The DFR-Algebra for Poisson Vector Bundles

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Abstract

The aim of the present paper is to present the construction of a general family of C^*-algebras that includes, as a special case, the "quantum space-time algebra" first introduced by Doplicher, Fredenhagen and Roberts. To this end, we first review, within the C^*-algebra context, the Weyl-Moyal quantization procedure on a fixed Poisson vector space (a vector space equipped with a given bivector, which may be degenerate). We then show how to extend this construction to a Poisson vector bundle over a general manifold M, giving rise to a C^*-algebra which is also a module over $C_0(M)$. Apart from including the original DFR-model, this method yields a "fiberwise quantization" of general Poisson manifolds.

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1 Introduction

In a seminal paper published in 1995 [3], Doplicher, Fredenhagen and Roberts (DFR) have introduced a special C^*-algebra to provide a model for space-time in which localization of events can no longer be performed with arbitrary precision: they refer to it as a model of "quantum space-time". Apart from being beautifully motivated, their construction is mathematically simple: it starts from a symplectic form σ on Minkowski space and considers the corresponding canonical commutation relations (CCRs), which can be viewed as a representation of a well-known finite-dimensional nilpotent Lie algebra, the Heisenberg (Lie) algebra. More precisely, the CCRs appear in the Weyl form, i.e., we are really dealing with an irreducible unitary representation of the Heisenberg group. This in turn generates a C^*-algebra that we propose to call the Heisenberg C^*-algebra, related to the original representation through Weyl quantization, that is, via the Weyl-Moyal star product. Due to von Neumann's theorem about the uniqueness of the Schödinger representation, it is unique up to isomorphism.

The main novelty in the DFR construction is that the symplectic form σ defining the Heisenberg algebra is treated as a *variable*. In this way, one is able to reconcile the construction with the principle of relativistic invariance: since Minkowski space $\mathbb{R}^4$ has no distinguished symplectic structure, the only way out is to consider, simultaneously, *all* possible symplectic structures on Minkowski space that can be obtained from a fixed one, say σ_0, by the action of the Lorentz group. Now the orbit of σ_0 under the action of the Lorentz group is $TS^2 \times \mathbb{Z}_2$, thus explaining the origin of the extra dimensions that appear in this approach.[1]

Assuming the symplectic form σ to vary over the orbit of some fixed representative σ_0 produces not just a single Heisenberg C^*-algebra but an entire bundle of C^*-algebras over that orbit, with the Heisenberg C^*-algebra for σ_0 as typical fiber. The continuous sections of that bundle vanishing at infinity then define a "section" C^*-algebra, which carries a natural action of the Lorentz group induced from its natural action on the underlying bundle of C^*-algebras (which moves base points as well as fibers). Besides, this "section" C^*-algebra is also a C^*-module over the "scalar" C^*-algebra $C_0(M)$ of continuous functions on M vanishing at infinity. In the special case considered by DFR, the underlying bundle turns out to be globally trivial, which in view of von Neumann's theorem implies a classification result on irreducible as well as on Lorentz covariant representations of the DFR-algebra.

In retrospect, it is clear that when formulated in this geometrically inspired language, the results of [3] yearn for generalization – even if only for purely mathematical reasons.

From a more physical side, one of the original motivations of the present work was an idea of J.C.A. Barata, who proposed to look for a clearer geometrical interpretation of the classical limit of the DFR-algebra in terms of coherent states, as developed by K. Hepp [6].

[1] Note that the factor $\mathbb{Z}_2$ comes from the fact that we are dealing with the full Lorentz group; it would be absent if we dropped (separate) invariance under parity P or time reversal T.

This led the second author to investigate possible generalizations of the DFR-construction to other vector spaces than four-dimensional Minkowski space and other Lie groups than the Lorentz group in four dimensions. Isolating the essential hypotheses underlying the DFR-construction from the marginal ones and using the reformulation contained in [11], we finally came up with the construction presented in this paper.[2] As it turned out, we were even able to eliminate the hypothesis of nondegeneracy of the underlying symplectic form on each fiber and thus end up with a version that works even for general Poisson tensors, which we consider to be a rather dramatic generalization, taking into account the degree to which Poisson manifolds are more general than symplectic manifolds.

The original question of how to define the classical limit in the general context outlined below will be pursued elsewhere, but we believe that the mathematical construction as presented here is of independent interest, going beyond the physical motivations of the original DFR paper.

