial X^{i,\mu_1\cdots\mu_{s-1}\mu_{s+1}\cdots\mu_r}}{\partial p_j^\nu}p_i^{\mu_s}\right)\mathrm{d}p_j^\nu\wedge\mathrm{d}^n x_{\mu_1\cdots\mu_r} \\
& + \frac{1}{r!}\left(\frac{\partial X^{\mu_1\cdots\mu_r}}{\partial p}p + \sum_{s=1}^{r}(-1)^{s-1}\frac{\partial X^{i,\mu_1\cdots\mu_{s-1}\mu_{s+1}\cdots\mu_r}}{\partial p}p_i^{\mu_s}\right)\mathrm{d}p\wedge\mathrm{d}^n x_{\mu_1\cdots\mu_r},
\end{aligned}
$$

where we have omitted four terms that vanish because X is locally Hamiltonian. Moreover, using the expressions derived above for locally Hamiltonian multivector fields, we see that the other terms vanish as well if and only if we have

$$
(\Sigma^{-1}-1)\left(\sum_{s=1}^{r}(-1)^{s-1}p_j^{\mu_s}\frac{\partial X^{j,\mu_1\cdots\mu_{s-1}\mu_{s+1}\cdots\mu_r}}{\partial q^i}\right)=0,
$$

$$
(\Sigma^{-1}-1)\left(p_i^\mu\frac{\partial X^{i,\mu_2\cdots\mu_r}}{\partial x^\mu} - \sum_{s=2}^{r}p_i^{\mu_s}\frac{\partial X^{i,\mu_2\cdots\mu_{s-1}\nu\mu_{s+1}\cdots\mu_r}}{\partial x^\nu}\right)=0,
$$

and

$$
\frac{\partial X_-^{\mu_1\cdots\mu_r}}{\partial q^i}=0, \qquad \frac{\partial X_-^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu}=0.
$$

But this means that the coefficients of the multimomentum variables in the above expressions must be independent of the multimomentum variables and that the coefficients $X_-^{\mu_1\cdots\mu_r}$ can without loss of generality be assumed to vanish, which completes the proof of the theorem. □

Proof of Theorem 1.3, part 1: The statements of Theorem 1.3 about multivector fields are, in their local form, based on the local decomposition given in Proposition 2.1, taking into account the scaling behavior of the coefficient functions that follows from Theorem 2.4, together with that of the coordinate vector fields. The global version of these statements can be obtained by glueing together such local decompositions using appropriate partitions of unity. To see that the homogeneous components X_s, $s=-1,\ldots,r-1$, of a fiberwise polynomial locally Hamiltonian r-multivector field X are locally Hamiltonian, we compute

$$
0=(L_\Sigma)^k\,\mathrm{d}i_X\,\omega=\sum_{s=0}^{r-1}(s+1)^k\,\mathrm{d}i_{X_s}\,\omega, \quad k=1,\ldots,r-1.
$$

Together with $\mathrm{d}i_X\omega=0$, this leads to a Vandermonde matrix equation with entries $0,2,3,\ldots,r-1$ annihilating the vector $(\mathrm{d}i_{X_{-1}}\omega,\ldots,\mathrm{d}i_{X_{r-1}}\omega)^T$. As the determinant of a Vandermonde matrix does not vanish, the above vector must vanish. □

The following proposition clarifies the interpretation of homogeneous locally Hamiltonian multivector fields.

Proposition 2.5: Let X be a locally Hamiltonian r-multivector field on P. Then

(1) *X is exact Hamiltonian iff* $[\Sigma, X]$ *takes values in the kernel of* ω.

(2) *If* $[\Sigma, X] - sX$ *takes values in the kernel of* ω, *for some integer s between 0 and* $r-1$, *then X is globally Hamiltonian with associated Poisson form*

$$\frac{(-1)^{r-1}}{s+1} i_X \theta.$$

(3) *If* $[\Sigma, X] + X$ *takes values in the kernel of* ω, *then* $i_X \theta = 0$.

Proof: The first statement follows immediately from Eq. (25). Similarly, the second claim can be proved by multiplying Eq. (25) by $(-1)^{r-1}/(s+1)$ and combining it with Eq. (1) and Eq. (4) to give

$$\mathrm{d}\left(\frac{(-1)^{r-1}}{s+1} i_X \theta\right) = \frac{(-1)^{r-1}}{s+1} L_X \theta + \frac{1}{s+1} i_X \omega = \frac{1}{s+1} i_{[\Sigma,X]+X} \omega,$$

which equals $i_X \omega$ since, by hypothesis, $i_{[\Sigma,X]-sX} \omega = 0$. Finally, the third statement follows by observing that the kernel of ω is contained in the kernel of θ and hence according to the hypothesis made,

$$0 = i_{[\Sigma,X]+X} \theta = L_\Sigma i_X \theta - i_X L_\Sigma \theta + i_X \theta = L_\Sigma i_X \theta,$$

where we have used the invariance of θ under Σ. Therefore, according to Proposition A.1, $i_X \theta$ is the pull-back to P of an n-form on E via the projection that defines P as a vector bundle over E, which in turn can be obtained as the pull-back to E of $i_X \theta$ via the zero section of P over E. But this pull-back is zero, since θ vanishes along the zero section of P over E. □

It may be instructive to spell all this out more explicitly for locally Hamiltonian vector fields ($r=1$).

We begin by writing down the general form of a locally Hamiltonian vector field X: in adapted local coordinates, it has the representation

$$X = X^\mu \frac{\partial}{\partial x^\mu} + X^i \frac{\partial}{\partial q^i} + X_i^\mu \frac{\partial}{\partial p_i^\mu} + \tilde{X} \frac{\partial}{\partial p},$$

where according to Theorem 2.4, the coefficient functions X^μ and X^i depend only on the local coordinates x^ρ for M and on the local fiber coordinates q^r for E (the X^μ being independent of the latter as soon as $N > 1$), whereas the coefficient functions X_i^μ and $\tilde{X}$ are explicitly given by

$$X_i^\mu = -p \frac{\partial X^\mu}{\partial q^i} + p_i^\nu \frac{\partial X^\mu}{\partial x^\nu} - p_i^\mu \frac{\partial X^\nu}{\partial x^\nu} - p_j^\mu \frac{\partial X^j}{\partial q^i} + \frac{\partial X_-^\mu}{\partial q^i} \tag{43}$$

(the first term being absent as soon as $N > 1$) and

$$\tilde{X} = -p \frac{\partial X^\nu}{\partial x^\nu} - p_i^\mu \frac{\partial X^i}{\partial x^\mu} + \frac{\partial X_-^\nu}{\partial x^\nu} \tag{44}$$

with coefficient functions X_-^μ that once again depend only on the local coordinates x^ρ for M and on the local fiber coordinates q^r for E. Regarding the decomposition (19), the situation here is particularly interesting and somewhat special since ω is nondegenerate on vector fields, so there are no nontrivial vector fields taking values in the kernel of ω and hence the decomposition (19) can be improved.