2 The Heisenberg C^*-Algebra for Poisson Vector Spaces

Let V be a Poisson vector space, i.e., a real vector space of dimension n, say, equipped with a fixed bivector σ of rank $2r$; in other words, the dual V^* of V is a presymplectic vector space.[3] It gives rise to an $(n+1)$-dimensional Lie algebra $\mathfrak{h}_\sigma$ which is a one-dimensional central extension of the abelian Lie algebra V^* defined by the cocycle σ and will be called the *Heisenberg algebra* or, more explicitly, the *Heisenberg Lie algebra* (associated to V^* and σ): as a vector space, $\mathfrak{h}_\sigma = V^* \oplus \mathbb{R}$, with commutator given by

$$[(\xi,\lambda),(\eta,\mu)] \;=\; (0\,,\,\sigma(\xi,\eta)) \qquad \text{for} \;\; \xi,\eta \in V^*\,,\; \lambda,\mu \in \mathbb{R}\;. \tag{2.1}$$

Associated with this Lie algebra is the *Heisenberg group* H_σ: as a manifold, $H_\sigma = V^* \times \mathbb{R}$, with product (written additively) given by

$$(\xi,\lambda)\,(\eta,\mu) \;=\; \big(\xi+\eta\,,\,\lambda+\mu-\tfrac{1}{2}\,\sigma(\xi,\eta)\big) \qquad \text{for} \;\; \xi,\eta \in V^*\,,\; \lambda,\mu \in \mathbb{R}\;. \tag{2.2}$$

As is well known, the *canonical commutation relations*

$$\big[\mathbf{P}^j,\mathbf{Q}^k\big] \;=\; -\,i\,\delta^{jk} \qquad \text{for} \;\; 1 \leqslant j,k \leqslant r \tag{2.3}$$

can formally be viewed as a representation $\dot{\pi}_\sigma$ of the Heisenberg algebra $\mathfrak{h}_\sigma$ by essentially skew adjoint operators in a certain Hilbert space, namely $L^2(\mathbb{R}^{n-r})$, as follows. First,

[2] As was kindly pointed out by D. Bahns, our construction may be reduced in some sense to Rieffel's theory of strict deformations, as has happened with the original DFR construction [12]. Since such a reduction has not yet been worked ou explicitly and since the methods used here are quite different, we still believe our work may be useful.

[3] Note that we do *not* require σ to be non-degenerate.

introduce a *Darboux basis* for σ, that is, a basis $\{v_1, \ldots, v_n\}$ of V, with corresponding dual basis $\{v^1, \ldots, v^n\}$ of V^*, in which σ takes the form represented by the matrix

$$\begin{pmatrix} 0 & 1_r & 0 \\ -1_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{2.4}$$

that is, the only non-vanishing matrix elements of σ are

$$\sigma(v^j, v^{r+k}) \;=\; \delta^{jk} \;=\; -\,\sigma(v^{r+j}, v^k) \qquad \text{for } \; 1 \leqslant j, k \leqslant r \; . \tag{2.5}$$

Such a basis can always be constructed by a symplectic analogue of Gram-Schmidt orthogonalization; see, for example, [1, Prop. 3.1.2, pp. 162-164]. Then, abbreviating $\dot{\pi}_\sigma(\xi, 0)$ to $\dot{\pi}_\sigma(\xi)$, set

$$\begin{aligned} \dot{\pi}_\sigma(v^j) \;&=\; i\,\mathbf{P}^j \qquad \text{for } \; 1 \leqslant j \leqslant r \; , \\ \dot{\pi}_\sigma(v^{r+j}) \;&=\; i\,\mathbf{Q}^j \qquad \text{for } \; 1 \leqslant j \leqslant r \; , \\ \dot{\pi}_\sigma(v^{2r+k}) \;&=\; i\,\mathbf{Q}^{r+k} \qquad \text{for } \; 1 \leqslant k \leqslant n-2r \; , \\ \dot{\pi}_\sigma(0,1) \;&=\; i\,1 \; , \end{aligned} \tag{2.6}$$

where the $\mathbf{P}$'s and $\mathbf{Q}$'s are the usual position and momentum operators of quantum mechanics in $L^2(\mathbb{R}^{n-r})$. Since these operators have a common dense invariant domain of analytic vectors [10] (take, for example, the Schwartz space $\mathscr{S}(\mathbb{R}^{n-r})$), this representation can be exponentiated to a strongly continuous, unitary representation of the Heisenberg group which will be denoted by π_σ: these are the canonical commutation relations in *Weyl form* which, after abbreviating $\pi_\sigma(\xi, 0)$ to $\pi_\sigma(\xi)$, can be written as

$$\pi_\sigma(\xi)\,\pi_\sigma(\eta) \;=\; e^{-\frac{i}{2}\sigma(\xi,\eta)}\,\pi_\sigma(\xi+\eta) \; . \tag{2.7}$$

Note that although this explicit definition of $\dot{\pi}_\sigma$ and (hence) of π_σ depends on the choice of Darboux basis, the resulting representation itself does not, up to unitary equivalence. Moreover, both representations are faithful, since they are restrictions of faithful representations of a larger (non-degenerate) Heisenberg algebra/group to a subalgebra/subgroup.

In a second step, we extend this whole construction to the C^*-algebra setting. To this end, we construct a linear map

$$\begin{array}{rccc} W_\sigma : & \mathscr{S}(V) & \longrightarrow & B(L^2(\mathbb{R}^{n-r})) \\ & f & \longmapsto & W_\sigma f \end{array} \tag{2.8}$$

from the space of Schwartz test functions on V to the space of bounded linear operators on the Hilbert space $L^2(\mathbb{R}^{n-r})$ by setting

$$W_\sigma f \;=\; \int_{V^*} d\xi \; \check{f}(\xi)\,\pi_\sigma(\xi) \; , \tag{2.9}$$

which is to be compared with

$$f(x) \;=\; \int_{V^*} d\xi \; \check{f}(\xi)\,e^{i\langle\xi,x\rangle} \; , \tag{2.10}$$

where $\check{f}$ is the inverse Fourier transform of f,

$$\check{f}(\xi) \;=\; \frac{1}{(2\pi)^n} \int_V dx\; f(x)\, e^{-i\langle \xi, x\rangle} \;. \tag{2.11}$$

An argument analogous to the one that proves injectivity of the Fourier transform (on Schwartz space) can be used to show that W_σ is faithful, and an explicit calculation gives

$$W_\sigma f\; W_\sigma g \;=\; W_\sigma(f \star_\sigma g) \qquad \text{for} \;\; f, g \in \mathscr{S}(V) \;, \tag{2.12}$$

where $\star_\sigma$ denotes the *Weyl-Moyal star product*, given by

$$\begin{aligned} (f \star_\sigma g)(x) \;=\; & \int_{V^*} d\xi\; e^{i\langle \xi, x\rangle} \int_{V^*} d\eta\; \check{f}(\eta)\, \check{g}(\xi - \eta)\, e^{-\frac{i}{2}\sigma(\xi,\eta)} \\ & \text{for} \;\; f, g \in \mathscr{S}(V) \;. \end{aligned} \tag{2.13}$$

In other words, using the Weyl-Moyal star product, together with the standard involution of pointwise complex conjugation, we can turn $\mathscr{S}(V)$ into a $*$-algebra.

This $*$-algebra admits various norms. The naive choice would be the sup norm or L^∞ norm, since obviously, $\mathscr{S}(V) \subset C_0(V)$. But this is a C^*-norm for the usual pointwise product, not the Weyl-Moyal star product. However, the above construction shows that there is a natural C^*-norm (which by general nonsense is unique), namely the operator norm for the above *Weyl quantization map* W_σ: the completion of $\mathscr{S}(V)$ with respect to this C^*-norm will be denoted by $\mathcal{E}_\sigma$. Observing that this is still a C^*-algebra without unit, the final step consists in passing to its *multiplier algebra* $\mathcal{H}_\sigma = M(\mathcal{E}_\sigma)$: these are the algebras that we propose to call the *Heisenberg C^*-algebra*, *without unit* or *with unit*, respectively, (associated to V^* and σ). It is then clear that π_σ yields an embedding of the Heisenberg group H_σ into the group of unitaries of the Heisenberg C^*-algebra $\mathcal{H}_\sigma$.

For later use, we note that $\mathcal{E}_\sigma$ contains two natural dense subspaces which do not depend on σ: one is of course the Schwartz space $\mathscr{S}(V)$ that we started with, while the other is its completion $\check{L}^1(V)$ with respect to the $\check{L}^1$-norm, that is, the L^1-norm of the inverse Fourier transform: due to the inequalities $\|f\|_\infty \leqslant \|\check{f}\|_1$ and $\|W_\sigma f\| \leqslant \|\check{f}\|_1$ which follow immediately from equations (2.10) and (2.9), this norm induces a finer topology on $\mathscr{S}(V)$ than that of $C_0(V)$ and also than that of $\mathcal{E}_\sigma$, so that by completion we get the inclusions $\check{L}^1(V) \subset C_0(V)$ and $\check{L}^1(V) \subset \mathcal{E}_\sigma$. Moreover, with respect to its own norm (and with respect to the Weyl-Moyal star product and the standard involution of pointwise complex conjugation), $\check{L}^1(V)$ is even a Banach $*$-algebra, and $\mathcal{E}_\sigma$ is its C^*-completion.

Turning to representations of the various mathematical entities involved, we mention first of all that representations of $\mathcal{H}_\sigma$ correspond uniquely to non-degenerate representations of $\mathcal{E}_\sigma$.[4] Similarly, strongly continuous unitary representations of H_σ correspond

[4] A representation of an algebra A on a vector space V is called non-degenerate if there is no non-zero vector in V that is annihilated by all elements of A. Obviously, if A has a unit, every representation of A is non-degenerate. Also, irreducible representations are always non-degenerate. Finally, it can be proved that any non-degenerate representation of an algebra A extends uniquely to a representation of its multiplier algebra $M(A)$.