Corollary 2.6: Any locally Hamiltonian vector field X on P can be uniquely decomposed into the sum of two terms,

$$X = X_- + X_+, \tag{45}$$

where

- *(i)* X_- *has scaling degree* -1, *i.e.,* $[\Sigma, X_-] = -X_-$, *and is vertical with respect to the projection onto E.*
- *(ii)* X_+ *has scaling degree 0, i.e.,* $[\Sigma, X_+] = 0$, *is exact Hamiltonian, is projectable onto E and coincides with the canonical lift of its projection onto E.*

Proof: In adapted local coordinates, the two contributions to X are, according to Eqs. (43) and (44), given by

$$X_- = \frac{\partial X_-^\mu}{\partial q^i} \frac{\partial}{\partial p_i^\mu} + \frac{\partial X_-^\nu}{\partial x^\nu} \frac{\partial}{\partial p}$$

and

$$X_+ = X^\mu \frac{\partial}{\partial x^\mu} + X^i \frac{\partial}{\partial q^i} - \left(\frac{\partial X^j}{\partial q^i} p_j^\mu - \frac{\partial X^\mu}{\partial x^\nu} p_i^\nu + \frac{\partial X^\nu}{\partial x^\nu} p_i^\mu + \frac{\partial X^\mu}{\partial q^i} p \right) \frac{\partial}{\partial p_i^\mu} - \left(\frac{\partial X^i}{\partial x^\mu} p_i^\mu + \frac{\partial X^\nu}{\partial x^\nu} p \right) \frac{\partial}{\partial p}.$$

Thus all statements of the corollary follow from what has already been shown, except for the very last one, which is based on the following remark. □

Remark: Every bundle automorphism of E (as a fiber bundle over M) admits a canonical lift to a bundle automorphism of its first order jet bundle JE (as an affine bundle over E) and, by appropriate (twisted affine) dualization, to the extended multiphase space P (as a vector bundle over E). Similarly, passing to generators of one-parameter groups, one sees that every vector field X_E on E that is projectable to a vector field X_M on M admits a canonical lift to a vector field X_{JE} on JE and, by appropriate (twisted affine) dualization, to a vector field X_P on P. (See, for example, Ref. 7, Sec. 4B.) When $N=1$, lifting to P is even possible for arbitrary diffeomorphisms of E and arbitrary vector fields on E, since in this case P can be identified with the nth exterior power of the cotangent bundle of E. Explicitly, in terms of adapted local coordinates (x^μ, q^i, p_i^μ, p), we may write

$$X_M = X^\mu \frac{\partial}{\partial x^\mu} \quad \text{and } X_E = X^\mu \frac{\partial}{\partial x^\mu} + X^i \frac{\partial}{\partial q^i},$$

where, except for $N=1$, the X^μ do not depend on the q^r; then the coordinate expression for the lifted vector field, X_P, is precisely given by the expression for X_+ above. Obviously, X_P has scaling degree 0 and hence is not only locally but even exact Hamiltonian. Conversely, starting with an exact Hamiltonian vector field X_+, we can obtain X_M and X_E by projection onto M and E, respectively. Thus, the coordinate expression for X_+ shows that *precisely all exact Hamiltonian vector fields are obtained by this lifting procedure*. Similarly, one can show that all diffeomorphisms of P that preserve the multicanonical form θ are obtained by lifting of automorphisms or, for $N=1$, diffeomorphisms of E, this is the field theoretical analog of a well-known theorem in geometric mechanics, according to which all diffeomorphisms of a cotangent bundle that preserve the canonical form θ are induced by diffeomorphisms of its base manifold.

To conclude this section, let us note that the definition of projectability of vector fields can be immediately generalized to multivector fields: an r-multivector field X_E on the total space E of a fiber bundle over a manifold M with bundle projection $\pi: E \to M$ is called *projectable* if for any two points e_1 and e_2 in E,

$$\Lambda^r T_{e_1}\pi \cdot X_E(e_1) = \Lambda^r T_{e_2}\pi \cdot X_E(e_2) \quad \text{if } \pi(e_1) = \pi(e_2),$$

or in other words, if there exists an r-multivector field X_M on M such that

$$\Lambda^r T\pi \circ X_E = X_M \circ \pi.$$

In adapted local coordinates, this amounts to requiring that if we write

$$X_E = \frac{1}{r!} X^{\mu_1 \cdots \mu_r} \frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} + \cdots ,$$

where the dots denote 1-vertical terms, the coefficients $X^{\mu_1 \cdots \mu_r}$ should depend only on the local coordinates x^ρ for M but not on the local fiber coordinates q^r for E. Now we introduce the following terminology.

Definition 2.7: An r-multivector field on P is called projectable if it is projectable with respect to any one of the three projections from P: to P_0, to E, and to M.

With this terminology, Theorem 2.4 states that for $0 < r < n$, locally Hamiltonian r-multivector fields on P are projectable as soon as $N > 1$ and are projectable to E but not necessarily to P_0 or to M when $N=1$. [Inspection of Eq. (32) shows, however, that they are projectable to P_0 if and only if they are projectable to M.]

Considering the special case of vector fields ($r=1$), we believe that vector fields on the total space of a fiber bundle over space-time which are not projectable should be regarded as pathological, since they generate transformations which do not induce transformations of space-time. It is hard to see how such transformations might be interpreted as candidates for symmetries of a physical system. By analogy, we shall adopt the same point of view regarding multivector fields of higher degree, since although these do not generate diffeomorphisms of E as a manifold, they may perhaps allow for an interpretation as generators of superdiffeomorphisms of an appropriate supermanifold built over E as its even part.

III. POISSON FORMS AND HAMILTONIAN FORMS

Our aim in this section is to give an explicit construction of Poisson $(n-r)$-forms and, more generally, of Hamiltonian $(n-r)$-forms on the extended multiphase space P, where $0 \leqslant r \leqslant n$. [Note that Eq. (11) only makes sense for r in this range.] A special role is played by closed forms, since closed forms are always Hamiltonian and closed forms that vanish on the kernel of ω are always Poisson, these are in a sense the trivial examples. In other words, the main task is to understand the extent to which general Hamiltonian forms deviate from closed forms and general Poisson forms deviate from closed forms that vanish on the kernel of ω.

As a warm-up exercise, we shall settle the extreme cases of tensor degree 0 and n. The case $r=n$ has already been analyzed in Ref. 2, so we just quote the result.

Proposition 3.1: A function f on P, regarded as a 0-form, is always Hamiltonian and even Poisson. Moreover, its associated Hamiltonian n-multivector field X is, in adapted local coordinates and modulo terms taking values in the kernel of ω, given by Eq. (29).

The case $r=0$ is equally easy.

Proposition 3.2: An n-form f on P is Hamiltonian or Poisson if and only if it can be written as the sum of a constant multiple of θ with a closed form which is arbitrary if f is Hamiltonian and vanishes on the kernel of ω if f is Poisson.

Indeed, if f is a Hamiltonian n-form, the multivector field X that appears in Eq. (11) will in fact be a function which must be locally Hamiltonian and hence, by Proposition 2.2, constant. Thus $\mathrm{d}f$ must be proportional to ω and so f must be the sum of some constant multiple of θ and a closed form.

The intermediate cases $(0 < r < n)$ are much more interesting. To handle them, the first step is to identify the content of the kernel condition (13) in adapted local coordinates (for completeness, we also include the two extreme cases).

Proposition 3.3: An $(n-r)$-form f on P, with $0 \leqslant r \leqslant n$, vanishes on the kernel of ω if and only if, in adapted local coordinates, it can be written in the form

$$\begin{aligned} f = {} & \frac{1}{r!} f^{\mu_1 \cdots \mu_r} \mathrm{d}^n x_{\mu_1 \cdots \mu_r} + \frac{1}{(r+1)!} f_i^{\mu_0 \cdots \mu_r} \, \mathrm{d}q^i \wedge \mathrm{d}^n x_{\mu_0 \cdots \mu_r} + \frac{1}{r!} f^{i,\mu_1 \cdots \mu_r} \, \mathrm{d}p_i^\mu \wedge \mathrm{d}^n x_{\mu \mu_1 \cdots \mu_r} \\ & + \frac{1}{(r+1)!} f'^{\mu_0 \cdots \mu_r} (\mathrm{d}p \wedge \mathrm{d}^n x_{\mu_0 \cdots \mu_r} - \mathrm{d}q^i \wedge \mathrm{d}p_i^\mu \wedge \mathrm{d}^n x_{\mu_0 \cdots \mu_r \mu}), \end{aligned} \tag{46}$$

where the second term in the last parentheses is to be omitted if $r = n-1$ *whereas only the first term remains if* $r = n$.