uniquely to representations of $\mathfrak{h}_\sigma$ which in [3] are called regular: according to Nelson's theorem [10], these are representations by essentially skew adjoint operators with a common dense invariant domain of analytic vectors. Finally, it is obvious that the former induce the latter (just by restriction from $\mathcal{H}_\sigma$ to H_σ), while the converse follows from an argument similar to one already used above: given a strongly continuous unitary representation π of H_σ on some Hilbert space $\mathfrak{H}$, the linear map

$$\begin{array}{rccc} W_\pi : & \mathscr{S}(V) & \longrightarrow & B(\mathfrak{H}) \\ & f & \longmapsto & W_\pi f \end{array} \tag{2.14}$$

defined by

$$W_\pi f \;=\; \int_{V^*} d\xi \; \check{f}(\xi)\, \pi(\xi) \;, \tag{2.15}$$

extends to a representation of $\mathcal{E}_\sigma$ which, using an argument similar to the one that guarantees injectivity of the map (2.8), can be shown to be non-degenerate.

All these constructions seem to be well known when σ is *non-degenerate*: in that case, the representation π_σ is *irreducible* and, according to one of von Neumann's famous theorems, is the *unique* such representation, generally known as the *Schrödinger representation of the canonical commutation relations.*

In the degenerate case, i.e., when σ has a non-trivial null space, denoted by $\ker\sigma$, we can use the following trick: choose a subspace V' of V^* complementary to $\ker\sigma$, so that the restriction σ' of σ to $V'\times V'$ is non-degenerate, and introduce the corresponding Heisenberg algebra $\mathfrak{h}_{\sigma'} = V'\oplus\mathbb{R}$ and Heisenberg group $H_{\sigma'} = V'\times\mathbb{R}$ to decompose the original ones into the direct sum $\mathfrak{h}_\sigma = \ker\sigma\oplus\mathfrak{h}_{\sigma'}$ of two commuting ideals and $H_\sigma = \ker\sigma\times H_{\sigma'}$ of two commuting normal subgroups.[5] It follows that every (strongly continuous unitary) representation of H_σ is the tensor product of a (strongly continuous unitary) representation of $\ker\sigma$ and a (strongly continuous unitary) representation of $H_{\sigma'}$, where the first is irreducible if and only if each of the last two is irreducible. Now since $\ker\sigma$ is abelian, its irreducible representations are one-dimensional and given by their character, which proves the following

Theorem 1 (Classification of irreducible representations)**.** *With the notation above, the strongly continuous, unitary, irreducible representations of the Heisenberg group H_σ, or equivalently, the irreducible representations of the Heisenberg C^*-algebras without unit $\mathcal{E}_\sigma$ or with unit $\mathcal{H}_\sigma$, are classified by their* ***highest weight****, which is a vector v in V, or more precisely, its class $[v]$ in the quotient space $V/(\ker\sigma)^\perp$, such that*

$$\pi_{[v]}(\xi,\eta) \;=\; e^{i\langle\xi,v\rangle}\,\pi_{\sigma'}(\eta) \qquad \text{for} \;\; \xi\in\ker\sigma\,,\; \eta\in H_{\sigma'}\;,$$

where $\pi_{\sigma'}$ is of course the Schrödinger representation of $H_{\sigma'}$.

It should be noted, however, that the representation π_σ used to construct the Heisenberg C^*-algebras is of course very far from being irreducible.

[5] As is common practice in the abelian case, we consider the same vector space $\ker\sigma$ as an abelian Lie algebra in the first case and as an (additively written) abelian Lie group in the second case, so that the exponential map becomes the identity.

3 The DFR-Algebra for Poisson Vector Bundles

Let E be a Poisson vector bundle, i.e., a (smooth) real vector bundle of fiber dimension n, say, over a (smooth) manifold M, with typical fiber $\mathbb{E}$, equipped with a fixed (smooth) bivector field σ; in other words, the dual E^* of E is a (smooth) presymplectic vector bundle.[6] Then it is clear that we can apply all constructions of the previous section to each fiber. The question to be addressed in this section is how the results can be glued together along the base manifold M and to describe the resulting global objects.

Starting with the collection of Heisenberg algebras $\mathfrak{h}_{\sigma(m)}$ $(m \in M)$, we note first of all that these fit together into a (smooth) real vector bundle over M, which is just the direct sum of E^* and the trivial line bundle $M \times \mathbb{R}$ over M. The non-trivial part is the commutator, which is defined by equation (2.1) applied to each fiber, turning this vector bundle into a *totally intransitive Lie algebroid* [8, Def. 3.3.1, p. 100] which we shall call the *Heisenberg algebroid* associated to (E, σ) and denote by $\mathfrak{h}(E, \sigma)$: it will even be a *Lie algebra bundle* [8, Def. 3.3.8, p. 104] if and only if σ has constant rank. Of course, spaces of sections (with certain regularity properties) of $\mathfrak{h}(E, \sigma)$ will then form (infinite-dimensional) Lie algebras with respect to the (pointwise defined) commutator, but the correct choice of regularity conditions is a question of functional analytic nature to be dictated by the problem at hand.

Similarly, considering the collection of Heisenberg groups $H_{\sigma(m)}$ $(m \in M)$, we note that these fit together into a (smooth) real fiber bundle over M, which is just the fiber product of E^* and the trivial line bundle $M \times \mathbb{R}$. Again, the non-trivial part is the product, which is defined by equation (2.2) applied to each fiber, turning this fiber bundle into a *totally intransitive Lie groupoid* [8, Def. 1.1.3, p. 5 & Def. 1.5.9, p. 32] which we shall call the *Heisenberg groupoid* associated to (E, σ) and denote by $H(E, \sigma)$: it will even be a *Lie group bundle* [8, Def. 1.1.19, p. 11] if and only if σ has constant rank. And again, spaces of sections (with certain regularity properties) of $H(E, \sigma)$ will form (infinite-dimensional) Lie groups with respect to the (pointwise defined) product, but the correct choice of regularity conditions is a question of functional analytic nature to be dictated by the problem at hand.

An analogous strategy can be applied to the collection of Heisenberg C^*-algebras $\mathcal{E}_{\sigma(m)}$ and $\mathcal{H}_{\sigma(m)}$ $(m \in M)$, but the details are somewhat intricate since the fibers are now (infinite-dimensional) C^*-algebras which may depend on the base point in a discontinuous way, since the rank of σ is allowed to jump. Concretely, our goal is to fit the collections of Heisenberg C^*-algebras $\mathcal{E}_{\sigma(m)}$ and $\mathcal{H}_{\sigma(m)}$ into continuous fields of C^*-algebras over M [2, Def. 10.3.1, p. 218], [5, Def. 2.10, pp. 68-69] whose continuous sections (subject to appropriate decay or boundedness conditions at infinity) yield spaces $\mathscr{E}(E, \sigma)$ and $\mathscr{H}(E, \sigma)$

[6] Note that we do *not* require σ to be non-degenerate or even to have constant rank.

which are not only again C^*-algebras but also modules over the function algebras $C_0(M)$ and/or $C_b(M)$.[7]

To do so, we start by noting that there is a naturally defined smooth vector bundle over M which we shall denote by $\check{\mathcal{L}}^1$: its fibers are just the Banach spaces of $\check{L}^1$-functions on the fibers of E, i.e.

$$\check{\mathcal{L}}^1 \;=\; \dot{\bigcup_{m\in M}} \check{\mathcal{L}}^1_m \qquad \text{where} \qquad \check{\mathcal{L}}^1_m \;=\; \check{L}^1(E_m)\;. \tag{3.1}$$

(See [7] for the theory of vector bundles whose typical fiber is a fixed Banach space.) Note that as a vector bundle, $\check{\mathcal{L}}^1$ does *not* depend on σ and is locally trivial, but of course, the fiberwise Weyl-Moyal star product on this vector bundle does depend on σ and thus will be locally trivial, turning $\check{\mathcal{L}}^1$ into a smooth bundle of Banach $*$-algebras, if and only if σ has constant rank.[8]

But at any rate, it is clear that $\check{\mathcal{L}}^1$ is a continuous field of Banach $*$-algebras over M and that the space $C_0(\check{\mathcal{L}}^1)$ of continuous sections of $\check{\mathcal{L}}^1$ vanishing at infinity is a Banach $*$-algebra which we shall denote by $\mathscr{E}^0(E,\sigma)$, with norm given by

$$\|\varphi\|_{\mathscr{E}^0} \;=\; \sup_{m\in M} \|\varphi(m)\|_{\check{L}^1(E_m)} \qquad \text{for} \;\; \varphi\in\mathscr{E}^0(E,\sigma)\;. \tag{3.2}$$

Equally clear is that $\mathscr{E}^0(E,\sigma)$ is not only a Banach $*$-algebra but also a module over the function algebra $C_0(M)$ (and even over the function algebra $C_b(M)$), with

$$\|f\varphi\|_{\mathscr{E}^0} \;\leqslant\; \|f\|_\infty\,\|\varphi\|_{\mathscr{E}^0} \qquad \text{for} \;\; f\in C_b(M)\,,\; \varphi\in\mathscr{E}^0(E,\sigma)\;, \tag{3.3}$$

and for every point m in M, the evaluation map (Dirac delta function) at m,

$$\begin{array}{rccc} \delta_m: & \mathscr{E}^0(E,\sigma) = C_0(\check{\mathcal{L}}^1) & \longrightarrow & \check{L}^1(E_m) \\ & \varphi & \longmapsto & \varphi(m) \end{array} \tag{3.4}$$

is a Banach $*$-algebra homomorphism which, due to local triviality, is onto. Composing it with the Weyl quantization map $W_{\sigma(m)}$ (naturally extended from $\mathscr{S}(E_m)$ to $\check{L}^1(E_m)$, by continuity) gives a $*$-representation $W_m = W_{\sigma(m)}\circ\delta_m$ of $\mathscr{E}^0(E,\sigma)$ by bounded operators on some Hilbert space, and since the family of all these $*$-representations is separating (i.e., $\bigcap_{m\in M}\ker W_m = \{0\}$), it provides a C^*-norm on $\mathscr{E}^0(E,\sigma)$, explicitly given by

$$\begin{array}{c} \|\varphi\|_{\mathscr{E}} \;=\; \sup_{m\in M}\|\varphi(m)\|_{\mathcal{E}_{\sigma(m)}} \;=\; \sup_{m\in M}\|W_{\sigma(m)}(\varphi(m))\| \\ \text{for} \;\; \varphi\in\mathscr{E}^0(E,\sigma)\;. \end{array} \tag{3.5}$$