Note that for one-forms (just as for functions), the kernel condition (13) is void, since ω is nondegenerate. Also, it is in this case usually more convenient to replace Eq. (46) by the standard local coordinate representation

$$f = f_\mu \,\mathrm{d}x^\mu + f_i \,\mathrm{d}q^i + f^i_\mu \,\mathrm{d}p^\mu_i + f_0 \,\mathrm{d}p. \tag{47}$$

Proof: From the particular expression for ω in adapted local coordinates, we see first of all that forms of degree $n-r$ vanishing on the kernel of ω must be $(n-r-2)$-horizontal (since they vanish on 3-vertical multivector fields) and that the only term which is not $(n-r-1)$-horizontal is

$$\mathrm{d}q^i \wedge \mathrm{d}p^\kappa_k \wedge \mathrm{d}^n x_{\mu_0 \cdots \mu_r \mu}.$$

Furthermore, f must vanish on the bivectors

$$\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^\kappa_k} + \delta^k_i \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial x^\kappa} \quad \text{and} \quad \frac{\partial}{\partial p^\mu_i} \wedge \frac{\partial}{\partial x^\nu} + \frac{\partial}{\partial p^\nu_i} \wedge \frac{\partial}{\partial x^\mu}$$

which yields the statement of the proposition. For more details, see Ref. 9 □

The proposition above can be used to prove the following interesting and useful fact.

Proposition 3.4: An $(n-r)$*-form* f *on* P, *with* $0 \leqslant r \leqslant n$, *vanishes on the kernel of* ω *if and only if there exists an* $(r+1)$*-multivector field* X *on* P *such that*

$$f = i_X \omega. \tag{48}$$

Then obviously,

$$\mathrm{d}f = L_X \omega. \tag{49}$$

In particular, f *is closed if and only if* X *is locally Hamiltonian*.

At every point of P, the statement that the inclusion of the kernel of ω in the kernel of f implies that there is a multivector Y such that $i_Y \omega = f$ at this point, can be shown without reference to the particular form of ω.[10] However, the expression for ω in adapted local coordinates shows that we can even obtain a multivector field Y with this property.

Proof: The "if" part being obvious, observe that it suffices to prove the "only if" part locally, in the domain of definition of an arbitrary system of adapted local coordinates, by constructing the coefficients of X from those of f. [Indeed, since the relation between f and X postulated in Eq. (48) is purely algebraic, i.e., it does not involve derivatives, we can construct a global solution patching together local solutions with a partition of unity.] A comparison of $i_X \omega$, where X is an $(r+1)$-multivector field [!] given by Eq. (26), with (46) shows that when $r < n$, this can be achieved by setting

$$X^{\mu_0 \cdots \mu_r} = (-1)^r f'^{\,\mu_0 \cdots \mu_r}, \quad X^{i,\mu_1 \cdots \mu_r} = (-1)^r f^{i,\mu_1 \cdots \mu_r}, \tag{50}$$

$$X^{\mu_0 \cdots \mu_r}_i = (-1)^{r+1} f^{\mu_0 \cdots \mu_r}_i, \quad \tilde{X}^{\mu_1 \cdots \mu_r} = -f^{\mu_1 \cdots \mu_r}, \tag{51}$$

while for $r = n$, only the last equation is pertinent [for $r = n-1$, the same conclusion can also be reached by comparing (28) and (47)]. □

Corollary 3.5: An $(n-r)$*-form* f *on* P, *with* $0 \leqslant r \leqslant n$, *is a Hamiltonian form if and only if* df *vanishes on the kernel of* ω *and is a Poisson form if and only if both* df *and* f *vanish on the kernel of* ω.

With these preliminaries out of the way, we can proceed to the construction of Poisson forms which are not closed. As we shall see, there are two such constructions which, taken together, will be sufficient to handle the general case.

The first construction is a generalization of the universal multimomentum map of Ref. 2, which to each exact Hamiltonian r-multivector field F on P associates a Poisson $(n-r)$-form $J(F)$ on P defined by Eq. (52) below. What remained unnoticed in Ref. 2 is that this construction works even when X is only locally Hamiltonian. In fact, we have the following generalization of Proposition 4.3 of Ref. 2.

Proposition 3.6: For every locally Hamiltonian r-multivector field F on P, with $0 \leqslant r \leqslant n$, *the formula*

$$J(F) = (-1)^{r-1} i_F \theta \tag{52}$$

defines a Poisson $(n-r)$*-form* $J(F)$ *on P whose associated Hamiltonian multivector field is* $F + [\Sigma, F]$, *that is, we have*

$$\mathrm{d}(J(F)) = i_{F+[\Sigma,F]}\omega. \tag{53}$$

Proof: Obviously, $J(F)$ vanishes on the kernel of ω since this is contained in the kernel of θ. Moreover, since $L_F\omega$ is supposed to vanish, we can use the algebraic relations for the Lie derivative along multivector fields and $\theta = -i_\Sigma \omega$ to compute

$$\mathrm{d}(J(F)) = (-1)^{r-1}\mathrm{d}(i_F\theta) = (-1)^{r-1}L_F\theta - i_F\,\mathrm{d}\theta = (-1)^r L_F i_\Sigma\omega + i_\Sigma L_F\omega + i_F\omega = -\,i_{[F,\Sigma]}\omega + i_F\omega.$$

□

The second construction uses differential forms on E, pulled back to differential forms on P via the target projection $\tau: P \to E$. Characterizing which of these are Hamiltonian forms and which are Poisson forms is a simple exercise.

Proposition 3.7: Let f_0 *be an* $(n-r)$*-form on E, with* $0 < r < n$. *Then*

- *(i)* $\tau^* f_0$ *is a Hamiltonian form on P if and only if* $\mathrm{d}f_0$ *is* $(n-r)$*-horizontal.*
- *(ii)* $\tau^* f_0$ *is a Poisson form on P if and only if* f_0 *is* $(n-r-1)$*-horizontal and* $\mathrm{d}f_0$ *is* $(n-r)$*-horizontal.*

Proof: In adapted local coordinates (x^μ, q^i) for E and (x^μ, q^i, p_i^μ, p) for P, we can write

$$f_0 = \frac{1}{r!} f_0^{\mu_1\cdots\mu_r}\mathrm{d}^n x_{\mu_1\cdots\mu_r} + \frac{1}{(r+1)!}(f_0)_i^{\mu_0\cdots\mu_r}\,\mathrm{d}q^i \wedge \mathrm{d}^n x_{\mu_0\cdots\mu_r} + \cdots\ , \tag{54}$$

where the dots denote higher order terms containing at least two $\mathrm{d}q$'s. Now applying Proposition 3.3 to $\tau^* f_0$, we see that $\tau^* f_0$ will vanish on the kernel of ω if and only if the terms denoted by the dots all vanish, i.e., if f_0 can be written in the form

$$f_0 = \frac{1}{r!} f_0^{\mu_1\cdots\mu_r}\mathrm{d}^n x_{\mu_1\cdots\mu_r} + \frac{1}{(r+1)!}(f_0)_i^{\mu_0\cdots\mu_r}\,\mathrm{d}q^i \wedge \mathrm{d}^n x_{\mu_0\cdots\mu_r}. \tag{55}$$

But this is precisely the condition for the $(n-r)$-form f_0 to be $(n-r-1)$-horizontal. (Note that this equivalence holds even when $r = n-1$, provided we understand the condition of being 0-horizontal to be empty.) Similarly, since Proposition 3.4 implies that a form on P is Hamiltonian if and only if its exterior derivative vanishes on the kernel of ω, the same argument applied to $\mathrm{d}(\tau^* f_0) = \tau^*\,\mathrm{d}f_0$ shows that, irrespectively of whether $\tau^* f_0$ itself vanishes on the kernel of ω or not and hence whether we use Eq. (54) or Eq. (55) as our starting point, $\tau^* f_0$ will be Hamiltonian if and only if

$$\mathrm{d}f_0 = \frac{1}{(r-1)!}\frac{\partial f_0^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu}\mathrm{d}^n x_{\mu_2\cdots\mu_r} + \frac{1}{r!}\left(\frac{\partial f_0^{\mu_1\cdots\mu_r}}{\partial q^i} - \frac{\partial (f_0)_i^{\mu_1\cdots\mu_r\nu}}{\partial x^\nu}\right)\mathrm{d}q^i \wedge \mathrm{d}^n x_{\mu_1\cdots\mu_r}.$$