[7] As usual, we denote by $C_0(M)$ the commutative C^*-algebra of continuous functions on M that vanish at infinity and by $C_b(M)$ the commutative C^*-algebra of bounded continuous functions on M; as is well known, the latter is the multiplier algebra of the former.

[8] This is exactly the same situation as for the fiberwise commutator in the Heisenberg algebroid or the fiberwise product in the Heisenberg grupoid.

The completion of $\mathscr{E}^0(E,\sigma)$ with respect to this C^*-norm will be denoted by $\mathscr{E}(E,\sigma)$, and we have

$$\|\varphi\|_{\mathscr{E}} \;\leqslant\; \|\varphi\|_{\mathscr{E}^0} \qquad \text{for} \;\; \varphi \in \mathscr{E}^0(E,\sigma) \;, \tag{3.6}$$

as well as

$$\begin{gathered} \|\varphi\|_{\mathscr{E}} \;=\; \sup_{m\in M} \|\varphi(m)\|_{\mathcal{E}_{\sigma(m)}} \;=\; \sup_{m\in M} \|W_{\sigma(m)}(\varphi(m))\| \\ \text{for} \;\; \varphi \in \mathscr{E}(E,\sigma) \;. \end{gathered} \tag{3.7}$$

Again, $\mathscr{E}(E,\sigma)$ is not only a C^*-algebra but also a module over the function algebra $C_0(M)$ (and even over the function algebra $C_b(M)$), with

$$\|f\varphi\|_{\mathscr{E}} \;\leqslant\; \|f\|_\infty \, \|\varphi\|_{\mathscr{E}} \qquad \text{for} \;\; f \in C_b(M) \,,\, \varphi \in \mathscr{E}(E,\sigma) \;, \tag{3.8}$$

and for every point m in M, the evaluation map (Dirac delta function) at m,

$$\begin{array}{rccc} \delta_m : & \mathscr{E}(E,\sigma) & \longrightarrow & \mathcal{E}_{\sigma(m)} \\ & \varphi & \longmapsto & \varphi(m) \end{array} \tag{3.9}$$

is a C^*-algebra homomorphism which, having a dense image, is onto.[9] Finally, observing that $\mathscr{E}(E,\sigma)$ is still a C^*-algebra without unit, the final step consists in passing to its multiplier algebra $\mathscr{H}(E,\sigma) = M(\mathscr{E}(E,\sigma))$: then, once more, $\mathscr{H}(E,\sigma)$ is not only a C^*-algebra but also a module over the function algebra $C_0(M)$ (and even over the function algebra $C_b(M)$), with

$$\|f\varphi\|_{\mathscr{H}} \;\leqslant\; \|f\|_\infty \, \|\varphi\|_{\mathscr{H}} \qquad \text{for} \;\; f \in C_b(M) \,,\, \varphi \in \mathscr{H}(E,\sigma) \;. \tag{3.10}$$

and for every point m in M, the evaluation map (Dirac delta function) at m,

$$\begin{array}{rccc} \delta_m : & \mathscr{H}(E,\sigma) & \longrightarrow & \mathcal{H}_{\sigma(m)} \\ & \varphi & \longmapsto & \varphi(m) \end{array} \tag{3.11}$$

is a C^*-algebra homomorphism which, having a dense image, is onto.[10]

Noting that the module structure and the evaluation maps are related by the obvious formula

$$\begin{gathered} \delta_m(f\varphi) \;=\; f(m)\, \delta_m(\varphi) \\ \text{for} \;\; f \in C_0(M) \;\; \text{and} \;\; \varphi \in \mathscr{E}^0(E,\sigma) \text{ or } \mathscr{E}(E,\sigma) \text{ or } \mathscr{H}(E,\sigma) \;, \end{gathered} \tag{3.12}$$

we see that $(\mathscr{E}^0(E,\sigma), (\delta_m)_{m\in M}, M)$ is a field of Banach $*$-algebras over M whereas $(\mathscr{E}(E,\sigma), (\delta_m)_{m\in M}, M)$ and $(\mathscr{H}(E,\sigma), (\delta_m)_{m\in M}, M)$ are fields of C^*-algebras over M. As we have seen before, the first is certainly a *continuous* field of Banach $*$-algebras, while

[9] These statements are easily derived by noting that, for any $f \in C_b(M)$ and any point m in M, multiplication by f as a linear map from $\mathscr{E}^0(E,\sigma)$ to $\mathscr{E}^0(E,\sigma)$ and evaluation at m as a linear map from $\mathscr{E}^0(E,\sigma)$ to $\check{L}^1(E_m)$ are continuous not only in their own topologies, but, according to equation (3.5), also with respect to their C^*-norms: this implies that they admit the required unique continuous linear extensions.