But this is precisely the condition for the $(n-r+1)$-form $\mathrm{d}f_0$ to be $(n-r)$-horizontal. Moreover, it is easy to write down an associated Hamiltonian r-multivector field X_0,

$$X_0 = \frac{(-1)^r}{r!}\left(\frac{\partial f_0^{\mu_1\cdots\mu_r}}{\partial q^i} - \frac{\partial (f_0)_i^{\mu_1\cdots\mu_r\nu}}{\partial x^\nu}\right)\frac{\partial}{\partial p_i^{\mu_1}} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}}$$

$$-\frac{1}{(r-1)!}\frac{\partial f_0^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu}\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}}.$$

□

Note also that if f_0 is $(n-r-1)$-horizontal and thus has the form stated in Eq. (55), $\mathrm{d}f_0$ would contain just one additional higher order term, namely

$$\frac{1}{(r+1)!}\frac{\partial (f_0)_j^{\mu_0\cdots\mu_r}}{\partial q^i}\mathrm{d}q^i \wedge \mathrm{d}q^j \wedge \mathrm{d}^n x_{\mu_0\cdots\mu_r}.$$

Its absence means that

$$\frac{\partial (f_0)_j^{\mu_0\cdots\mu_r}}{\partial q^i} = \frac{\partial (f_0)_i^{\mu_0\cdots\mu_r}}{\partial q^j},$$

so there exist local functions $f_0^{\mu_0\cdots\mu_r}$ on E such that

$$(f_0)_i^{\mu_0\cdots\mu_r} = \frac{\partial f_0^{\mu_0\cdots\mu_r}}{\partial q^i}.$$

This implies that f_0 can be written as the sum

$$f_0 = f_h + f_c \tag{56}$$

of a horizontal form f_h and a closed form f_c, defined by setting

$$f_h = \frac{1}{r!}\left(f_0^{\mu_1\cdots\mu_r} - \frac{\partial f_0^{\mu_1\cdots\mu_r\nu}}{\partial x^\nu}\right)\mathrm{d}^n x_{\mu_1\cdots\mu_r}$$

and

$$f_c = \frac{1}{r!}\frac{\partial f_0^{\mu_1\cdots\mu_r\nu}}{\partial x^\nu}\mathrm{d}^n x_{\mu_1\cdots\mu_r} + \frac{1}{(r+1)!}\frac{\partial f_0^{\mu_0\cdots\mu_r}}{\partial q^i}\mathrm{d}q^i \wedge \mathrm{d}^n x_{\mu_0\cdots\mu_r}.$$

The same kind of local decomposition into the sum of a horizontal form and a closed form can also be derived if f_0 is arbitrary and thus has the form stated in Eq. (54); this case can be handled by decreasing induction on the number of $\mathrm{d}q$'s that appear in the higher order terms denoted by the dots in Eq. (54). We shall refrain from working this out in detail, since unfortunately the decomposition (56) depends on the system of adapted local coordinates used in its construction: under coordinate transformations, the terms f_h and f_c mix. Therefore, this decomposition has no coordinate independent meaning and is in general valid only locally.

Finally, we note that in the above discussion, we have deliberately excluded the extreme cases $r=0$ (n-forms) and $r=n$ (functions). For n-forms, the equivalences stated above would be incorrect since if f_0 has tensor degree n and hence X_0 has tensor degree 0, $i_{X_0}\omega$ would by Proposition 2.2 be a constant multiple of ω whereas $\mathrm{d}(\tau^* f_0)$ would be reduced to a linear combination of terms of the form $\mathrm{d}q^i \wedge \mathrm{d}^n x$, implying that $\tau^* f_0$ can only be Hamiltonian if it is closed. For functions, the construction is uninteresting since according to Proposition 3.1, all functions on P are Poisson, and not just the ones lifted from E.

Now we are ready to state our main decomposition theorem. (In what follows, we shall simply write f_0 instead of $\tau^* f_0$ when there is no danger of confusion, the main exception being the proof of Theorem 3.8 below.)

Theorem 3.8: *Any Hamiltonian* $(n-r)$*-form and, in particular, any Poisson* $(n-r)$*-form* f *on* P, *with* $0<r<n$, *admits a unique decomposition*

$$f = f_0 + f_+ + f_c \quad \textit{with } f_+ = \sum_{s=1}^{r} f_s, \tag{57}$$

where

(1) f_0 *is (the pull-back to* P *of) an* $(n-r)$*-form on* E *whose exterior derivative is* $(n-r)$*-horizontal and which is otherwise arbitrary iff* f *is Hamiltonian whereas it is restricted to be* $(n-r-1)$*-horizontal iff* f *is Poisson.*

(2) f_+ *is of the form*

$$f_+ = J(F) = (-1)^{r-1} i_F \theta \quad \textit{with } F = (1 + L_\Sigma)^{-1} X_+, \tag{58}$$

and correspondingly, for $s = 1, \ldots, r$, f_s *is of the form*

$$f_s = \frac{(-1)^{r-1}}{s} i_{X_{s-1}} \theta, \tag{59}$$

where X *is any fiberwise polynomial Hamiltonian* r*-multivector field associated with* f*, decomposed according to Eq. (19).*

(3) f_c *is a closed* $(n-r)$*-form on* P *which vanishes on the zero section of* P *(as a vector bundle over* E*) and which is otherwise arbitrary iff* f *is Hamiltonian whereas it is restricted to vanish on the kernel of* ω *iff* f *is Poisson.*

We shall refer to Eq. (57) and to Eq. (60) below as the canonical decomposition of Hamiltonian forms or Poisson forms on P.

Proof: Let f be a Poisson $(n-r)$-form and X be a Hamiltonian r-multivector field associated with f. As already mentioned in the introduction, we may without loss of generality assume X to be fiberwise polynomial and decompose it into homogeneous components with respect to scaling degree, according to Eq. (19),

$$X = X_- + X_+ + \xi \quad \textit{with } X_+ = \sum_{s=1}^{r} X_{s-1}.$$

Then defining F as in the theorem, or equivalently, by

$$F = \sum_{s=1}^{r} F_{s-1} \quad \textit{with } F_{s-1} = \frac{1}{s} X_{s-1},$$

we obtain

$$F + [\Sigma, F] = X_+,$$

and hence according to Eq. (53), the exterior derivative of the difference $f - J(F)$ is given by

$$\mathrm{d}(f - J(F)) = \mathrm{d}f - \mathrm{d}(J(F)) = i_X \omega - i_{X_+} \omega = i_{X_-} \omega.$$

Applying the equivalence stated in Eq. (23), we see that since X_- has scaling degree -1, $i_{X_-}\omega$ must have scaling degree 0 and hence, according to Proposition A.1, is the pull-back to P of some $(n-r)$-form f_0' on E,

$$\mathrm{d}(f - J(F)) = i_{X_-} \omega = \tau^* f_0'.$$

Next, we define f_0 to be the restriction of $f - J(F)$ to the zero section of P, or more precisely, its pull-back to E with the zero section $s_0 : E \to P$,

$$f_0 = s_0^*(f - J(F)),$$

and set

$$f_c = f - \tau^* f_0 - J(F).$$

Then

$$\mathrm{d}f_c = \mathrm{d}(f - J(F)) - \mathrm{d}(\tau^* s_0^*(f - J(F))) = \mathrm{d}(f - J(F)) - \tau^* s_0^* \mathrm{d}(f - J(F)) = \tau^* f_0' - \tau^* s_0^* \tau^* f_0' = 0,$$

and

$$s_0^* f_c = s_0^*(f - J(F)) - s_0^* \tau^* f_0 = f_0 - s_0^* \tau^* f_0 = 0,$$

showing that indeed, f_c is closed and vanishes on the zero section of P. ☐

Proof of Theorem 1.3, part 2: The statements of Theorem 1.3 about differential forms are immediate consequences of Theorem 3.8. ☐

Remark: It should be noted that despite appearances, the decompositions (57) of Theorem 3.8 and (21) of Theorem 1.3 are not necessarily identical: for $s = 1, \ldots, r$, the f_s of Eq. (57) and the f_s of Eq. (21) may differ by homogeneous closed $(n-r)$-forms of scaling degree s. But the decomposition (57) of Theorem 3.8 seems to be the more natural one.

Theorem 3.8 implies that Poisson forms have a rather intricate local coordinate representation, involving two locally Hamiltonian multivector fields. Indeed, if we take f to be a general Poisson $(n-r)$-form on P, with $0 < r < n$, we can apply Propositions 3.4 and 3.6 to rewrite Eq. (57) in the form

$$f = f_0 + (-1)^{r-1} i_F \theta + (-1)^r i_{F_c} \omega, \tag{60}$$

where f_0 is as before while F and F_c are two locally Hamiltonian multivector fields on P of tensor degree r and $r+1$, respectively, satisfying $F_- = 0$ and $(F_c)_- = 0$. [The condition $(F_c)_- = 0$ will guarantee that $i_{F_c}\omega$ vanishes on the zero section of P.] In terms of the standard local coordinate representations (46) for f, (55) for f_0, (26) for F and for F_c, and for θ and ω, Eqs. (2) and (3), we obtain

$$f^{\mu_1 \cdots \mu_r} = (-1)^{r-1} p F^{\mu_1 \cdots \mu_r} + \sum_{s=1}^{r} (-1)^{r-s} p_i^{\mu_s} F^{i,\mu_1 \cdots \mu_{s-1}\mu_{s+1} \cdots \mu_r} + f_0^{\mu_1 \cdots \mu_r} + (-1)^{r-1} (\tilde{F}_c)^{\mu_1 \cdots \mu_r}, \tag{61}$$

$$f_i^{\mu_0 \cdots \mu_r} = -\sum_{s=0}^{r} (-1)^s p_i^{\mu_s} F^{\mu_0 \cdots \mu_{s-1}\mu_{s+1} \cdots \mu_r} + (f_0)_i^{\mu_0 \cdots \mu_r} - (F_c)_i^{\mu_0 \cdots \mu_r}, \tag{62}$$

$$f^{i,\mu_1 \cdots \mu_r} = (F_c)^{i,\mu_1 \cdots \mu_r}, \tag{63}$$

$$f'^{\mu_0 \cdots \mu_r} = (F_c)^{\mu_0 \cdots \mu_r}, \tag{64}$$

where the coefficients of F and of F_c are subject to the constraints listed in Theorem 2.4; in particular, the coefficients $(F_c)_i^{\mu_0 \cdots \mu_r}$ and $(\tilde{F}_c)^{\mu_1 \cdots \mu_r}$ can be completely expressed in terms of the coefficients $(F_c)^{\mu_0 \cdots \mu_r}$ and $(F_c)^{i,\mu_1 \cdots \mu_r}$, according to Eqs. (32) and (33) (with r replaced by $r+1$, X replaced by F_c, and X_- replaced by 0). In particular, we see that the coefficients $f^{\mu_1 \cdots \mu_r}$ are "antisymmetric polynomials in the multimomentum variables" of degree r. More explicitly, we can rewrite Eq. (61) in the form

$$f^{\mu_1\cdots\mu_r} = (-1)^{r-1} p F^{\mu_1\cdots\mu_r} + \sum_{s=1}^{r} f_s^{\mu_1\cdots\mu_r} + f_0^{\mu_1\cdots\mu_r} + (-1)^{r-1} (\widetilde{F}_c)^{\mu_1\cdots\mu_r},$$

where inserting the expansion (31) [with X replaced by F, X_{s-1} replaced by F_{s-1} and Y_{s-1} replaced by $G_{s-1}=(1/s)g_s$] gives, after a short calculation,

$$f_s^{\mu_1\cdots\mu_r} = (-1)^{r-1} \frac{1}{s!} \frac{1}{(r-s)!} \sum_{\pi\in S_r} (-1)^{\pi} p_{i_1}^{\mu_{\pi(1)}} \cdots p_{i_s}^{\mu_{\pi(s)}} g_s^{i_1\cdots i_s,\mu_{\pi(s+1)}\cdots\mu_{\pi(r)}}.$$

Finally, we want to clarify the relation between Poisson forms and Hamiltonian multivector fields in terms of their standard local coordinate representations.

Theorem 3.9: *Let f be a Poisson $(n-r)$-form and X be a Hamiltonian r-multivector field on P associated with f. Assume that, in adapted local coordinates, f and X are given by Eqs* (46) and (26), *respectively. Then*

$$X^{\mu_1\cdots\mu_r} = (-1)^{r-1} \left(\frac{\partial f^{\mu_1\cdots\mu_r}}{\partial p} - \frac{\partial f'^{\mu_1\cdots\mu_r\nu}}{\partial x^\nu} \right), \tag{65}$$

$$X^{i,\mu_2\cdots\mu_r} = \frac{1}{n-r+1} \frac{\partial f^{\mu_2\cdots\mu_r\mu}}{\partial p_i^\mu}, \tag{66}$$

$$X_i^{\mu_1\cdots\mu_r} = (-1)^r \left(\frac{\partial f^{\mu_1\cdots\mu_r}}{\partial q^i} - \frac{\partial f_i^{\mu_1\cdots\mu_r\nu}}{\partial x^\nu} \right), \tag{67}$$

$$\widetilde{X}^{\mu_2\cdots\mu_r} = -\frac{\partial f^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu}, \tag{68}$$

that is, locally and modulo terms taking values in the kernel of ω, X is given by

$$\begin{aligned} X = & -\frac{1}{(r-1)!} \left(\frac{\partial f^{\mu_2\cdots\mu_r\mu}}{\partial x^\mu} \frac{\partial}{\partial p} - \frac{1}{r} \frac{\partial f^{\mu_2\cdots\mu_r\mu}}{\partial p} \frac{\partial}{\partial x^\mu} + \frac{1}{r} \frac{\partial f'^{\mu_2\cdots\mu_r\mu\nu}}{\partial x^\nu} \frac{\partial}{\partial x^\mu} \right) \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} \\ & + \frac{1}{(r-1)!} \left(\frac{1}{n-r+1} \frac{\partial f^{\mu_2\cdots\mu_r\mu}}{\partial p_i^\mu} \frac{\partial}{\partial q^i} - \frac{1}{r} \frac{\partial f^{\mu_2\cdots\mu_r\mu}}{\partial q^i} \frac{\partial}{\partial p_i^\mu} + \frac{1}{r} \frac{\partial f_i^{\mu_2\cdots\mu_r\mu\nu}}{\partial x^\nu} \frac{\partial}{\partial p_i^\mu} \right) \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}}. \end{aligned} \tag{69}$$

If, in the canonical decomposition (57) and (60) *of f, the closed term $f_c=(-1)^r i_{F_c}\omega$ is absent, then $f'^{\mu_0\cdots\mu_r}=0$. If f is horizontal with respect to the projection onto M, then $f_i^{\mu_0\cdots\mu_r}=0$. In these cases, the above formulas simplify accordingly.*

Proof: There are several methods for proving this, with certain overlaps. Let us begin with the "trivial" case of closed forms f, for which we must have $X=0$. Assuming f to be of the form $f_c=(-1)^r i_{F_c}\omega$ and using Eqs. (61)–(64) to rewrite the expressions on the rhs of the above equations in terms of the components of F_c, we must show that

$$\frac{\partial(\widetilde{F}_c)^{\mu_1\cdots\mu_r}}{\partial p} + (-1)^r \frac{\partial(F_c)^{\mu_1\cdots\mu_r\nu}}{\partial x^\nu} = 0, \qquad \frac{\partial(\widetilde{F}_c)^{\mu_2\cdots\mu_r\mu}}{\partial p_i^\mu} = 0,$$

$$\frac{\partial(\widetilde{F}_c)^{\mu_1\cdots\mu_r}}{\partial q^i} - (-1)^r \frac{\partial(F_c)_i^{\mu_1\cdots\mu_r\nu}}{\partial x^\nu} = 0, \qquad \frac{\partial(\widetilde{F}_c)^{\mu_2\cdots\mu_r\nu}}{\partial x^\nu} = 0.$$