[10] The argumentation that permits these extensions remains to be completed.

the question whether the last two are also *continuous* fields of C^*-algebras, which would provide the disjoint unions

$$\mathcal{E} \;=\; \dot{\bigcup_{m\in M}} \mathcal{E}_{\sigma(m)} \qquad \text{and} \qquad \mathcal{H} \;=\; \dot{\bigcup_{m\in M}} \mathcal{H}_{\sigma(m)} \tag{3.13}$$

with natural topologies so as to turn them into total spaces of Fell bundles such that $\mathscr{E}(E,\sigma)$ becomes the space of continuous sections of $\mathcal{E}$ that vanish at infinity and $\mathscr{H}(E,\sigma)$ becomes the space of bounded continuous sections of $\mathcal{H}$ [4, Sect. II.13.18, pp. 132-134], is still under investigation.

Inspired by the construction in [11], we propose to call the C^*-algebra modules $\mathscr{E}(E,\sigma)$ and $\mathscr{H}(E,\sigma)$ the *DFR-algebra* (associated to (E,σ)), *without unit* or *with unit*, respectively.

Concerning irreducible representations, we have

Theorem 2. *Every irreducible representation of $\mathscr{E}(E,\sigma)$, or equivalently, of $\mathscr{H}(E,\sigma)$, is of the form $\pi_{[v_m]} \circ \delta_m$, where m is some point in M and $[v_m]$ is the class of a vector $v_m \in E_m$ in the quotient space $E_m/\ker\sigma(m)^\perp$, as in Theorem 1.*

Proof. Let π be a irreducible representation of $\mathscr{E}(E,\sigma)$, which extends uniquely to an irreducible representation of its multiplier algebra $\mathscr{H}(E,\sigma)$. Then since $\mathscr{E}(E,\sigma)$ is a module over $C_b(M)$, we have a canonical C^*-algebra homomorphism

$$C_b(M) \;\longrightarrow\; Z\big(\mathscr{H}(E,\sigma)\big)$$

of $C_b(M)$ into the center $Z\big(\mathscr{H}(E,\sigma)\big)$ of $\mathscr{H}(E,\sigma)$. Thus π restricts to an irreducible representation of $C_b(M)$, which must be one-dimensional and given by a character. But the characters of $C_b(M)$ are the points of M, so there is a point $m \in M$ such that

$$\pi(f) \;=\; f(m)\,1 \qquad \text{for} \;\; f \in C_b(M)\;,$$

and hence

$$\pi(f\varphi) \;=\; f(m)\,\pi(\varphi) \qquad \text{for} \;\; f \in C_b(M)\,,\; \varphi \in \mathscr{E}(E,\sigma)\;,$$

which implies that π must vanish on the kernel of δ_m. Therefore, π factors over $\ker\delta_m$, that is, $\pi = \pi_m \circ \delta_m$ where π_m is an irreducible representation of $\mathcal{E}_{\sigma(m)}$. Now apply Theorem 1. □

4 Examples

4.1 Homogeneous Vector Bundles

A first important special case of the construction outlined in the previous section occurs when the underlying manifold M and Poisson vector bundle (E,σ) are homogeneous.

More specifically, assume that G is a Lie group which acts transitively (and properly) on M as well as on E and such that σ is G-invariant: this means that writing

$$\begin{array}{ccc} G \times M & \longrightarrow & M \\ (g,m) & \longmapsto & g \cdot m \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G \times E & \longrightarrow & E \\ (g,u) & \longmapsto & g \cdot u \end{array} \tag{4.1}$$

for the respective actions, where the latter is linear along the fibers and hence induces an action

$$\begin{array}{ccc} G \times \bigwedge^2 E & \longrightarrow & \bigwedge^2 E \\ (g,u) & \longmapsto & g \cdot u \end{array} \tag{4.2}$$

so that

$$\sigma(g \cdot m) \;=\; g \cdot \sigma(m) \qquad \text{for} \;\; g \in G \,,\, m \in M \;. \tag{4.3}$$