But this follows directly from the pertinent relations for locally Hamiltonian multivector fields derived in the proof of Theorem 1.3 which hold since F_c is locally Hamiltonian. To handle the remaining cases where f is of the form $f=f_0+(-1)^{r-1}i_F\theta$, it is easier to proceed by direct inspection of Eq. (11). Indeed, we may for a general Poisson form f apply the exterior derivative to Eq. (46) and compare the result with the expression for $i_X\omega$. In this way, Eqs. (68), (67), and (65) can be obtained directly by equating the coefficients of $\mathrm{d}^n x_{\mu_2\cdots\mu_r}$, of $\mathrm{d}q^i \wedge \mathrm{d}^n x_{\mu_1\cdots\mu_r}$ and of $\mathrm{d}p \wedge \mathrm{d}^n x_{\mu_1\cdots\mu_r}$, respectively. The only case which requires an additional argument is Eq. (66), since collecting terms proportional to $\mathrm{d}p_i^\mu \wedge \mathrm{d}^n x_{\mu_1\cdots\mu_r}$ leads to

$$\begin{aligned}
&\frac{(-1)^{r-1}}{(r-1)!} X^{i,\mu_2\cdots\mu_r}\, \mathrm{d}p_i^\mu \wedge \mathrm{d}^n x_{\mu\mu_2\cdots\mu_r} \\
&\quad = \frac{1}{r!}\frac{\partial f^{\mu_1\cdots\mu_r}}{\partial p_i^\mu} \mathrm{d}p_i^\mu \wedge \mathrm{d}^n x_{\mu_1\cdots\mu_r} - \frac{1}{(r-1)!}\frac{\partial f^{i,\mu_2\cdots\mu_r\nu}}{\partial x^\nu} \mathrm{d}p_i^\mu \wedge \mathrm{d}^n x_{\mu\mu_2\cdots\mu_r} \\
&\quad - \frac{(-1)^r}{r!}\frac{\partial f^{i,\mu_1\cdots\mu_r}}{\partial x^\mu} \mathrm{d}p_i^\mu \wedge \mathrm{d}^n x_{\mu_1\cdots\mu_r}.
\end{aligned}$$

But when f is of the form $f=f_0+(-1)^{r-1}i_F\theta$, Eq. (63) implies that the last two terms on the rhs of the equation above vanish. Moreover, since F is Hamiltonian, we know from Theorem 2.4 that the $F^{\mu_1\cdots\mu_r}$ depend on the p_i^μ only if $\mu \in \{\mu_1, \ldots, \mu_r\}$, and hence according to Eq. (61), the same is true for the $f^{\mu_1\cdots\mu_r}$. This reduces the first term on the rhs of the above equation to an expression which, when compared with the lhs, leads to the conclusion that for any choice of mutually different indices μ and $\mu_2, \ldots, \mu_r$, we have

$$X^{i,\mu_2\cdots\mu_r} = \frac{\partial f^{\mu_2\cdots\mu_r\mu}}{\partial p_i^\mu} \quad \text{if } \mu \notin \{\mu_2, \ldots, \mu_r\} \quad (\text{no sum over } \mu).$$

Summing over μ gives Eq. (66). □

IV. POISSON BRACKETS

In the characterization of locally Hamiltonian multivector fields and of Poisson forms derived in the preceding two sections, the decomposition into homogeneous terms with respect to scaling degree plays a central role. It is therefore natural to ask how this decomposition complies with the Schouten bracket of Hamiltonian multivector fields and with the Poisson bracket of Poisson forms. To this end, let us first recall the definition of the Poisson bracket between Poisson forms given in Ref. 1 for $(n-1)$-forms and in Ref. 2 for forms of arbitrary degree.

Definition 4.1: Let f and g be Poisson forms of tensor degree $n-r$ and $n-s$ on P, respectively. Their Poisson bracket is the Poisson form of tensor degree $n-r-s+1$ on P defined by

$$\{f,g\} = (-1)^{r(s-1)} i_Y i_X \omega + \mathrm{d}((-1)^{(r-1)(s-1)} i_Y f - i_X g - (-1)^{(r-1)s} i_Y i_X \theta), \tag{70}$$

where X and Y are Hamiltonian multivector fields associated with f and g, respectively.

We find the following properties of the two mentioned bracket operations with respect to scaling degree.

Proposition 4.2: Let X and Y be homogeneous multivector fields on P of scaling degree k and l, respectively. Then their Schouten bracket $[X,Y]$ is of scaling degree $k+l$,

$$L_\Sigma X = kX, \quad L_\Sigma Y = lY \quad \Rightarrow \quad L_\Sigma [X,Y] = (k+l)[X,Y]. \tag{71}$$

Proof: The proposition is a consequence of the graded Jacobi identity for multivector fields,[8] which can be rewritten as the statement that the Schouten bracket with a given multivector field Z of odd/even tensor degree acts as an even/odd superderivation,

$$[Z,[X,Y]] = [[Z,X],Y] + (-1)^{(t-1)(r-1)}[X,[Z,Y]].$$

In particular, since Σ has tensor degree 1,

$$[\Sigma,[X,Y]] = [[\Sigma,X],Y] + [X,[\Sigma,Y]],$$

from which the proposition follows immediately. □

Corollary 4.3: Let X and Y be locally Hamiltonian multivector fields on P of scaling degree -1. Then their Schouten bracket $[X,Y]$ takes values in the kernel of ω.

Proof: From the preceding proposition, $[X,Y]$ is a locally Hamiltonian multivector field of scaling degree -2. and hence, by Theorem 1.3, must take values in the kernel of ω. □

For the Poisson bracket of Poisson forms, we have the following property.

Proposition 4.4: Let f and g be homogeneous Poisson forms on P of scaling degree k and l, respectively. Then their Poisson bracket $\{f,g\}$ is of scaling degree $k+l-1$:

$$L_\Sigma f = kf, \quad L_\Sigma g = lg \quad \Rightarrow \quad L_\Sigma\{f,g\} = (k+l-1)\{f,g\}. \tag{72}$$

Proof: As explained in the last paragraph of Sec. I [see, in particular, Eq. (23)], we can find homogeneous Hamiltonian multivector fields X of scaling degree $k-1$ and Y of scaling degree $l-1$ such that $i_X\omega = \mathrm{d}f$ and $i_Y\omega = \mathrm{d}g$. We shall consider each of the terms in the definition of the Poisson bracket separately. We find

$$L_\Sigma(i_Y i_X\omega) = i_Y L_\Sigma i_X\omega + i_{[\Sigma,Y]} i_X\omega = i_Y i_X L_\Sigma\omega + i_Y i_{[\Sigma,X]}\omega + i_{[\Sigma,Y]} i_X\omega = (k+l-1) i_Y i_X\omega.$$

The same calculation works with ω replaced by θ, so that, since L_Σ commutes with d,

$$L_\Sigma(\mathrm{d}(i_Y i_X\theta)) = (k+l-1)\mathrm{d}(i_Y i_X\theta).$$

Moreover,

$$L_\Sigma(\mathrm{d}(i_Y f)) = \mathrm{d}(L_\Sigma i_Y f) = \mathrm{d}(i_Y L_\Sigma f + i_{[\Sigma,Y]} f) = (k+l-1)\mathrm{d}(i_Y f),$$

and similarly $L_\Sigma(\mathrm{d}(i_X g)) = (k+l-1)\mathrm{d}(i_X g)$. Putting the pieces together, the proposition follows. □

Having shown in what sense both the Schouten bracket and the Poisson bracket respect scaling degree, let us use the canonical decomposition of Poisson forms to express their Poisson bracket in terms of known operations on the simpler objects from which they can be constructed. To start with, we settle the case of homogeneous Poisson forms of positive scaling degree.

Proposition 4.5: Let X_{k-1} be a homogeneous locally Hamiltonian r-multivector field on P of scaling degree $k-1$, with $1 \leqslant k \leqslant r$, and Y_{l-1} be a homogeneous locally Hamiltonian s-multivector field on P of scaling degree $l-1$, with $1 \leqslant l \leqslant s$. Set

$$f_k = \frac{(-1)^{r-1}}{k} i_{X_{k-1}}\theta, \quad g_l = \frac{(-1)^{s-1}}{l} i_{Y_{l-1}}\theta.$$

Then

$$\{f_k, g_l\} = \frac{(-1)^{r+s}}{k+l-1} i_{[Y_{l-1},X_{k-1}]}\theta - (-1)^{(r-1)s}\frac{(k-1)(l-1)(k+l)}{kl(k+l-1)}\mathrm{d}(i_{X_{k-1}} i_{Y_{l-1}}\theta).$$

Proof: From the defining equation (70) for the Poisson bracket, we find

$$\{f_k,g_l\} = (-1)^{r(s-1)} i_{Y_{l-1}} i_{X_{k-1}} \omega + \mathrm{d}\left(\frac{(-1)^{(r-1)s}}{k} i_{Y_{l-1}} i_{X_{k-1}} \theta - \frac{(-1)^{(s-1)}}{l} i_{X_{k-1}} i_{Y_{l-1}} \theta - (-1)^{(r-1)s} i_{Y_{l-1}} i_{X_{k-1}} \theta \right)$$

$$= (-1)^{r(s-1)} i_{Y_{l-1}} i_{X_{k-1}} \omega + (-1)^{(r-1)s} \left(\frac{1}{k} + \frac{1}{l} - 1 \right) \mathrm{d}(i_{Y_{l-1}} i_{X_{k-1}} \theta).$$

On the other hand, one verifies that $\mathrm{d} i_{X_{k-1}} \theta = k(-1)^{r-1} i_{X_{k-1}} \omega$ and hence

$$i_{[Y_{l-1},X_{k-1}]}\theta = (-1)^{(s-1)r} L_{Y_{l-1}} i_{X_{k-1}} \theta - i_{X_{k-1}} L_{Y_{l-1}} \theta$$

$$= (-1)^{(s-1)r}\, \mathrm{d} i_{Y_{l-1}}\, i_{X_{k-1}} \theta + (-1)^{s(r-1)} (k+l-1) i_{Y_{l-1}} i_{X_{k-1}} \omega.$$

Thus,

$$\{f_k,g_l\} = \frac{(-1)^{r+s}}{k+l-1} i_{[Y_{l-1},X_{k-1}]}\theta - \frac{(-1)^{(r-1)s}}{k+l-1} \mathrm{d}(i_{Y_{l-1}} i_{X_{k-1}} \theta) + (-1)^{(r-1)s} \left(\frac{1}{k} + \frac{1}{l} - 1 \right) \mathrm{d}(i_{Y_{l-1}} i_{X_{k-1}} \theta), \tag{73}$$

which implies the asserted relation. □

As a special case, consider homogeneous Poisson forms of scaling degree 1, which arise by contracting θ with a Hamiltonian multivector field of scaling degree 0, that is, with an exact Hamiltonian multivector field (see the first statement in Proposition 2.5). These Poisson forms have been studied in Ref. 2 under the name "universal multimomentum map."

Corollary 4.6: The space of homogeneous Poisson forms on P of scaling degree 1 closes under the Poisson bracket.

Obviously, it also follows from the proposition that no such statement holds for homogeneous Poisson forms of scaling degree >1, since the second term in the expression in Proposition 4.5 vanishes only for $k=1$ or $l=1$.

Turning to homogeneous Poisson forms on P of scaling degree 0, which come from forms on E by pull-back, we have the following.

Proposition 4.7: The space of homogeneous Poisson forms on P of scaling degree 0 is Abelian under the Poisson bracket,

$$\{f_0,g_0\} = 0. \tag{74}$$

Proof: Without loss of generality, we may assume the Hamiltonian multivector fields X_- and Y_- associated with f_0 and with g_0, respectively, to be homogeneous of scaling degree -1. Therefore, using the fact that if a multivector field X is homogeneous of scaling degree k and a differential form α is homogeneous of scaling degree l, then the differential form $i_X\alpha$ is homogeneous of scaling degree $k+l$,

$$L_\Sigma X = kX, \quad L_\Sigma \alpha = l\alpha \quad \Rightarrow \quad L_\Sigma i_X \alpha = (k+l) i_X \alpha,$$

which follows immediately from the formula $L_\Sigma i_X \alpha = i_X L_\Sigma \alpha + i_{[\Sigma,X]}\alpha$, we see that all four terms in the definition (70) of the Poisson bracket between f_0 and g_0 are differential forms of scaling degree -1 and hence must vanish. □

For the mixed case of the Poisson bracket between a homogeneous Poisson form of strictly positive scaling degree with one of scaling degree zero, we find the following result.

Proposition 4.8: Let X_{k-1} be a homogeneous locally Hamiltonian r-multivector field on P of scaling degree $k-1$, with $1 \leq k \leq r$, and let g_0 be a homogeneous Poisson $(n-s)$-form on P of scaling degree zero, with associated Hamiltonian s-multivector field Y_-. Set

$$f_k = \frac{(-1)^{r-1}}{k} i_{X_{k-1}} \theta. \tag{75}$$

Then

$$\{f_k,g_0\} = -L_{X_{k-1}}g_0. \tag{76}$$

Proof: By Proposition 2.5, $i_{Y_-}\theta$ vanishes. Hence only two of the four terms in the defining equation (70) for the Poisson bracket survive,

$$\{f_k,g_0\} = (-1)^{r(s-1)}i_{Y_-}i_{X_{k-1}}\omega - \mathrm{d}i_{X_{k-1}}g_0 = -(\mathrm{d}i_{X_{k-1}}g_0 - (-1)^r i_{X_{k-1}}\,\mathrm{d}g_0) = -L_{X_{k-1}}g_0.$$

□

Finally, let us consider closed Poisson forms, whose associated Hamiltonian multivector fields vanish. Still, the Poisson bracket of a closed Poisson form with an arbitrary Poisson form does not vanish, but it is once again a closed Poisson form.

Proposition 4.9: Let f *be a Poisson* $(n-r)$*-form on* P, *with associated Hamiltonian* r*-multivector field* X, *and let* g *be a closed Poisson* $(n-s)$*-form on* P. *Set*

$$g = (-1)^s i_{G_c}\omega. \tag{77}$$

Then

$$\{f,g\} = (-1)^{r+s-1}i_{[G_c,X]}\omega. \tag{78}$$

Proof: As the Hamiltonian multivector field associated with g vanishes, only one of the four terms in the defining equation (70) for the Poisson bracket survives,

$$\{f,g\} = -\mathrm{d}(i_X g) = (-1)^{s-1}\mathrm{d}(i_X i_{G_c}\omega) = (-1)^{rs-1}i_{[X,G_c]}\omega = (-1)^{r+s-1}i_{[G_c,X]}\omega.$$

(For the penultimate equation, see, e.g., Proposition 3.3 of Ref. 2.) □

In view of the canonical decomposition for Poisson forms stated in Theorem 3.8, the above propositions exhaust the possible combinations for the computation of Poisson brackets.

V. CONCLUSIONS AND OUTLOOK

In this paper, we have achieved three goals. First, we have determined the general structure of locally Hamiltonian multivector fields on the extended multiphase space of classical first order field theories. According to Theorem 2.4, the basic structure that arises from explicit calculations in adapted local coordinates is the decomposition of any such multivector field X, of tensor degree r $(0<r<n)$, into a sum of terms of homogeneous scaling degree plus a remainder ξ which is a multivector field taking values in the kernel of ω,

$$X = X_{-1} + X_0 + \cdots + X_{r-1} + \xi \quad \text{with } L_\Sigma X_k = kX_k. \tag{79}$$

Moreover, according to Proposition 2.5, all homogeneous locally Hamiltonian multivector fields of non-negative scaling degree are in fact globally Hamiltonian, and they are exact Hamiltonian if and only if they have zero scaling degree. At the level of local coefficient functions, this decomposition arises because the coefficient functions must be antisymmetric polynomials in the multimomentum variables; see Eqs. (30) and (31).