Choosing a reference point m_0 in M and denoting by H the stability group of m_0 in G, by $\mathbb{E}$ the fiber of E over m_0 and by σ_0 the value of the bivector field σ at m_0, we can identify M with the homogeneous space G/H, E with the vector bundle $G \times_H \mathbb{E}$ associated to G, viewed as a principal H-bundle over G/H, and the representation of H on $\mathbb{E}$ obtained from the action of G on E by appropriate restriction, and σ with the bivector field obtained from σ_0 by the association process. Explicitly, for example, we identify the left coset $gH \in G/H$ with the point $g \cdot m_0 \in M$ and the equivalence class $[g, u_0] = [gh, h^{-1} \cdot u_0] \in G \times_H \mathbb{E}$ with the vector $g \cdot u_0 \in E$. As a result, we see that if the representation of H on $\mathbb{E}$ extends to a representation of G, then the associated bundle $G \times_H \mathbb{E}$ is globally trivial: an explicit trivialization is given by

$$\begin{array}{ccc} G \times_H \mathbb{E} & \longrightarrow & M \times \mathbb{E} \\ [g, u_0] = [gh, h^{-1} \cdot u_0] & \longmapsto & (gH, g^{-1} \cdot u_0) \end{array} \tag{4.4}$$

Of course, in this case, G-invariance implies that σ has constant rank and hence the Heisenberg algebroid becomes a Lie algebra bundle, the Heisenberg groupoid becomes a Lie group bundle and the fields of C^*-algebras $(\mathscr{E}(E,\sigma), (\delta_m)_{m \in M}, M)$ and $(\mathscr{H}(E,\sigma), (\delta_m)_{m \in M}, M)$ over M are associated to locally trivial bundles of C^*-algebras $\mathcal{E}$ and $\mathcal{H}$ over M (see equation (3.13)). And if the representation of H on $\mathbb{E}$ extends to a representation of G, all these bundles will even be globally trivial.

This special situation prevails in the case of the original DFR-algebra, where $\mathbb{E}$ is four-dimensional Minkowski space $\mathbb{R}^{1,3}$, G is (possibly up to a two-fold covering) the Lorentz group $\mathrm{O}(1,3)$, H is its intersection with the symplectic group $\mathrm{Sp}(4,\mathbb{R})$, so H is an abelian Lie group of type $\mathbb{R} \times \mathrm{U}(1)$, and σ_0 is the standard symplectic form, defined by the matrix $\left(\begin{smallmatrix} 0 & 1_2 \\ -1_2 & 0 \end{smallmatrix}\right)$.

4.2 Poisson Manifolds

Let M be a Poisson manifold with Poisson tensor σ. In this case, it is natural to assume E to be the tangent bundle TM of M. Here, we have one piece of additional information:

the requirement that the Poisson tensor should be integrable (i.e., that the Schouten bracket of σ with itself should vanish) implies that M admits a *foliation* into symplectic submanifolds, called its *symplectic leaves* [13]. Then since the Poisson tensor has constant (and maximal) rank along each leaf, it is clear from the construction of the previous section that, upon restriction to each leaf S, the Heisenberg algebroid becomes a Lie algebra bundle, the Heisenberg groupoid becomes a Lie group bundle and the fields of C^*-algebras $(\mathscr{E}(TM|_S, \sigma_S), (\delta_s)_{s\in S}, S)$ and $(\mathscr{H}(TM|_S, \sigma_S), (\delta_s)_{s\in S}, S)$ over S are associated to locally trivial bundles of C^*-algebras $\mathcal{E}_S$ and $\mathcal{H}_S$ over S.

5 Outlook

Our first goal when starting this investigation was to find an appropriate mathematical setting for geometrical generalizations of the DFR model. Here, we report on first results in this direction, but there are of course various steps that still have to be taken, for example:

- Establish closer contact between the Heisenberg Lie algebroid and Heisenberg groupoid on the one hand and the fields of C^*-algebras that give rise to the DFR-algebra on the other hand. This will most likely require extending the concepts of unitaries and of affiliated unbounded elements from C^*-algebras to fields of C^*-algebras and/or C^*-modules over $C_0(M)$ and/or $C_b(M)$.
- The same goes for the concept of states.
- Another important question is how this construction relates to *deformation quantization*, both with regard to its formal version, such as Fedosov's construction for symplectic manifolds or Kontsevich's theorem for Poisson maifolds, and with regard to Rieffel's strict deformation quantization.

We are fully aware of the fact that these questions are predominantly of mathematical nature: the physical interpretation is quite another matter. But to a certain extent this applies even to the original DFR-model, since it is not clear how to extend the interpretation of the commutation relations postulated in [3], in terms of uncertainty relations, to other space-time manifolds, or even to Minkowski space in dimensions $\neq 4$. In addition, it should not be forgotten that, even classically, space-time coordinates are *not* observables: this means that the basic axiom of algebraic quantum field theory according to which observables should be described by (local) algebras of a certain kind (such as C^*-algebras or von Neumann algebras) does *not at all* imply that in quantum gravity one should replace classical space-time coordinate functions by noncommuting operators. To us, the basic question seems to be: *How can we formulate space-time uncertainty relations, in the sense of obstructions to the possibility of localizing events with arbitrary precision, in terms of* ***observables****?* That of course stirs up the question: *How do we actually* ***measure*** *the geometry of space-time when quantum effects become strong?*

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