Second, we have extended the scaling degree analysis to the study of Hamiltonian forms by means of the formula

$$L_\Sigma i_X\omega = i_{X+[\Sigma,X]}\omega.$$

As shown in Theorem 3.8, this leads to a canonical decomposition of any Hamiltonian $(n-r)$-form f $(0<r<n)$ into a sum of terms of homogeneous scaling degree plus a remainder f_c which is a closed form,

$$f = f_0 + f_1 + \cdots + f_r + f_c \quad \text{with } L_\Sigma f_s = sf_s. \tag{80}$$

Moreover, if X is a Hamiltonian multivector field associated with f, then

$$f_s = \frac{(-1)^{r-1}}{s} i_{X_{s-1}} \theta \quad \text{for } s > 0, \tag{81}$$

where the X_{s-1} are the homogeneous components of X of non-negative scaling degree as described before, whereas f_0 arises by pull-back from a form on the total space of the configuration bundle of the theory. Locally, this form can be decomposed into the sum of a horizontal form and a closed form (we prove this explicitly only for Poisson forms), but this decomposition has no global, coordinate invariant meaning. The canonical decomposition of Poisson forms is also useful for deriving local formulas for X in terms of f; these are given in Theorem 3.9. They clearly show that the situation in multisymplectic geometry resembles that encountered in symplectic geometry but exhibits a significantly richer structure. In particular, the notion of conjugate variables requires a conceptual extension.

Third, we have used the canonical decomposition of Poisson forms to derive explicit formulas for the Poisson bracket between Poisson forms. The resulting Lie algebra shows an interesting and nontrivial structure. It has a trivial part, namely the space of closed Poisson forms, which constitutes an ideal that one might wish to divide out; this ideal is Abelian but not central. It commutes with the most interesting and useful part, namely the subalgebra of homogeneous Poisson forms of scaling degree 1, which by means of Eq. (81), specialized to the case $s=1$, correspond to the exact Hamiltonian multivector fields, and in such a way that the Poisson bracket on this subalgebra corresponds to the Schouten bracket for exact Hamiltonian multivector fields (up to signs). The nontrivial mixing occurs through the spaces of homogeneous Poisson forms of scaling degree 0 and of scaling degree >1, they close under the operation of taking the Poisson bracket with a homogeneous Poisson forms of scaling degree 1 but not under the operation of taking mutual Poisson brackets, since these contain contributions lying in the ideal of closed Poisson forms.

An important aspect of our results is that they confirm, once again, the apparently unavoidable appearance of strong constraints on the dependence of Hamiltonian multivector fields and Hamiltonian forms on the multimomentum variables and the energy variable in extended multiphase space, expressed through the "antisymmetric polynomial" structure of their coefficient functions. This strongly suggests that there should be some product structure complementing the Poisson bracket operation. So far, such a structure seems to exist only for a very restricted class of Poisson forms, namely the horizontal forms studied by Kanatchikov.[11] Also, one might wonder whether the structural properties derived here still hold in the multisymplectic formulation of higher order field theories.[6]

Finally, a central question that remains is how the various proposals of Poisson brackets in the multisymplectic formalism that can be found in the literature, including the one proposed in Refs. 1 and 2, relates to the Peierls-DeWitt bracket that comes from the functional approach based on the concept of covariant phase space. Briefly, covariant phase space is defined as the space $\mathcal{S}$ of solutions of the equations of motion and, formally viewed as an infinite-dimensional manifold, carries a naturally defined symplectic form Ω.[12–14] A systematic general investigation of the Peierls-DeWitt bracket in the multisymplectic framework, including a proof of the fact that it is precisely the canonical Poisson bracket for functionals on $\mathcal{S}$ derived from the symplectic form Ω on $\mathcal{S}$, has been carried out recently.[15,16] In order to establish the desired relation, we must restrict this bracket to a certain class of functionals, namely functionals F obtained by using fields to pull Hamiltonian forms or Poisson forms f on extended multiphase space back to space-time and then integrate over submanifolds Σ of the corresponding dimension. Explicitly, using the notation of Ref. 16, we have

$$\mathsf{F}[\phi] = \int_\Sigma (\mathbb{F}\mathcal{L} \circ (\varphi, \partial\varphi))^* f \tag{82}$$

in the Lagrangian framework and

$$\mathsf{F}[\phi] = \int_{\Sigma} (\mathcal{H} \circ (\varphi, \pi))^* f \tag{83}$$

in the Hamiltonian framework. Now using the classification of Hamiltonian vector fields and Hamiltonian $(n-1)$-forms obtained in this paper, it has been shown recently that the Peierls-DeWitt bracket $\{\mathsf{F},\mathsf{G}\}$ between two functionals F and G derived from Hamiltonian $(n-1)$-forms f and g, respectively, is the functional derived from the Hamiltonian $(n-1)$-form $\{f,g\}$;[17] details will be published elsewhere. The question of how to extend this result to Poisson forms of other degree is currently under investigation.

ACKNOWLEDGMENTS

Part of this work has been financially supported by CNPq ("Conselho Nacional de Desenvolvimento Científico e Tecnológico"), Brazil (M.F.), by DFG ("Deutsche Forschungsgemeinschaft"), Germany, under the Emmy Noether Programme (C.P.), and by FAPESP ("Fundação de Amparo à Pesquisa do Estado de São Paulo"), Brazil (H.R.).

APPENDIX

Proposition A.1: Let V be a vector bundle over a manifold M with projection π and let Σ be the scaling or Euler vector field on V. A differential form α on the total space V will be the pull-back of a differential form α_0 on the base space M to V via π if and only if it is scale invariant,

$$\alpha = \pi^* \alpha_0 \quad \Leftrightarrow \quad L_\Sigma \alpha = 0.$$

Proof: Assume first that the form α on V is the pull-back of a form α_0 on M; then $\alpha = \pi^* \alpha_0$ and hence $\mathrm{d}\alpha = \pi^* \, \mathrm{d}\alpha_0$. Therefore, α and $\mathrm{d}\alpha$ are both horizontal. This means that for any vertical vector field X on V, including Σ, we have $i_X \alpha = 0$ as well as $i_X \, \mathrm{d}\alpha = 0$, so $L_X \alpha = 0$.

Conversely, assume that the form α on V, of degree r, say, satisfies $L_\Sigma \alpha = 0$, so α is invariant under the flow F of Σ,[9]

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} F_\lambda^* \alpha = 0.$$

This means that given $v \in V$ and $w_1, \ldots, w_r \in T_v V$, the expression

$$(F_\lambda^* \alpha)_v (w_1, \ldots, w_r) = \alpha_{F_\lambda(v)} (T_v F_\lambda \cdot w_1, \ldots, T_v F_\lambda \cdot w_r)$$

does not depend on λ, so its value $\alpha_v(w_1, \ldots, w_r)$ at $\lambda = 0$ is equal to its value $\alpha_{v_0}((w_1)_0, \ldots, (w_r)_0)$ obtained in the limit $\lambda \to -\infty$, where v_0 denotes the zero vector in the fiber of v. But this means that α is equal to $\pi^* \alpha_0$ where α_0 is defined with the help of the zero section $i_0 : M \to V$ of V as $\alpha_0 = i_0^* \alpha$. □

The following lemma is used in the proof of Theorem 2.4.

Lemma A.2: Let f be a polynomial of degree s in a set of variables x^μ. Let f_r, $0 \leqslant r \leqslant s$, be the homogeneous component of degree r of f. Then

$$f = \sum_{r=1}^{s} \frac{1}{r} x^\mu \frac{\partial f_r}{\partial x^\mu} + f_0.$$

Proof: For the operator $x^\mu (\partial / \partial x^\mu)$, the homogeneous polynomials f_r are eigenvectors with eigenvalue r. Writing this as $f_r = (1/r) x^\mu (\partial f_r / \partial x^\mu)$ for $0 < r \leqslant s$, we obtain

$$f = \sum_{r=0}^{s} f_r = f_0 + \sum_{r=1}^{s} \frac{1}{r} x^\mu \frac{\partial f_r}{\partial x^\mu}.$$

□

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